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Galerkin methods for natural frequencies of high-speed axially moving beams

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ABSTRACT

In this paper, natural frequencies of planar vibration of axially moving beams are numerically investigated in the supercritical ranges. In the supercritical transport speed regime, the straight equilibrium configuration becomes unstable and bifurcate in multiple equilibrium positions. The governing equations of coupled planar is reduced to two nonlinear models of transverse vibration. For motion about each bifurcated solution, those nonlinear equations are cast in the standard form of continuous gyroscopic systems by introducing a coordinate transform. The natural frequencies are investigated for the beams via the Galerkin method to truncate the corresponding governing equations without nonlinear parts into an infinite set of ordinary-differential equations under the simple support boundary. Numerical results indicate that the nonlinear coefficient has little effects on the natural frequency, and the three models predict qualitatively the same tendencies of the natural frequencies with the changing parameters and the integro-partial-differential equation yields results quantitatively closer to those of the coupled equations.

1. Introduction

Axially moving beams are involved in many engineering devices, such as power transmission belt, band saws and aerial cable tramways. The wide diffusion of axially moving systems in industrial processes has motivated intense research activity. In 1965, Mote [1] studied the fundamental frequency approximations via the Galerkin method, and supported by the experiment [2]. Wickert and Mote [3] presented a classical vibration theory for the traveling string and the traveling beam where natural frequencies and modes associated with free vibration serve as a basis for analysis. Wickert [4] used a perturbation method to investigate the natural frequency of an axially moving beam under the simple support boundary conditions in the supercritical regime. Öz and Pakdemirli [5] and Öz [6] calculated the first two natural frequencies values in the cases of pinned–pinned ends and clamped–clamped ends, respectively. Özkaya and Öz [7] applied artificial neural networks to determine the natural frequencies of axially moving beams and gave a comparison of analytical and ANN results for the first two natural frequencies. Öz [8] computed natural frequencies of an axially moving beam in contact with a small stationary mass under pinned–pinned or clamped–clamped boundary conditions. Chen and Yang [9] gave the first two frequencies of axially moving elastic and viscoelastic beams on simple supports with torsion springs. Wang and Ni [10] presented numerical results for natural frequency for an axially moving beam in fluid based on the differential quadrature method. Ghayesh and Khadem [11] investigated natural frequency for the first two modes in free nonlinear transverse

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vibration of an axially moving beam in which rotary inertia and temperature variation effects have been considered. All of above literatures except [4], the natural frequency of axially moving beams were calculated from governing equation of transverse vibration in the subcritical ranges. And Wickert [4] only investigated the fundamental natural frequency of transverse vibration of an axially moving beam in the supercritical regime.

Under certain conditions, an axially moving beam may undergo transverse and longitudinal motions that are usually coupled if the geometrical nonlinearity has to be considered. Thurman and Mote [12] first developed the full governing equations of planar motion and calculated the nonlinear fundamental frequency from simplified governing equations. Wang and Mote [13] studied the linear coupling between the transverse and longitudinal motions due to the finite equilibrium curvatures. Riedel and Tan [14] studied the forced response of a nonlinear axially moving beam with coupled transverse and longitudinal motion. Sze et al. [15] applied the incremental harmonic balance method for transverse and longitudinal motions for nonlinear phenomena of axially moving beams. The literature regarding the coupled equations of axially moving beams is wide [10]. However, there have been no investigations about the natural frequency of planar vibration of axially moving beams in the supercritical regime. Here, we focus on the first and second natural frequencies of planar vibration of axially moving beams in the supercritical regime.

The transverse motion can be decoupled from the longitudinal motion so that the nonlinear integro-partial-differential equation and the nonlinear partial-differential equation are obtained to govern the transverse motion. Both models for transverse vibration of axially moving beams have been widely used as summarized in [16]. Approximate analytical results based on the two models were compared for parametric vibration [17], free vibration [18] of axially moving beams. It is found that the predictions made by the two models are qualitatively the same, but quantitatively different. The transverse responses calculated numerically from the two models were, respectively, compared with the transverse component calculated from the coupled equation for free vibration [19] and forced vibration [20] of axially moving beams in the subcritical ranges. Both models yield almost the same precision results and the integro-partial-differential equation gives better results. However, so far it has not been clear which model yields better outcomes. And there are no works for this issue in the supercritical ranges.

