Journal of Sound and Vibration 329 (2010) 2603–2615

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

A general solution procedure for the forced vibrations of a system with cubic nonlinearities: Three-to-one internal resonances with external excitation

B. Burak Özhan, Mehmet Pakdemirli *

Department of Mechanical Engineering, Celal Bayar University 45140, Muradiye, Manisa, Türkiye

ARTICLE INFO

Article history:
Received 11 July 2009
Received in revised form
11 December 2009
Accepted 8 January 2010
Handling editor: M.P. Cartmell
Available online 9 February 2010

ABSTRACT

A general vibrational model of a continuous system with arbitrary linear and cubic operators is considered. Approximate analytical solutions are found using the method of multiple scales. The primary resonances of the external excitation and three-to-one internal resonances between two arbitrary natural frequencies are treated. The amplitude and phase modulation equations are derived. The steady-state solutions and their stability are discussed. The solution algorithm is applied to two specific problems: (1) axially moving Euler–Bernoulli beam, and (2) axially moving viscoelastic beam.

1. Introduction

Many nonlinear models for vibrations of continuous systems contain cubic nonlinearities. Nonlinearities add to the complexity of the systems, which make exact analytical solutions impossible to obtain. Approximate analytical solutions are sought for such system as a second best alternative and perturbation methods are widely used for this task. The nonlinearities may appear in a variety of forms including algebraic, differential or integral structures. Regardless of the specific forms of the nonlinearities, they can be classified with respect to their common natures such as quadratic or cubic nonlinearities. To understand and analyze the effects of arbitrary quadratic and cubic nonlinearities on the solutions, an operator notation suitable for perturbation analysis was developed [1]. The motivation behind the study was to compare the direct-perturbation methods with discretization–perturbation methods. The discussion was limited to single mode approximations of free vibrations. Later the analysis was generalized to infinite number of modes for forced vibrations [2]. The advantages of direct-perturbation methods were discussed in detail. Comparison of both methods for a parametically excited linear system expressed by arbitrary linear operators was also done [3]. It is concluded that while infinite mode results agree with each other, finite mode truncations of both methods yield different results with direct-perturbation method producing more precise solutions. In order to compare results of different versions of method of multiple scales and decide which method yields better steady-state solutions, an arbitrary cubic nonlinear system was treated [4]. For coupled partial differential systems, a solution procedure for one-to-one internal resonances was developed [5]. Using the same model of [5], possible internal resonances were classified [6]. For a general cubic nonlinear system, three-to-one internal resonances were further considered [7]. Two-to-one internal resonances were analysed for arbitrary quadratic

* Corresponding author. Tel.: +90 236 2412146; fax: +90 236 2412143.
E-mail addresses: mpak@bayar.edu.tr, pakdemirli@gmail.com, pakdemirli@yahoo.com (M. Pakdemirli).

0022-460X/$ - see front matter
doi:10.1016/j.jsvi.2010.01.010

nonlinearities [8]. The new operator notation developed and used in the previous studies [1–8] was adopted to understand the effects of nonlinearities by others also (see [9,10] for example).

In a very recent work [11], additional linear and cubic operators with time derivatives were incorporated into the model. In contrast to the mentioned previous studies with cubic nonlinearities, which employ only spatial operators, additional linear and cubic operators containing spatial as well as time derivatives were included. This enables analyzing a more general class of continuous systems such as gyroscopic systems, which are encountered in axially moving media or pipes conveying fluids.

In this work, the work of [11] is extended to include three-to-one internal resonances. A general solution procedure is developed for arbitrary operators first. Method of multiple scales, a perturbation technique is used in the analysis. Amplitude and phase modulation equations, steady-state solutions and their stability are discussed in a general sense. The solution algorithm is applied to two different problems: (1)nonlinear vibrations of stretched axially moving Euler–Bernoulli beam, and (2) nonlinear vibrations of axially moving viscoelastic beam. While the first application contain integro-differential type and the second differential type, both models possess the common feature of being cubic nonlinear, which enables the algorithm developed to be applied directly to these equations. The application problems are discussed in detail. Stability analysis is presented and frequency-response curves are drawn to depict the effects of various parameters on the vibrations of the system. Energy transfer between the modes are displayed in the figures. Apart from the examples considered, the general solution procedure developed may be applied to a wide range of physical problems.

2. Equation of motion

The dimensionless model considered is

$$\ddot{w}+\mathbf{L}_1(w)+\mathbf{L}_2(\dot{w})+\varepsilon\mathbf{L}_3(\dot{w})=\varepsilon F\cos\Omega t+\varepsilon\{\mathbf{C}_1(w,w,w)+\mathbf{C}_2(\dot{w},w,w)\} \tag{1}$$

$$\mathbf{B}_1(w)=0 \quad x=0, \quad \mathbf{B}_2(w)=0 \quad x=1 \tag{2}$$

with dependent variable $w(x, t)$ representing deflection, and independent variables x and t being the spatial and time variables, respectively. Note that there may be more than one spatial variable and a 3-D problem in spatial variables x, y and z has not been excluded from the analysis. $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ are linear differential and/or integral operators. $\mathbf{C}_1$ and $\mathbf{C}_2$ are cubic nonlinear operators. F and Ω represents external excitation amplitude and external excitation frequency, respectively. $\mathbf{B}_1$ and $\mathbf{B}_2$ are linear operators of boundary conditions. The representation of boundary conditions should be expressed in a modified form for a 3-D problem.ε is a small dimensionless physical parameter. Dot denotes partial differentiation with respect to time. To capture the effects of gyroscopic systems, additional linear and cubic operators (i.e. $\mathbf{L_2}$, $\mathbf{L_3}$ and $\mathbf{C_2}$) containing time derivatives are included in the model.

Note that model (1) is a fairly general model and any vibration problem that can be cast into the formalism of Eq. (1) can be solved approximately through the algorithm developed in the following analysis. A restriction of the boundary value problem comes from the boundary conditions, i.e. they are assumed to be linear. Furthermore, operator $\mathbf{L_1}$ is symmetric and $\mathbf{L_2}$ is skew symmetric with respect to boundary conditions. This introduces some simplifications when calculating solvability conditions [12]. If the specific problem contains nonlinear boundary conditions or the operators $\mathbf{L_1}$ and $\mathbf{L_2}$ do not possess the mentioned properties with respect to boundary conditions, the general solution algorithm cannot be directly applied to it. This case needs further analysis since the solvability condition brings more terms for nonlinear boundary conditions, which are hard to express in a general way. Although the boundary conditions given here represent a 1-D problem such as normalized length, in fact the solution algorithm is more general than that and can be successfully applied to 2 or 3-D problems in spatial variables. Note that both equations of motion and boundary conditions should be expressed first in a non-dimensional form for applications.

The cubic operators $\mathbf{C_1}$ and $\mathbf{C_2}$ may not be symmetric with respect to the inner variables and possesses the property of being multilinear. See Ref. [11] for details.

3. Perturbation solution

The method of multiple scales [13] is applied directly to the model to find the general solution of Eq. (1). The following expansion for $w(x, t)$ is assumed

$$w(x,T_0,T_1;\varepsilon)=w_0(x,T_0,T_1)+\varepsilon w_1(x,T_0,T_1)+\cdots \tag{3}$$

where T_0=t is the usual fast time scale and T_1=εt is the slow time scale. Time derivatives are expressed in terms of fast and slow time scales as follows

$$\frac{d}{dt}=D_0+\varepsilon D_1+\cdots \tag{4}$$

$$\frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0D_1+\cdots \tag{5}$$

where $D_k=\partial/\partial T_k$. Inserting Eqs. (3)–(5) into Eqs. (1) and (2) and separating at each order of ε, one obtains

$$\underline{O(\varepsilon^0)}D_0^2w_0+\mathbf{L}_1(w_0)+\mathbf{L}_2(D_0w_0)=0 \tag{6}$$

$$\mathbf{B}_1(w_0)=0 \text{ at } x=0, \quad \mathbf{B}_2(w_0)=0 \text{ at } x=1 \tag{7}$$

$$\underline{O(\varepsilon^1)}D_0^2w_1+\mathbf{L}_1(w_1)+\mathbf{L}_2(D_0w_1)=-2D_0D_1w_0-\mathbf{L}_2(D_1w_0)-\mathbf{L}_3(D_0w_0)+F\cos\Omega T_0+\mathbf{C}_1(w_0,w_0,w_0)+\mathbf{C}_2(D_0w_0,w_0,w_0) \tag{8}$$

$$\mathbf{B}_1(w_1)=0 \text{ at } x=0, \quad \mathbf{B}_2(w_1)=0 \text{ at } x=1 \tag{9}$$

At $O(\varepsilon^0)$, two arbitrary modes of frequencies ω_n and ω_m ($\omega_m>\omega_n$) are assumed to interact through internal resonances

$$w_0(x,T_0,T_1)=A_n(T_1)e^{i\omega_nT_0}Y_n(x)+\overline{A}_n(T_1)e^{-i\omega_nT_0}\overline{Y}_n(x)+A_m(T_1)e^{i\omega_mT_0}Y_m(x)+\overline{A}_m(T_1)e^{-i\omega_mT_0}\overline{Y}_m(x) \tag{10}$$

where A_n and $\overline{A}_n$ are complex amplitudes and their conjugates, respectively. $Y_n(x)$ satisfy the following equations and boundary conditions,

$$\mathbf{L}_1(Y_n)-\omega_n^2Y_n+i\omega_n\mathbf{L}_2(Y_n)=0, \quad n=1,2,\ldots \tag{11}$$

$$\mathbf{B}_1(Y_n)=0 \text{ at } x=0, \quad \mathbf{B}_2(Y_n)=0 \text{ at } x=1 \tag{12}$$

Due to the dissipative term at the zeroth order, $Y_n(x)$ may not be real and the complex conjugate of the function is incorporated in the zeroth-order solution (10). The above equation and boundary conditions constitute an eigenvalue–eigenfunction problem. ω_n are the eigenvalues and $Y_n(x)$ are the eigenfunctions of the system, respectively. For continuous system, it is clear that there are infinite number of eigenvalues and corresponding eigenfunctions. Note that under some circumstances more complex mode interactions may occur for the system considered. This work, however, is limited to external excitation of one of the modes and excitation of another mode via 3:1 internal resonance.

Upon substitution of solution (10) to the right-hand side of (8) yields

$$\begin{aligned}
D_0^2w_1+\mathbf{L}_1(w_1)+\mathbf{L}_2(D_0w_1)=&-2D_1(i\omega_nA_ne^{i\omega_nT_0}Y_n-i\omega_n\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+i\omega_mA_me^{i\omega_mT_0}Y_m-i\omega_m\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m)\\
&-\mathbf{L}_2(D_1(A_ne^{i\omega_nT_0}Y_n+\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+A_me^{i\omega_mT_0}Y_m+\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m))\\
&-\mathbf{L}_3(i\omega_nA_ne^{i\omega_nT_0}Y_n-i\omega_n\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+i\omega_mA_me^{i\omega_mT_0}Y_m-i\omega_m\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m)+\frac{1}{2}F(e^{i\Omega T_0}+e^{-i\Omega T_0})\\
&+\mathbf{C}_1(A_ne^{i\omega_nT_0}Y_n+\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+A_me^{i\omega_mT_0}Y_m+\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m,\\
&A_ne^{i\omega_nT_0}Y_n+\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+A_me^{i\omega_mT_0}Y_m+\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m, A_ne^{i\omega_nT_0}Y_n+\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+A_me^{i\omega_mT_0}Y_m+\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m)\\
&+\mathbf{C}_2(i\omega_nA_ne^{i\omega_nT_0}Y_n-i\omega_n\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+i\omega_mA_me^{i\omega_mT_0}Y_m-i\omega_m\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m,\\
&A_ne^{i\omega_nT_0}Y_n+\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+A_me^{i\omega_mT_0}Y_m+\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m, A_ne^{i\omega_nT_0}Y_n+\overline{A}_ne^{-i\omega_nT_0}\overline{Y}_n+A_me^{i\omega_mT_0}Y_m+\overline{A}_me^{-i\omega_mT_0}\overline{Y}_m)
\end{aligned} \tag{13}$$

$$\mathbf{B}_1(w_1)=0 \text{ at } x=0, \quad \mathbf{B}_2(w_1)=0 \text{ at } x=1 \tag{14}$$

Under the assumed resonances

$$\Omega=\omega_n+\varepsilon\sigma_n \tag{15}$$

$$\omega_m=3\omega_n+\varepsilon\rho_n \tag{16}$$

where the external excitation frequency is near to the nth natural frequency and the mth natural frequency is excited via 3:1 internal resonance. σ_n and ρ_n are detuning parameters of order 1. Using the multilinearity properties of the cubic operators with the resonance conditions, Eq. (13) resumes the form

$$\begin{aligned}
D_0^2w_1+\mathbf{L}_1(w_1)+\mathbf{L}_2(D_0w_1)=&-2i\omega_nD_1A_ne^{i\omega_nT_0}Y_n-2i\omega_mD_1A_me^{i\omega_mT_0}Y_m-D_1A_ne^{i\omega_nT_0}\mathbf{L}_2(Y_n)-D_1A_me^{i\omega_mT_0}\mathbf{L}_2(Y_m)\\
&-i\omega_nA_ne^{i\omega_nT_0}\mathbf{L}_3(Y_n)-i\omega_mA_me^{i\omega_mT_0}\mathbf{L}_3(Y_m)+\frac{1}{2}Fe^{i\omega_nT_0}e^{i\sigma_nT_1}\\
&+A_n^3e^{i\omega_mT_0}e^{-i\rho_nT_1}\{\mathbf{C}_1(Y_n,Y_n,Y_n)+i\omega_n\mathbf{C}_2(Y_n,Y_n,Y_n)\}\\
&+\overline{A}_n^2A_me^{i\omega_nT_0}e^{i\rho_nT_1}\{\mathbf{C}_1(Y_m,\overline{Y}_n,\overline{Y}_n)+\mathbf{C}_1(\overline{Y}_n,Y_m,\overline{Y}_n)\\
&+\mathbf{C}_1(\overline{Y}_n,\overline{Y}_n,Y_m)-i\omega_n(\mathbf{C}_2(\overline{Y}_n,Y_m,\overline{Y}_n)+\mathbf{C}_2(\overline{Y}_n,\overline{Y}_n,Y_m))+i\omega_m\mathbf{C}_2(Y_m,\overline{Y}_n,\overline{Y}_n)\}\\
&+A_n^2\overline{A}_ne^{i\omega_nT_0}\{\mathbf{C}_1(Y_n,Y_n,\overline{Y}_n)+\mathbf{C}_1(Y_n,\overline{Y}_n,Y_n)+\mathbf{C}_1(\overline{Y}_n,Y_n,Y_n)\\
&+i\omega_n(\mathbf{C}_2(Y_n,Y_n,\overline{Y}_n)+\mathbf{C}_2(Y_n,\overline{Y}_n,Y_n)-\mathbf{C}_2(\overline{Y}_n,Y_n,Y_n))\}+A_m^2\overline{A}_me^{i\omega_mT_0}\{\mathbf{C}_1(Y_m,Y_m,\overline{Y}_m)\\
&+\mathbf{C}_1(Y_m,\overline{Y}_m,Y_m)+\mathbf{C}_1(\overline{Y}_m,Y_m,Y_m)+i\omega_m(\mathbf{C}_2(Y_m,Y_m,\overline{Y}_m)+\mathbf{C}_2(Y_m,\overline{Y}_m,Y_m)-\mathbf{C}_2(\overline{Y}_m,Y_m,Y_m))\}\\
&+A_n\overline{A}_nA_me^{i\omega_mT_0}\{\mathbf{C}_1(Y_n,\overline{Y}_n,Y_m)+\mathbf{C}_1(Y_n,Y_m,\overline{Y}_n)+\mathbf{C}_1(Y_m,Y_n,\overline{Y}_n)\\
&+\mathbf{C}_1(\overline{Y}_n,Y_n,Y_m)+\mathbf{C}_1(\overline{Y}_n,Y_m,Y_n)+\mathbf{C}_1(Y_m,\overline{Y}_n,Y_n)+i\omega_n(\mathbf{C}_2(Y_n,\overline{Y}_n,Y_m)+\mathbf{C}_2(Y_n,Y_m,\overline{Y}_n)-\mathbf{C}_2(\overline{Y}_n,Y_n,Y_m)\\
&-\mathbf{C}_2(\overline{Y}_n,Y_m,Y_n))+i\omega_m(\mathbf{C}_2(Y_m,Y_n,\overline{Y}_n)+\mathbf{C}_2(Y_m,\overline{Y}_n,Y_n))\}\\
&+A_nA_m\overline{A}_me^{i\omega_nT_0}\{\mathbf{C}_1(Y_m,\overline{Y}_m,Y_n)+\mathbf{C}_1(Y_m,Y_n,\overline{Y}_m)+\mathbf{C}_1(Y_n,Y_m,\overline{Y}_m)\\
&+\mathbf{C}_1(\overline{Y}_m,Y_m,Y_n)+\mathbf{C}_1(\overline{Y}_m,Y_n,Y_m)+\mathbf{C}_1(Y_n,\overline{Y}_m,Y_m)\\
&+i\omega_n(\mathbf{C}_2(Y_n,Y_m,\overline{Y}_m)+\mathbf{C}_2(Y_n,\overline{Y}_m,Y_m))+i\omega_m(\mathbf{C}_2(Y_m,\overline{Y}_m,Y_n)+\mathbf{C}_2(Y_m,Y_n,\overline{Y}_m)-\mathbf{C}_2(\overline{Y}_m,Y_m,Y_n)\\
&-\mathbf{C}_2(\overline{Y}_m,Y_n,Y_m))\}+\mathrm{cc}+\mathrm{NST}
\end{aligned} \tag{17}$$

where cc stands for complex conjugates of the preceding terms and NST stands for non-secular terms. A solution for w_1 may be assumed in the form

$$w_1(x,T_0,T_1) = \varphi_n(x,T_1)e^{i\omega_n T_0} + \varphi_m(x,T_1)e^{i\omega_m T_0} + \text{cc} + W(x,T_0,T_1) \tag{18}$$

where $W(x, T_0, T_1)$ represents solution associated with non-secular terms and φ_n and φ_m represent solutions associated with secular terms. Substituting solution (18) into Eq. (17) and (14) yields after separation to relevant modes

$$\begin{aligned}&-\omega_n^2\varphi_n + \mathbf{L}_1(\varphi_n) + i\omega_n\mathbf{L}_2(\varphi_n) = -2i\omega_n D_1A_nY_n - D_1A_n\mathbf{L}_2(Y_n) - i\omega_nA_n\mathbf{L}_3(Y_n)\\&+\frac{1}{2}Fe^{i\sigma_nT_1} + \overline{A}_n^2A_me^{i\rho_nT_1}\{\mathbf{C}_1(Y_m,\overline{Y}_n,\overline{Y}_n) + \mathbf{C}_1(\overline{Y}_n,Y_m,\overline{Y}_n) + \mathbf{C}_1(\overline{Y}_n,\overline{Y}_n,Y_m) - i\omega_n(\mathbf{C}_2(\overline{Y}_n,Y_m,\overline{Y}_n)\\&+\mathbf{C}_2(\overline{Y}_n,\overline{Y}_n,Y_m)) + i\omega_m\mathbf{C}_2(Y_m,\overline{Y}_n,\overline{Y}_n)\} + A_n^2\overline{A}_n\{\mathbf{C}_1(Y_n,Y_n,\overline{Y}_n) + \mathbf{C}_1(Y_n,\overline{Y}_n,Y_n)\\&+\mathbf{C}_1(\overline{Y}_n,Y_n,Y_n) + i\omega_n(\mathbf{C}_2(Y_n,Y_n,\overline{Y}_n) + \mathbf{C}_2(Y_n,\overline{Y}_n,Y_n) - \mathbf{C}_2(\overline{Y}_n,Y_n,Y_n))\}\\&+A_nA_m\overline{A}_m\{\mathbf{C}_1(Y_m,\overline{Y}_m,Y_n) + \mathbf{C}_1(Y_m,Y_n,\overline{Y}_m) + \mathbf{C}_1(Y_n,Y_m,\overline{Y}_m)\\&+\mathbf{C}_1(\overline{Y}_m,Y_m,Y_n) + \mathbf{C}_1(\overline{Y}_m,Y_n,Y_m) + \mathbf{C}_1(Y_n,\overline{Y}_m,Y_m) + i\omega_n(\mathbf{C}_2(Y_n,Y_m,\overline{Y}_m) + \mathbf{C}_2(Y_n,\overline{Y}_m,Y_m))\\&+i\omega_m(\mathbf{C}_2(Y_m,\overline{Y}_m,Y_n) + \mathbf{C}_2(Y_m,Y_n,\overline{Y}_m) - \mathbf{C}_2(\overline{Y}_m,Y_m,Y_n) - \mathbf{C}_2(\overline{Y}_m,Y_n,Y_m))\}\end{aligned} \tag{19}$$