It should be remarked that the literatures on axially moving materials in the supercritical ranges are rather limited. Wickert [4] noticed that the straight equilibrium configuration becomes unstable and bifurcates in multiple equilibrium positions of a translating beam above a certain critical velocity are analogous to those in a buckled beam problem from the nonlinear integro-partial-differential equation, and for each non-trivial equilibrium solutions, the governing equation is cast in the standard form of continuous gyroscopic systems by introducing a coordinate transform, and the natural frequency of an axially moving beam is calculated by perturbation method. Hwang and Perkins [21,22] investigated the effect of an initial curvature due to supporting wheels and pulleys on the bifurcation and stability of equilibrium in the supercritical speed regime; they underlined the system sensitivity to initial imperfections. Ravindra and Zhu [23] investigated a symmetric pitchfork bifurcation for a parametrically excited as the axial velocity of the beam is varied beyond a critical value. Pellicano and Vestroni [24] focused on exploring the dynamics of a traveling beam subjected to a transverse load with simple supported via the Galerkin method when its main parameters vary in the supercritical velocity range. Parker [25] discussed supercritical phenomena in moving strings and found elastically supported strings always exhibit divergence instability above the first critical speed. Pakdemirli and Öz [26] calculated the natural frequency values of axially moving beams under clamped–clamped boundary conditions in the supercritical regime from the governing equation in the subcritical range; they found the complex frequency values non-zero imaginary parts, i.e. the beam is unstable at those velocities. The present investigation focus on the natural frequency of an axially moving beam in the supercritical speed ranges.

However, there are no works on coupled vibration of axially moving beams in the supercritical speed ranges. To address the lacks of research in this aspect, the natural frequencies are calculated by the Galerkin truncation to the coupled longitudinal–transverse governing equations. It is also not clear which transverse models approximate coupled model better for beams moving at the supercritical speed. The additional goal of the present investigation is to examine the validity and the superiority of the two nonlinear transverse models. To compare the two nonlinear transverse models in the sense of approximating the nonlinear coupled governing equation, the natural frequencies in the supercritical regime calculated from the two transverse models via the Galerkin method, and the frequencies are contrasted with the results based on the coupled equations.

The present paper is organized as follows. Section 2 establishes the coupled governing equation and two equations for transverse motion of an axially moving beam in the supercritical regime. Section 3 develops the Galerkin truncation schemes to solve the natural frequency from the coupled equations of planar motion presented in Section 2. Section 4 compares the coupled equations of planar motion with two governing equations of transverse motion via the natural frequency. Section 5 ends the paper with the concluding remarks.

2. Mathematical models and equilibria

Consider a uniform axially moving beam travels at the uniform constant transport speed γ between two boundaries separated by distance l with density ρ, cross-sectional area A, moment of inertial I, initial tension P_0 and Young's modulus E. Assume that the deformation of the beam is confined to the vertical plane. For a slender beam, the linear moment–curvature relationship is sufficiently accurate. The fixed axial coordinate x measure the distance from the left boundary. The longitudinal displacement $u(x,t)$ related to coordinate translating at speed γ and the transverse displacement $v(x,t)$ related to a spatial frame specify the in-plane motion of the beam. The beam is subjected to no external loads. The coupled equation for transverse motion and longitudinal displacement of axially moving elastic beam can be cast into the dimensionless form [19]

$$\begin{aligned} &u_{,tt}+2\gamma u_{,xt}+\gamma^2 u_{,xx}-k_1^2 u_{,xx}=(k_1^2-1)v_{,x}[(1+u_{,x})v_{,xx}-u_{,xx}v_{,x}][(1+u_{,x})^2+v_{,x}^2]^{-3/2} \\ &v_{,tt}+2\gamma v_{,xt}+\gamma^2 v_{,xx}-k_1^2 v_{,xx}+k_f^2 v_{,xxxx}=-(k_1^2-1)(1+u_{,x})[(1+u_{,x})v_{,xx}-u_{,xx}v_{,x}][(1+u_{,x})^2+v_{,x}^2]^{-3/2} \end{aligned} \tag{1}$$