$$\mathbf{B}_1(\varphi_n) = 0 \quad x=0 \quad \mathbf{B}_2(\varphi_n) = 0 \quad x=1 \tag{20}$$

$$\begin{aligned}&-\omega_m^2\varphi_m + \mathbf{L}_1(\varphi_m) + i\omega_m\mathbf{L}_2(\varphi_m) = -2i\omega_m D_1A_mY_m - D_1A_m\mathbf{L}_2(Y_m) - i\omega_mA_m\mathbf{L}_3(Y_m)\\&+A_n^3e^{-i\rho_nT_1}\{\mathbf{C}_1(Y_n,Y_n,Y_n) + i\omega_n\mathbf{C}_2(Y_n,Y_n,Y_n)\} + A_m^2\overline{A}_m\{\mathbf{C}_1(Y_m,Y_m,\overline{Y}_m) + \mathbf{C}_1(Y_m,\overline{Y}_m,Y_m)\\&+\mathbf{C}_1(\overline{Y}_m,Y_m,Y_m) + i\omega_m(\mathbf{C}_2(Y_m,Y_m,\overline{Y}_m) + \mathbf{C}_2(Y_m,\overline{Y}_m,Y_m) - \mathbf{C}_2(\overline{Y}_m,Y_m,Y_m))\}\\&+A_n\overline{A}_nA_m\{\mathbf{C}_1(Y_n,\overline{Y}_n,Y_m) + \mathbf{C}_1(Y_n,Y_m,\overline{Y}_n) + \mathbf{C}_1(Y_m,Y_n,\overline{Y}_n) + \mathbf{C}_1(\overline{Y}_n,Y_n,Y_m)\\&+\mathbf{C}_1(\overline{Y}_n,Y_m,Y_n) + \mathbf{C}_1(Y_m,\overline{Y}_n,Y_n) + i\omega_n(\mathbf{C}_2(Y_n,\overline{Y}_n,Y_m) + \mathbf{C}_2(Y_n,Y_m,\overline{Y}_n) - \mathbf{C}_2(\overline{Y}_n,Y_n,Y_m)\\&-\mathbf{C}_2(\overline{Y}_n,Y_m,Y_n)) + i\omega_m(\mathbf{C}_2(Y_m,Y_n,\overline{Y}_n) + \mathbf{C}_2(Y_m,\overline{Y}_n,Y_n))\}\end{aligned} \tag{21}$$

$$\mathbf{B}_1(\varphi_m) = 0 \quad x=0 \quad \mathbf{B}_2(\varphi_m) = 0 \quad x=1 \tag{22}$$

Since the homogenous parts of Eqs. (19) and (21) have non-trivial solutions, non-homogenous equations have a solution only if a solvability condition is satisfied [13]. For the present model, with $\mathbf{L_1}$ symmetric and $\mathbf{L_2}$ skew symmetric with respect to linear boundary conditions (See the discussions in [12]), the solvability condition is

$$D_1A_n + k_{1n}A_n - f_ne^{i\sigma_nT_1} - k_{2n}A_n^2\overline{A}_n - k_{3nm}\overline{A}_n^2A_me^{i\rho_nT_1} - k_{4nm}A_nA_m\overline{A}_m = 0 \tag{23}$$

$$D_1A_m + k_{1m}A_m - k_{2m}A_m^2\overline{A}_m - k_{5mn}A_n^3e^{-i\rho_nT_1} - k_{4mn}A_n\overline{A}_nA_m = 0 \tag{24}$$

Note that if the mth mode is excited through external excitation, the complex amplitude modulation Eqs. (23) and (24) should be slightly modified with the forcing term appearing in (24) rather than in (23). The coefficients are

$$k_{1n} = \frac{i\omega_n\int_0^1\overline{Y}_n\mathbf{L}_3(Y_n)\,dx}{2i\omega_n\int_0^1Y_n\overline{Y}_n\,dx + \int_0^1\overline{Y}_n\mathbf{L}_2(Y_n)\,dx} \tag{25}$$

$$f_n = \frac{\frac{1}{2}\int_0^1F\overline{Y}_n\,dx}{2i\omega_n\int_0^1Y_n\overline{Y}_n\,dx + \int_0^1\overline{Y}_n\mathbf{L}_2(Y_n)\,dx} \tag{26}$$

$$k_{2n} = \frac{\int_0^1\overline{Y}_n\{\mathbf{C}_1(Y_n,Y_n,\overline{Y}_n) + \mathbf{C}_1(Y_n,\overline{Y}_n,Y_n) + \mathbf{C}_1(\overline{Y}_n,Y_n,Y_n) + i\omega_n(\mathbf{C}_2(Y_n,Y_n,\overline{Y}_n) + \mathbf{C}_2(Y_n,\overline{Y}_n,Y_n) - \mathbf{C}_2(\overline{Y}_n,Y_n,Y_n))\}\,dx}{2i\omega_n\int_0^1Y_n\overline{Y}_n\,dx + \int_0^1\overline{Y}_n\mathbf{L}_2(Y_n)\,dx} \tag{27}$$

$$k_{3nm} = \frac{\int_0^1\overline{Y}_n\{\mathbf{C}_1(Y_m,\overline{Y}_n,\overline{Y}_n) + \mathbf{C}_1(\overline{Y}_n,Y_m,\overline{Y}_n) + \mathbf{C}_1(\overline{Y}_n,\overline{Y}_n,Y_m) - i\omega_n(\mathbf{C}_2(\overline{Y}_n,Y_m,\overline{Y}_n) + \mathbf{C}_2(\overline{Y}_n,\overline{Y}_n,Y_m)) + i\omega_m\mathbf{C}_2(Y_m,\overline{Y}_n,\overline{Y}_n)\}\,dx}{2i\omega_n\int_0^1Y_n\overline{Y}_n\,dx + \int_0^1\overline{Y}_n\mathbf{L}_2(Y_n)\,dx} \tag{28}$$

$$k_{4nm} = \frac{\begin{aligned}&\int_0^1\overline{Y}_n\{\mathbf{C}_1(Y_m,\overline{Y}_m,Y_n) + \mathbf{C}_1(Y_m,Y_n,\overline{Y}_m) + \mathbf{C}_1(Y_n,Y_m,\overline{Y}_m) + \mathbf{C}_1(\overline{Y}_m,Y_m,Y_n) + \mathbf{C}_1(\overline{Y}_m,Y_n,Y_m)\\&+\mathbf{C}_1(Y_n,\overline{Y}_m,Y_m) + i\omega_n(\mathbf{C}_2(Y_n,Y_m,\overline{Y}_m) + \mathbf{C}_2(Y_n,\overline{Y}_m,Y_m)) + i\omega_m(\mathbf{C}_2(Y_m,\overline{Y}_m,Y_n)\\&+\mathbf{C}_2(Y_m,Y_n,\overline{Y}_m) - \mathbf{C}_2(\overline{Y}_m,Y_m,Y_n) - \mathbf{C}_2(\overline{Y}_m,Y_n,Y_m))\}\,dx\end{aligned}}{2i\omega_n\int_0^1Y_n\overline{Y}_n\,dx + \int_0^1\overline{Y}_n\mathbf{L}_2(Y_n)\,dx} \tag{29}$$

$$k_{5mn} = \frac{\int_0^1\overline{Y}_m\{\mathbf{C}_1(Y_n,Y_n,Y_n) + i\omega_n\mathbf{C}_2(Y_n,Y_n,Y_n)\}}{2i\omega_m\int_0^1Y_m\overline{Y}_m\,dx + \int_0^1\overline{Y}_m\mathbf{L}_2(Y_m)\,dx} \tag{30}$$

All coefficients have real and imaginary parts. Representing the complex amplitudes in polar form

$$A_n = \frac{1}{2}a_n(T_1)e^{i\beta_n(T_1)} \tag{31}$$

substituting into Eqs. (23) and (24), separating into real and imaginary parts, one finally has the amplitude and phase modulation equations

$$a_n' = -a_n k_{1nR} + 2(f_{nR}\cos\gamma_n - f_{nI}\sin\gamma_n) + \frac{1}{4}a_n^3 k_{2nR} + \frac{1}{4}a_n^2 a_m(k_{3nmR}\cos\lambda_{nm} - k_{3nmI}\sin\lambda_{nm}) + \frac{1}{4}a_n a_m^2 k_{4nmR} = F_1(a_n, a_m, \lambda_{nm}, \gamma_n) \tag{32}$$

$$a_m' = -a_m k_{1mR} + \frac{1}{4}a_m^3 k_{2mR} + \frac{1}{4}a_n^3(k_{5mnR}\cos\lambda_{nm} + k_{5mnI}\sin\lambda_{nm}) + \frac{1}{4}a_n^2 a_m k_{4mnR} = F_2(a_n, a_m, \lambda_{nm}, \gamma_n) \tag{33}$$

$$\gamma_n' = \sigma_n + k_{1nI} - \frac{2}{a_n}(f_{nR}\sin\gamma_n + f_{nI}\cos\gamma_n) - \frac{1}{4}a_n^2 k_{2nI} - \frac{1}{4}a_n a_m(k_{3nmR}\sin\lambda_{nm} + k_{3nmI}\cos\lambda_{nm}) - \frac{1}{4}a_m^2 k_{4nmI} = F_3(a_n, a_m, \lambda_{nm}, \gamma_n) \tag{34}$$

$$\begin{aligned}\lambda_{nm}' = {} & \rho_n - k_{1mI} + \frac{1}{4}a_m^2 k_{2mI} - \frac{1}{4}\frac{a_n^3}{a_m}(k_{5mnR}\sin\lambda_{nm} - k_{5mnI}\cos\lambda_{nm}) + \frac{1}{4}a_n^2 k_{4mnI} + 3k_{1nI} - \frac{6}{a_n}(f_{nR}\sin\gamma_n + f_{nI}\cos\gamma_n) \\ & - \frac{3}{4}a_n^2 k_{2nI} - \frac{3}{4}a_n a_m(k_{3nmR}\sin\lambda_{nm} + k_{3nmI}\cos\lambda_{nm}) - \frac{3}{4}a_m^2 k_{4nmI} = F_4(a_n, a_m, \lambda_{nm}, \gamma_n)\end{aligned} \tag{35}$$

where

$$\lambda_{nm} = \rho_n T_1 - 3\beta_n + \beta_m \tag{36}$$

$$\gamma_n = \sigma_n T_1 - \beta_n \tag{37}$$

For all relevant coefficients, the last subscript R denotes real part and I denotes imaginary part of these coefficients, respectively. For steady-state solutions, $a_n' = a_m' = \lambda_{nm}' = \gamma_n' = 0$

$$a_n k_{1nR} - 2(f_{nR}\cos\gamma_n - f_{nI}\sin\gamma_n) - \frac{1}{4}a_n^3 k_{2nR} - \frac{1}{4}a_n^2 a_m(k_{3nmR}\cos\lambda_{nm} - k_{3nmI}\sin\lambda_{nm}) - \frac{1}{4}a_n a_m^2 k_{4nmR} = 0 \tag{38}$$

$$a_n\sigma_n + a_n k_{1nI} - 2(f_{nR}\sin\gamma_n + f_{nI}\cos\gamma_n) - \frac{1}{4}a_n^3 k_{2nI} - \frac{1}{4}a_n^2 a_m(k_{3nmR}\sin\lambda_{nm} + k_{3nmI}\cos\lambda_{nm}) - \frac{1}{4}a_n a_m^2 k_{4nmI} = 0 \tag{39}$$

$$a_m k_{1mR} - \frac{1}{4}a_m^3 k_{2mR} - \frac{1}{4}a_n^3(k_{5mnR}\,cos\,\lambda_{nm} + k_{5mnI}\,sin\,\lambda_{nm}) - \frac{1}{4}a_n^2 a_m k_{4mnR} = 0 \tag{40}$$

$$a_m(3\sigma_n - \rho_n) + a_m k_{1mI} - \frac{1}{4}a_m^3 k_{2mI} + \frac{1}{4}a_n^3(k_{5mnR}\sin\lambda_{nm} - k_{5mnI}\cos\lambda_{nm}) - \frac{1}{4}a_n^2 a_m k_{4mnI} = 0 \tag{41}$$

From (40), if k_{5mn}=0, a natural consequence is a_m=0, and the second mode cannot be excited. Hence in addition to the necessary condition of mth mode being near to three times the nth mode, for 3:1 internal resonances to occur in such systems, the sufficiency condition is that

$$k_{5mn} = \frac{\int_0^1 \overline{Y}_m\{\mathbf{C}_1(Y_n, Y_n, Y_n) + i\omega_n \mathbf{C}_2(Y_n, Y_n, Y_n)\}}{2i\omega_m\int_0^1 Y_m\overline{Y}_m\,dx + \int_0^1 \overline{Y}_m \mathbf{L}_2(Y_m)\,dx} \neq 0 \tag{42}$$

The Jacobian matrix is evaluated to determine the stability of fixed points

$$\left.\begin{bmatrix} \frac{\partial F_1}{\partial a_n} & \frac{\partial F_1}{\partial a_m} & \frac{\partial F_1}{\partial \lambda_{nm}} & \frac{\partial F_1}{\partial \gamma_n} \\ \frac{\partial F_2}{\partial a_n} & \frac{\partial F_2}{\partial a_m} & \frac{\partial F_2}{\partial \lambda_{nm}} & \frac{\partial F_2}{\partial \gamma_n} \\ \frac{\partial F_3}{\partial a_n} & \frac{\partial F_3}{\partial a_m} & \frac{\partial F_3}{\partial \lambda_{nm}} & \frac{\partial F_3}{\partial \gamma_n} \\ \frac{\partial F_4}{\partial a_n} & \frac{\partial F_4}{\partial a_m} & \frac{\partial F_4}{\partial \lambda_{nm}} & \frac{\partial F_4}{\partial \gamma_n} \end{bmatrix}\right|_{\substack{a_n = a_{n0} \\ a_m = a_{m0} \\ \lambda_{nm} = \lambda_{nm0} \\ \gamma_n = \gamma_{n0}}} \tag{43}$$

Eigenvalues of the Jacobian matrix should not have positive real parts for stability. The approximate solution of the system is

$$w(x,t;\varepsilon)a_n\{\cos(\Omega t - \gamma_n)Y_{nR} - \sin(\Omega t - \gamma_n)Y_{nI}\} + a_m\{\cos(3\Omega t - 3\gamma_n + \lambda_{nm})Y_{mR} - \sin(3\Omega t - 3\gamma_n + \lambda_{nm})Y_{mI}\} + O(\varepsilon) \tag{44}$$

where Y_n can be decomposed into its real and imaginary parts. a_n, a_m, γ_n and λ_{nm} in the approximate solution are governed by Eqs. (32)–(35). Hence, for the general problem, an approximate solution algorithm is developed. The algorithm will be applied to two specific problems in the next section. Note that the approximate solution developed can trivially be extended to three spatial dimensions by expressing the eigenfunctions in three dimensions. A restriction of the solution is the involvement of only two modes linked to each other via 3:1 internal resonance and natural frequencies should be checked to avoid other modes of involvement through some kind of internal resonances.

4. Applications

In this section, the general solution algorithm will be applied to two specific vibration problems. Both problems are from axially moving continua. See references [14–22] for some example studies on axially moving continua vibrations. One of the problems contains integro-differential type nonlinearity and the other differential type nonlinearity. Although the nonlinearities are much different in nature, their common feature of being cubic nonlinearity makes them suitable applications for the general model considered.

4.1. Axially moving Euler–Bernoulli beam

For an axially moving Euler–Bernoulli beam, following [11,14], the dimensionless equation of motion is

$$\ddot{w}+(v_0^2-1)w''+2v_0\dot{w}'+v_f^2w^{IV}+\varepsilon\mu\dot{w}=\varepsilon F\cos\Omega t+\frac{1}{2}\varepsilon v_\ell^2 w''\int_0^1 w'^2\,dx \tag{45}$$

$$w(0,t)=w(1,t)=w''(0,t)=w''(1,t)=0 \tag{46}$$

where

$$w=\frac{w^*}{L},x=\frac{x^*}{L},t=t^*\sqrt{P/\rho AL^2}\quad v_0=v_0^*/\sqrt{P/\rho A},v_\ell=\sqrt{EA/P},v_f=\sqrt{EI/PL^2} \tag{47}$$

x^* is the coordinate along the axial direction, t^* is the time, L is the length, P is the axial tension force, EI is the flexural rigidity, A is the area of beam, ρ is the density, and w^* is the transverse displacement. Variables with asterisk denote dimensional quantities. Viscous damping and harmonic excitation are added to the equation with introducing a book-keeping small parameter ε to re-order the relative quantities of the terms.

For some example studies on axially moving Euler–Bernoulli beams see [14–18]. For this special problem, the operators are defined to be as follows

$$\mathbf{L}_1(w)=(v_0^2-1)w''+v_f^2w^{IV} \tag{48}$$

$$\mathbf{L}_2(\dot{w})=2v_0\dot{w}' \tag{49}$$

$$\mathbf{L}_3(\dot{w})=\mu\dot{w} \tag{50}$$

$$\mathbf{C}_1(w,w,w)=\frac{1}{2}v_\ell^2w''\int_0^1 w'^2\,dx \tag{51}$$

$$\mathbf{C}_2(\dot{w},w,w)=0 \tag{52}$$

The associated eigenfunction–eigenvalue problem given in Eqs. (11) and (12) reduces to

$$v_f^2Y_n^{IV}+(v_0^2-1)Y_n''+2v_0i\omega_nY_n'-\omega_n^2Y_n=0 \tag{53}$$

$$Y_n(0)=Y_n(1)=Y_n''(0)=Y_n''(1)=0 \tag{54}$$

The solution is [11,14]

$$\begin{aligned}Y_n(x)=c_1\Bigg\{&e^{i\beta_{1n}x}-\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})}e^{i\beta_{2n}x}-\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})}e^{i\beta_{3n}x}\\&+\left(-1+\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})}+\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})}\right)e^{i\beta_{4n}x}\Bigg\}\end{aligned} \tag{55}$$

where β_{in} satisfy the dispersive relation

$$v_f^2\beta_{in}^4+(1-v_0^2)\beta_{in}^2-2v_0\omega_n\beta_{in}-\omega_n^2=0\quad i=1,2,3,4\ldots,\quad n=1,2,\ldots. \tag{56}$$

associated with the support condition [11,14] as follows

$$\begin{aligned}&(e^{i(\beta_{1n}+\beta_{2n})}+e^{i(\beta_{3n}+\beta_{4n})})(\beta_{1n}^2-\beta_{2n}^2)(\beta_{3n}^2-\beta_{4n}^2)+(e^{i(\beta_{1n}+\beta_{3n})}+e^{i(\beta_{2n}+\beta_{4n})})(\beta_{2n}^2-\beta_{4n}^2)(\beta_{3n}^2-\beta_{1n}^2)\\&+(e^{i(\beta_{2n}+\beta_{3n})}+e^{i(\beta_{1n}+\beta_{4n})})(\beta_{1n}^2-\beta_{4n}^2)(\beta_{2n}^2-\beta_{3n}^2)=0\end{aligned} \tag{57}$$

ω_n and β_{in} can be numerically calculated using the dispersive relation and the support condition.

In Fig. 1, the fundamental frequency versus the axial transport velocity graphics are shown for different flexural stiffness values. Natural frequencies increase with increase in flexural stiffness values.

The next step is to calculate the coefficients in the amplitude and phase modulation equations. Substituting the operators (48)–(52) into (25)–(30) yields

$$k_{1n}=\frac{i\omega_n\mu\int_0^1 Y_n\overline{Y}_n\,dx}{2i\omega_n\int_0^1 Y_n\overline{Y}_n\,dx+2v_0\int_0^1 Y_n'\,\overline{Y}_n\,dx} \tag{58}$$

$$f_n=\frac{\frac{1}{2}\int_0^1 F\overline{Y}_n\,dx}{2i\omega_n\int_0^1 Y_n\overline{Y}_n\,dx+2v_0\int_0^1 Y_n'\,\overline{Y}_n\,dx} \tag{59}$$

$$k_{2n}=\frac{\frac{1}{2}v_\ell^2\{2\int_0^1\overline{Y}_nY_n''\,dx\int_0^1 Y_n'\,\overline{Y}_n'\,dx+\int_0^1\overline{Y}_n\overline{Y}_n''\,dx\int_0^1 Y_n'^2\,dx\}}{2i\omega_n\int_0^1 Y_n\overline{Y}_n\,dx+2v_0\int_0^1 Y_n'\,\overline{Y}_n\,dx} \tag{60}$$

$$k_{3nm}=\frac{\frac{1}{2}v_\ell^2\{\int_0^1\overline{Y}_nY_m''\,dx\int_0^1\overline{Y}_n'^2\,dx+2\int_0^1\overline{Y}_n\overline{Y}_n''\,dx\int_0^1 Y_m'\,\overline{Y}_n'\,dx\}}{2i\omega_n\int_0^1 Y_n\overline{Y}_n\,dx+2v_0\int_0^1 Y_n'\,\overline{Y}_n\,dx} \tag{61}$$

$$k_{4nm}=\frac{v_\ell^2\{\int_0^1\overline{Y}_nY_m''\,dx\int_0^1\overline{Y}_m'\,Y_n'\,dx+\int_0^1\overline{Y}_nY_n''\,dx\int_0^1 Y_m'\,\overline{Y}_m'\,dx+\int_0^1\overline{Y}_n\overline{Y}_m''\,dx\int_0^1 Y_m'\,Y_n'\,dx\}}{2i\omega_n\int_0^1 Y_n\overline{Y}_n\,dx+2v_0\int_0^1 Y_n'\,\overline{Y}_n\,dx} \tag{62}$$

$$k_{5mn}=\frac{\frac{1}{2}v_\ell^2\{\int_0^1\overline{Y}_mY_n''\,dx\int_0^1 Y_n'^2\,dx\}}{2i\omega_m\int_0^1 Y_m\overline{Y}_m\,dx+2v_0\int_0^1 Y_m'\,\overline{Y}_m\,dx} \tag{63}$$

Substitution of $Y_n(x)$ further into these equations yields the numerical values for the coefficients. For v_0=0.4, and v_f=0.2, the first two frequencies (n=1, m=2) are ω_1=3.39841, ω_2=9.78513, which are approximately 3:1 in ratio. If v_ℓ=0.2, μ=0.1, F=10 is further selected, the coefficients are k_{11}=0.04725+0.00001i, k_{12}=0.05010+0.00007i, k_{21}=0.02034+1.95820i, k_{22}=−0.15044+12.26200i, k_{312}=0.23640+0.03050i, k_{412}=0.02505+6.35310i, k_{421}=0.02165+2.20570i, k_{521}=−0.01297−0.02022i, and f_1=−0.36369+0.14938i.

In Fig. 2, the frequency-response curves for the first two modes are depicted. Through external excitation, energy is transferred to the first mode. Some of the energy gained is transferred further from the first mode to the second mode via 3:1 internal resonance. The energy transfer region is zoomed over the figure to show the details. In Fig. 3a, only the response of the first mode and in Fig. 3b, only that of the second mode is given in detail. Stable (solid) and unstable (dashed) regions can be seen better in the zoomed regions.

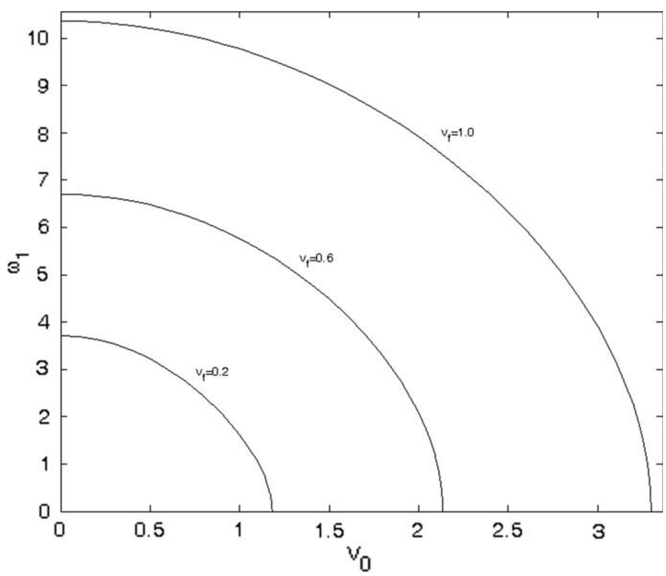

Fig. 1. Fundamental frequencies versus axial transport velocities for various flexural stiffness values.

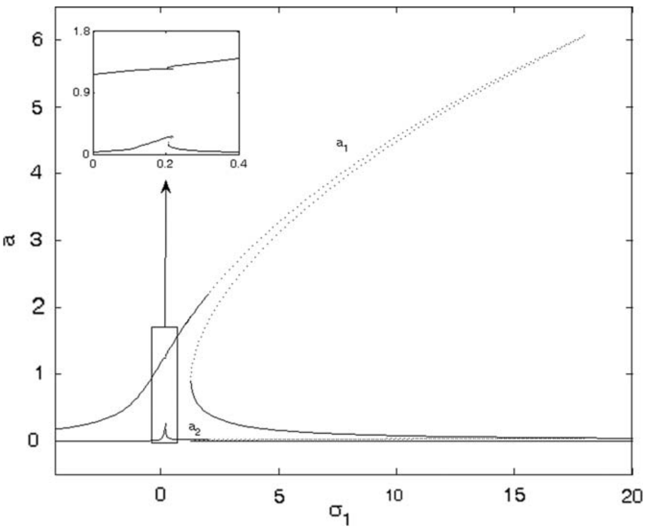

Fig. 2. Frequency-response curves for the first two modes (v_0=0.4, v_f=0.2, v_ℓ=0.2, μ=0.1, F=10, ω_1=3.39841, and ω_2=9.78513).

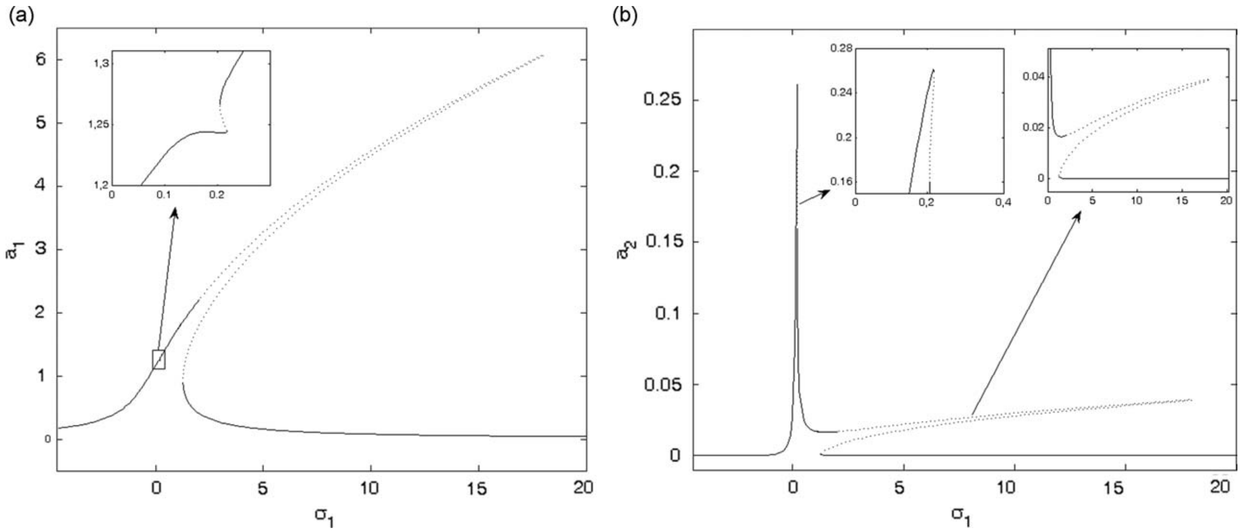

Fig. 3. (a) Frequency-response curve for the first mode (v_0=0.4, v_f=0.2, v_ℓ=0.2, μ=0.1, F=10, ω_1=3.39841, and ω_2=9.78513) and (b) frequency–response curve for the second mode (v_0=0.4, v_f=0.2, v_ℓ=0.2, μ=0.1, F=10, ω_1=3.39841, and ω_2=9.78513).

The plots shown in Figs. 4a and b demonstrate the influence of variation of the amplitude of the excitation on the amplitudes of the response with the frequency of the excitation held fixed (force-response curves, σ_1=2.5). In Fig. 4a, both modes are given in the same graph while in Fig. 4b, only the second mode response is given to show the details. Stable (solid) and unstable (dashed) solutions are shown on the graphs. Two saddle node bifurcation points one close to f_1=0 and the other greater than f_1=1 are observed.

The viscous damping coefficient is increased in Figs. 5–7 to μ=0.8. This affects the first two coefficients only, k_{11}=0.37797+0.00004i, and k_{12}=0.40084+0.00053i, with the remaining coefficients being identical. The frequency-response curve for the first mode is given in Fig. 5a and that of the second mode is given in Fig. 5b. All amplitudes decrease with an increase in viscosity.

In Fig. 6, force-response curves are given for the damped case for both modes of vibration. To see the details where jump occurs and stable and unstable solutions bifurcate, in Figs. 7a and b, the responses are shown seperately. The saddle node bifurcation points appear at f_1=0.4 and f_1=0.8 approximately. Note that within the parameter range considered, the response of the externally excited mode is much higher than that of the internally excited mode.

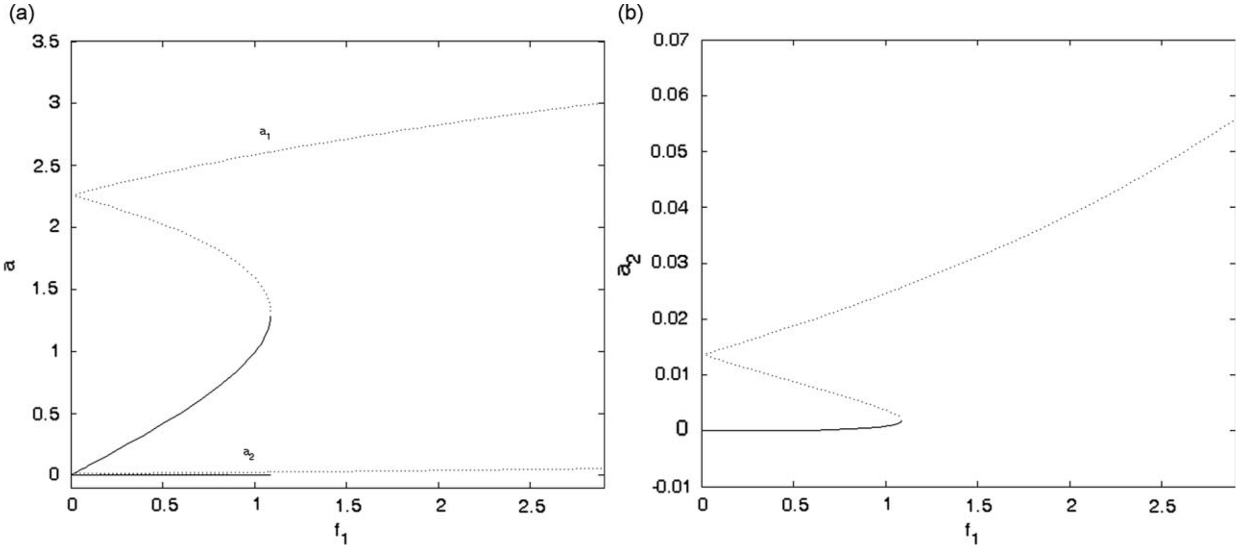

Fig. 4. (a) Force-response curves for the first and the second modes together (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.1, σ_1=2.5, ω_1=3.39841, and ω_2=9.78513) and (b) force response curve for the second mode only (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.1, σ_1=2.5, ω_1=3.39841, and ω_2=9.78513).

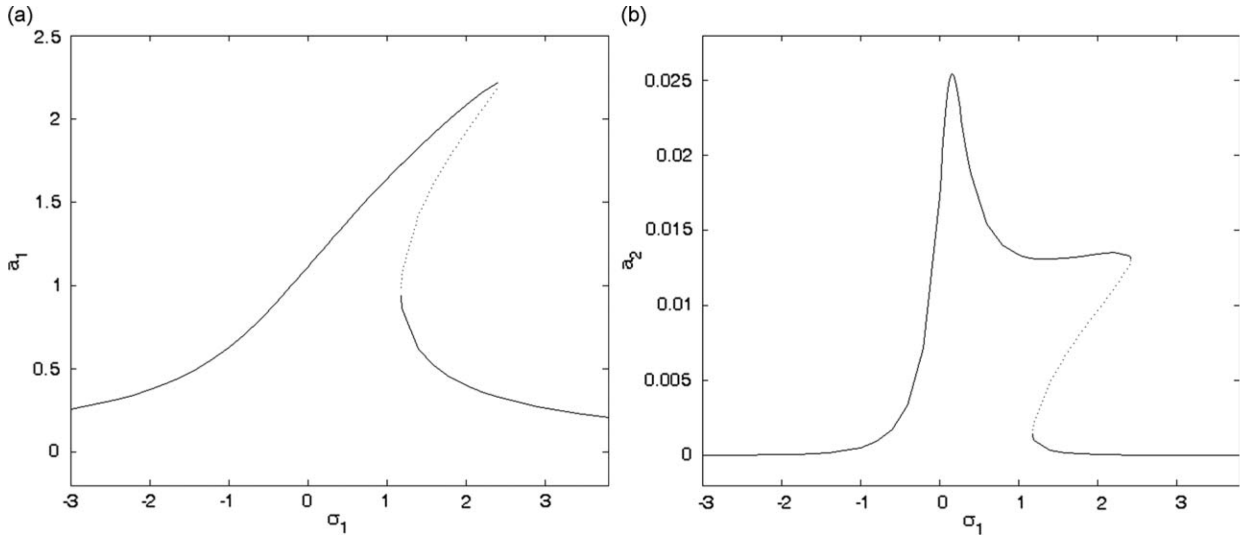

Fig. 5. (a) Frequency-response curve for the first mode (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.8, F=10, ω_1=3.39841, and ω_2=9.78513) and (b) frequency-response curve for the second mode (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.8, F=10, ω_1=3.39841, and ω_2=9.78513).

4.2. Axially moving viscoelastic beam

The second model represents nonlinear vibrations of an axially moving viscoelastic beam. Following [11,19], the equation of motion is

$$\ddot{w}+(\nu_0^2-1)w''+2\nu_0\dot{w}'+\nu_f^2w^{IV}+\varepsilon\alpha\dot{w}^{IV}+\varepsilon\mu\dot{w}=\varepsilon F\cos\Omega t+\varepsilon\left\{\frac{3}{2}\nu_\ell^2w''w'^2+2\alpha k\dot{w}'w'w''+\alpha k\dot{w}''w'^2\right\} \tag{64}$$

with boundary conditions

$$w(0,t)=w(1,t)=w''(0,t)=w''(1,t)=0 \tag{65}$$

where

$$\nu_\ell=\sqrt{\frac{EA}{P}},\quad \nu_f=\sqrt{\frac{EI}{PL^2}},\quad \alpha=\frac{I\eta}{L^3\sqrt{\rho AP}},\quad k=\frac{A\eta}{L\sqrt{\rho AP}} \tag{66}$$

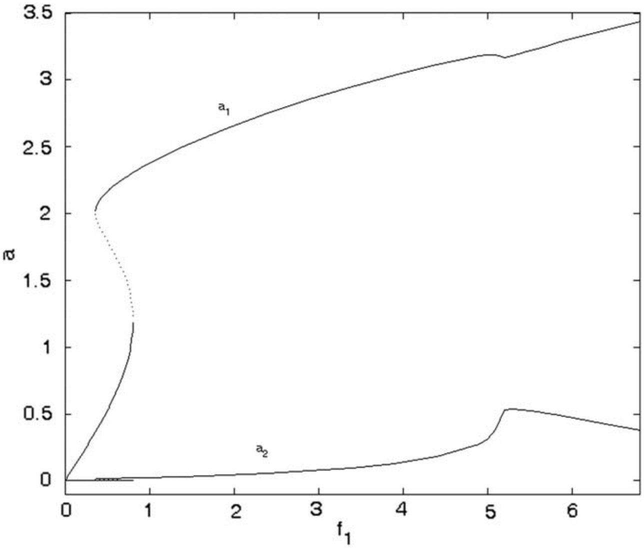

Fig. 6. Force-response curves for the first and the second modes together (υ_0=0.4, υ_f=0.2, υ_ℓ=0.2, μ=0.8, ω_1=3.39841, ω_2=9.78513, and σ_1=2).

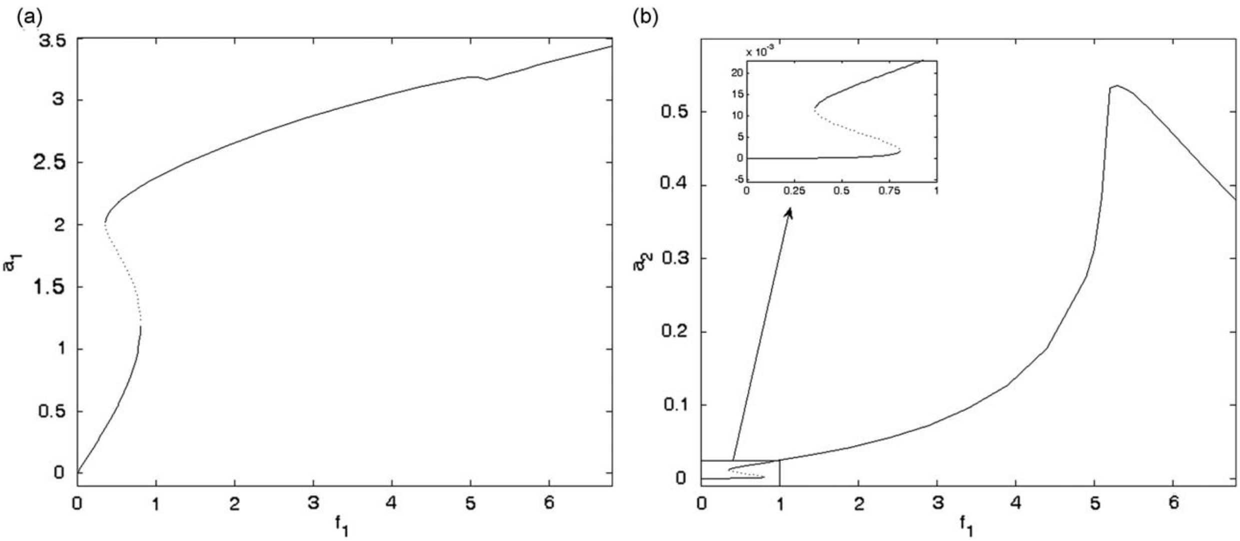

Fig. 7. (a) Force-response curve for the first mode only; υ_0=0.4, υ_f=0.2, υ_ℓ=0.2, μ=0.8, ω_1=3.39841, ω_2=9.78513, and σ_1=2 and (b) force response curve for the second mode only (υ_0=0.4, υ_f=0.2, υ_ℓ=0.2, μ=0.8, ω_1=3.39841, ω_2=9.78513, and σ_1=2).

η represents viscosity, and α and k are dimensionless parameters related to the viscosity. For similar studies on viscoelastic axially moving beams, the reader is referred to [19–22]. Here, the forced vibration case is considered whereas in [19–22] parametric resonances were considered.

For this specific problem, the general operators are defined as,

$$\mathbf{L}_1(w) = (\upsilon_0^2 - 1)w'' + \upsilon_f^2 w^{IV} \tag{67}$$

$$\mathbf{L}_2(\dot{w}) = 2\upsilon_0 \dot{w}' \tag{68}$$

$$\mathbf{L}_3(\dot{w}) = \alpha \dot{w}^{IV} + \mu \dot{w} \tag{69}$$

$$\mathbf{C}_1(w, w, w) = \frac{3}{2}\upsilon_\ell^2 w'' w'^2 \tag{70}$$

$$\mathbf{C}_2(\dot{w}, w, w) = 2\alpha k \dot{w}' w' w'' + \alpha k \dot{w}'' w'^2 \tag{71}$$

The associated eigenvalue–eigenfunction problem as given in Eqs. (11) and (12) reduces to

$$v_f^2 Y_n^{IV} + (v_0^2 - 1)Y_n'' + 2v_0 i\omega_n Y_n' - \omega_n^2 Y_n = 0 \tag{72}$$

$$Y_n(0) = Y_n(1) = Y_n''(0) = Y_n''(1) = 0 \tag{73}$$

for which the solution is

$$Y_n(x) = c_1 \left\{ e^{i\beta_{1n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})} e^{i\beta_{2n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})} e^{i\beta_{3n}x} \right.$$
$$\left. + \left(-1 + \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})} + \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})} \right) e^{i\beta_{4n}x} \right\} \tag{74}$$

where β_{in} satisfies the dispersive relation

$$v_f^2 \beta_{in}^4 + (1-v_0^2)\beta_{in}^2 - 2v_0\omega_n\beta_{in} - \omega_n^2 = 0 \quad i = 1,2,3,4\ldots, \quad n = 1,2,\ldots. \tag{75}$$

The support condition is also found similarly by application of the boundary conditions

$$(e^{i(\beta_{1n}+\beta_{2n})} + e^{i(\beta_{3n}+\beta_{4n})})(\beta_{1n}^2-\beta_{2n}^2)(\beta_{3n}^2-\beta_{4n}^2) + (e^{i(\beta_{1n}+\beta_{3n})} + e^{i(\beta_{2n}+\beta_{4n})})(\beta_{2n}^2-\beta_{4n}^2)(\beta_{3n}^2-\beta_{1n}^2)$$
$$+ (e^{i(\beta_{2n}+\beta_{3n})} + e^{i(\beta_{1n}+\beta_{4n})})(\beta_{1n}^2-\beta_{4n}^2)(\beta_{2n}^2-\beta_{3n}^2) = 0 \tag{76}$$

Numerical values of ω_n and β_{in} can be calculated by using the dispersive relation and the support condition in a similar way.

The coefficients of amplitude and phase modulation equations are found by substituting the operators (67)–(71) into (25)–(30)

$$k_{1n} = \frac{i\omega_n \int_0^1 (\alpha Y_n^{IV} + \mu Y_n)\overline{Y}_n\, dx}{2i\omega_n \int_0^1 Y_n \overline{Y}_n\, dx + 2v_0 \int_0^1 Y_n' \overline{Y}_n\, dx} \tag{77}$$

$$f_n = \frac{\frac{1}{2}\int_0^1 F\overline{Y}_n\, dx}{2i\omega_n \int_0^1 Y_n \overline{Y}_n\, dx + 2v_0 \int_0^1 Y_n' \overline{Y}_n\, dx} \tag{78}$$

$$k_{2n} = \frac{\frac{3}{2}v_\ell^2\{2\int_0^1 \overline{Y}_n Y_n'' Y_n \overline{Y}_n'\, dx + \int_0^1 \overline{Y}_n \overline{Y}_n'' Y_n'^2\, dx\} + i\omega_n\alpha k\{\int_0^1 \overline{Y}_n \overline{Y}_n'' Y_n'^2\, dx + 2\int_0^1 \overline{Y}_n Y_n'' Y_n' \overline{Y}_n'\, dx\}}{2i\omega_n \int_0^1 Y_n \overline{Y}_n\, dx + 2v_0 \int_0^1 Y_n' \overline{Y}_n\, dx} \tag{79}$$

$$k_{3nm} = \frac{\frac{3}{2}v_\ell^2\{\int_0^1 \overline{Y}_n Y_m'' \overline{Y}_n'^2\, dx + 2\int_0^1 \overline{Y}_n \overline{Y}_n'' Y_m' \overline{Y}_n'\, dx\} - 2i\omega_n\alpha k\{2\int_0^1 \overline{Y}_n \overline{Y}_n' \overline{Y}_n'' Y_m'\, dx + \int_0^1 \overline{Y}_n \overline{Y}_n'^2 Y_m''\, dx\} + i\omega_m\alpha k\{2\int_0^1 \overline{Y}_n \overline{Y}_n' \overline{Y}_n'' Y_m'\, dx + \int_0^1 \overline{Y}_n \overline{Y}_n'^2 Y_m''\, dx\}}{2i\omega_n \int_0^1 Y_n \overline{Y}_n\, dx + 2v_0 \int_0^1 Y_n' \overline{Y}_n\, dx} \tag{80}$$

$$k_{4nm} = \frac{\frac{3}{2}v_\ell^2\{2\int_0^1 \overline{Y}_n Y_m'' \overline{Y}_m' Y_n'\, dx + 2\int_0^1 \overline{Y}_n Y_n'' Y_m' \overline{Y}_m'\, dx + 2\int_0^1 \overline{Y}_n \overline{Y}_m'' Y_m' Y_n'\, dx\} + i\omega_n\alpha k\{2\int_0^1 \overline{Y}_n Y_n' Y_m' \overline{Y}_m''\, dx + 2\int_0^1 \overline{Y}_n Y_n'' Y_m' \overline{Y}_m'\, dx + 2\int_0^1 \overline{Y}_n Y_n' \overline{Y}_m' Y_m''\, dx\}}{2i\omega_n \int_0^1 Y_n \overline{Y}_n\, dx + 2v_0 \int_0^1 Y_n' \overline{Y}_n\, dx} \tag{81}$$

$$k_{5mn} = \frac{\frac{3}{2}v_\ell^2\{\int_0^1 \overline{Y}_m Y_n'' Y_n'^2\, dx\} + 3i\omega_n\alpha k\{\int_0^1 \overline{Y}_m Y_n'' Y_n'^2\, dx\}}{2i\omega_m \int_0^1 Y_m \overline{Y}_m\, dx + 2v_0 \int_0^1 Y_m' \overline{Y}_m\, dx} \tag{82}$$

By substituting $Y_n(x)$ further into these equations yields the numerical values for the coefficients. For v_0=0.4, and v_f=0.2, the first two frequencies are ω_1=3.39841, and ω_2=9.78513, which are approximately 3:1 in ratio. Selecting further α=.001, k=1, v_ℓ=0.2, μ=0.1, and F=10, the coefficients are k_{11}=0.10113+0.00082i, k_{12}=0.81902−0.00166i, k_{21}=−0.13657+2.95980i, k_{22}=−3.11260+19.16000i, k_{312}=0.98000−0.43575i, k_{412}=−0.67517+19.30800i, k_{421}=−1.01860+6.26380i, k_{521}=−0.04322−0.10754i, and f_1=−0.36369+0.14938i.

In Fig. 8, frequency-response curves are drawn. In Fig. 8a, both responses are shown on the same graph whereas, in Fig. 8b, response of the second mode is shown seperately to outline the details. Energy transfer from the excited first mode to the second mode via 3:1 internal resonance can be seen from the graphs.

The plots shown in Figs. 9a and b demonstrate the influence of the variation of the amplitude of the excitation on the amplitudes of the response with the frequency of the excitation held fixed (force–response curves, σ_1=2.5). In Fig. 9a, both modes are shown on the same graph and in Fig. 9b; only the energy transferred mode, which is the second mode is shown to outline the details. Note that, similar to the previous problem, within the parameter range considered, the response of the externally excited mode is always higher than that of the internally excited mode.

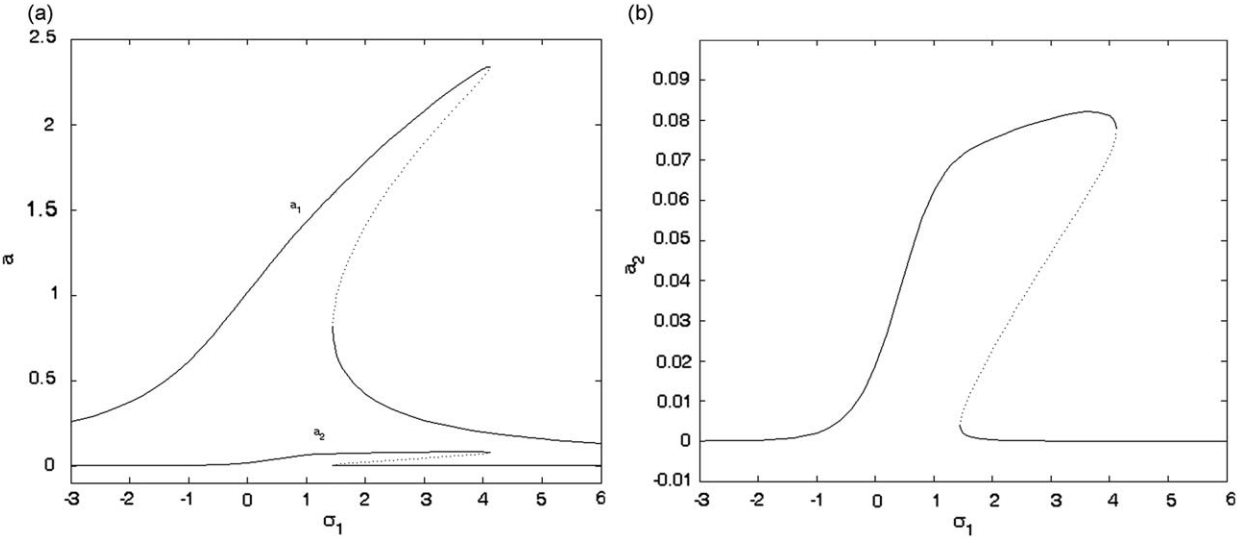

Fig. 8. (a) Frequency-response curves for the first and the second modes together (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.1, F=10, α=0.001, k=1, ω_1=3.39841, and ω_2=9.78513) and (b) frequency-response curve for the second mode only (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.1, F=10, α=0.001, k=1, ω_1=3.39841, and ω_2=9.78513).

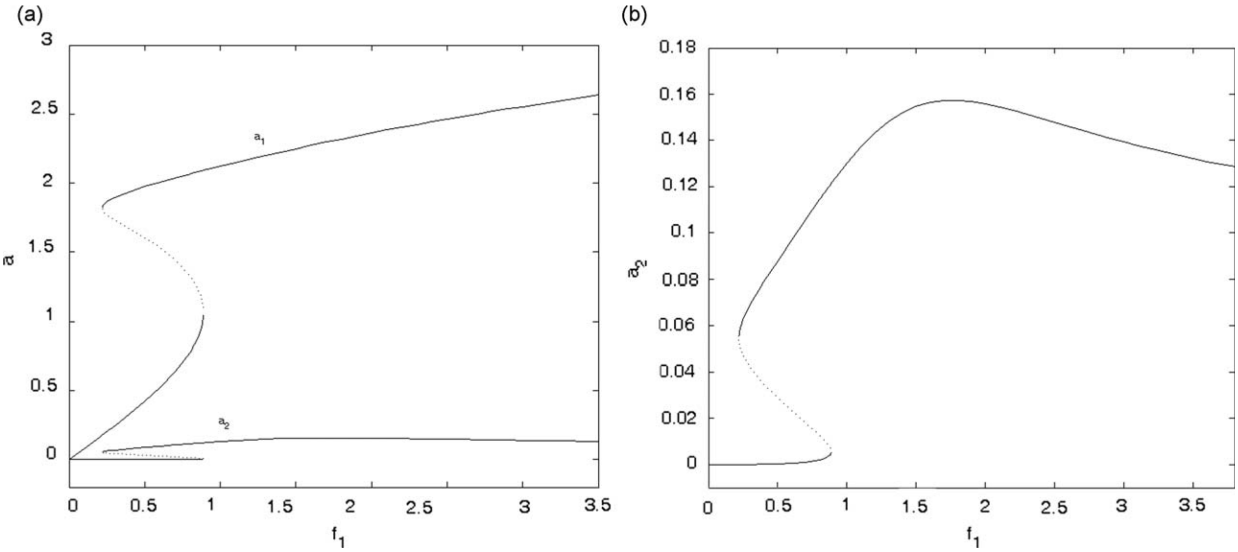

Fig. 9. (a) Force-response curves for the first and the second modes together (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.1, σ_1=2.5, α=0.001, k=1, ω_1=3.39841, and ω_2=9.78513) and (b) force-response curve for the second mode only (ν_0=0.4, ν_f=0.2, ν_ℓ=0.2, μ=0.1, σ_1=2.5, α=0.001, k=1, ω_1=3.39841, and ω_2=9.78513).

5. Concluding remarks

A general solution procedure is developed for vibrations of continuous systems with cubic nonlinearities. The arbitrary linear and cubic operators with spatial and time derivatives allow to generalize a wide range class of problems including gyroscopic systems. The approximate solutions, the amplitude and phase modulation equations are derived in terms of the operators. External excitation of one of the modes and excitation of another mode via 3:1 internal resonance case is considered. The method of multiple scales is employed in the analysis. Usually 3:1 ratio is satisfied between the natural modes yet the second mode cannot be excited via internal resonances. The sufficiency condition for such resonances to occur is derived for a general system. If the linear mode shapes and the form of the cubic nonlinearities are given, the existence of such internal resonances can easily be checked *a priori* from Eq. (42). The formalism developed is applied to two different problems namely the axially moving Euler–Bernoulli beam and the axially moving viscoelastic beam.

Frequency-response and force-response curves are presented and the energy transfer between the modes via internal resonance are outlined in the graphs. The algorithm developed may be applied to many more problems in nonlinear vibrations of continuous systems.

One further advantage of the study is that, mathematical models with cubic nonlinearities may be very complex. Especially, the nonlinear terms may be very lengthy and the algebra involved in search of approximate solutions increases tremendously for such systems. Here, in this study the solutions are given in a very compact form and the details of the specific calculations can be checked using our general solution and the general structure of the coefficients defined in the analysis.

References

[1] M. Pakdemirli, A comparison of two perturbation methods for vibrations of systems with quadratic and cubic nonlinearities, *Mechanics Research Communications* 21 (2) (1994) 203–208.
[2] M. Pakdemirli, H. Boyacı, Comparision of direct-perturbation methods with discretization-perturbation methods for nonlinear vibrations, *Journal of Sound and Vibration* 186 (1995) 837–845.
[3] M. Pakdemirli, H. Boyacı, The direct perturbation method versus the discretization perturbation method: Linear systems, *Journal of Sound and Vibration* 199 (5) (1997) 825–832.
[4] H. Boyacı, M. Pakdemirli, A comparison of different versions of the method of multiple scales for partial differential equations, *Journal of Sound and Vibration* 204 (4) (1997) 595–607.
[5] M. Pakdemirli, Vibrations of continuous systems with a general operator notation suitable for perturbative calculations, *Journal of Sound and Vibration* 246 (5) (2001) 841–851.
[6] M. Pakdemirli, A general solution procedure for coupled systems with arbitrary internal resonances, *Mechanics Research Communications* 28 (6) (2001) 522–617.
[7] M. Pakdemirli, E. Özkaya, Three-to-one internal resonances in a general cubic non-linear continuous system, *Journal of Sound and Vibration* 268 (2003) 543–553.
[8] M. Pakdemirli, E. Özkaya, Two-to-one internal resonances in continuous systems with arbitrary quadratic nonlinearities, *Arabian Journal for Science and Engineering* 29 (1C) (2004) 99–112 (Mechanical Vibration and Noise Theme Issue).
[9] W. Lacarbonara, Direct treatment and discretizations of non-linear spatially continuous systems, *Journal of Sound and Vibration* 221 (1999) 849–866.
[10] A.H. Nayfeh, Reduced order models of weakly nonlinear spatially continuous systems, *Nonlinear Dynamics* 16 (1998) 105–125.
[11] B.B. Özhan, M. Pakdemirli, A general solution procedure for the forced vibrations of a continuous system with cubic nonlinearities: Primary resonance case, *Journal of Sound and Vibration* 325 (2009) 894–906.
[12] L.Q. Chen, J.W. Zu, Solvability condition in multi-scale analysis of gyroscopic continua, *Journal of Sound and Vibration* 309 (2008) 338–342.
[13] A.H. Nayfeh, *Introduction to Perturbation Techniques*, John Wiley, New York, 1981.
[14] J.A. Wickert, Non-linear vibration of a traveling tensioned beam, *International Journal of Non-Linear Mechanics* 27 (1992) 503–517.
[15] H.R. Öz, M. Pakdemirli, H. Boyacı, Non-linear vibrations and stability of an axially moving beam with time-dependent velocity, *International Journal of Non-Linear Mechanics* 36 (2001) 107–115.
[16] J.A. Wickert, C.D. Mote Jr., Current research on the vibration and stability of axially moving materials, *Shock and Vibration Digest* 20 (5) (1988) 3–13.
[17] J.A. Wickert, C.D. Mote Jr., Classical vibration analysis of axially moving continua, *ASME Journal of Applied Mechanics* 57 (1990) 738–744.
[18] E.M. Mockenstrum, N.C. Perkins, A.G. Ulsoy, Stability and limit cycles of parametrically excited, axially moving strings, *ASME Journal of Vibration and Acoustics* 118 (1994) 346–350.
[19] L.Q. Chen, X.D. Yang, Steady-state response of axially moving viscoelastic beams with pulsating speed: comparison of two nonlinear models, *International Journal of Solids and Structures* 42 (1) (2005) 37–50.
[20] L.Q. Chen, X.D. Yang, Transverse nonlinear dynamics of axially accelerating viscoelastic beams based on 4-term Galerkin truncation, *Chaos, Solitons and Fractals* 27 (2006) 748–757.
[21] L.Q. Chen, X.D. Yang, Stability in parametric resonance of axially moving viscoelastic beams with time-dependent speed, *Journal of Sound and Vibration* 284 (2005) 879–891.
[22] X.D. Yang, L.Q. Chen, Bifurcation and chaos of an axially accelerating viscoelastic beam, *Chaos, Solitons and Fractals* 23 (2005) 249–258.