where a comma preceding x or t denotes partial differentiation with respect to x or t, and the dimensionless variables and parameters as follows:

$$v\leftrightarrow\frac{v}{l},\quad u\leftrightarrow\frac{u}{l},\quad x\leftrightarrow\frac{x}{l},\quad t\leftrightarrow t\sqrt{\frac{P_0}{\rho A l^2}},\quad \gamma\leftrightarrow\gamma\sqrt{\frac{\rho A}{P_0}},\quad k_1=\sqrt{\frac{EA}{P_0}},\quad k_f=\sqrt{\frac{EI}{P_0 l^2}} \tag{2}$$

Introduce an approximately transform by Taylor series expansion method

$$[(1+u_{,x})^2+v_{,x}^2]^{-3/2}\approx 1-\tfrac{3}{2}(2u_{,x}+u_{,x}^2+v_{,x}^2) \tag{3}$$

Substitution of Eq. (3) into Eq. (1) yield

$$\begin{aligned} &u_{,tt}+2\gamma u_{,xt}+\gamma^2 u_{,xx}=k_1^2 u_{,xx}+(k_1^2-1)v_{,x}[(1+u_{,x})v_{,xx}-u_{,xx}v_{,x}]\left[1-\frac{3}{2}(2u_{,x}+u_{,x}^2+v_{,x}^2)\right] \\ &v_{,tt}+2\gamma v_{,xt}+\gamma^2 v_{,xx}+k_f^2 v_{,xxxx}=k_1^2 v_{,xx}-(k_1^2-1)(1+u_{,x})[(1+u_{,x})v_{,xx}-u_{,xx}v_{,x}]\left[1-\frac{3}{2}(2u_{,x}+u_{,x}^2+v_{,x}^2)\right] \end{aligned} \tag{4}$$

Equilibrium solutions $\hat{v}(x)$ and $\hat{u}(x)$ of Eq. (4) satisfy

$$\begin{aligned} &\gamma^2\hat{u}''-k_1^2\hat{u}''=(k_1^2-1)\hat{v}'[(1+\hat{u}')\hat{v}''-\hat{u}''\hat{v}']\left[1-\frac{3}{2}(2\hat{u}'+\hat{u}'^2+\hat{v}'^2)\right], \\ &\gamma^2\hat{v}''-k_1^2\hat{v}''+k_f^2\hat{v}^{(4)}=-(k_1^2-1)(1+\hat{u}')[(1+\hat{u}')\hat{v}''-\hat{u}''\hat{v}']\left[1-\frac{3}{2}(2\hat{u}'+\hat{u}'^2+\hat{v}'^2)\right] \end{aligned} \tag{5}$$

where the prime indicates differentiation with respect to x and the superscript indicates the sense of the equilibrium displacement. In the present investigation, only the boundary conditions of the beam is simply supported at both ends are considered as follows:

$$u(0,t)=u(1,t)=0 \tag{6}$$

$$v(0,t)=v(1,t)=0, v_{,xx}(0,t)=v_{,xx}(1,t)=0 \tag{7}$$

Denote the function values $\hat{v}(x_i)$ and $\hat{u}(x_i)$ at (x_j) as $\hat{v}_i$ and $\hat{u}_i$. The trivial configuration $\hat{v}_0=0$, $\hat{u}_0=0$ is always an equilibrium solution. The numerical schemes are presented for Eq. (5) for pairs of non-trivial equilibrium solutions $\hat{v}_i^{\pm}$ and $\hat{u}_i^{\pm}$ in the supercritical regime via and differential quadrature method by modification of the weighting coefficient matrices to implement the simply supported boundary conditions (6) and (7). Then the nonlinear equations (5) can be solved using an iterative procedure, develop iterative schemes [19]

$$\begin{aligned} &\hat{v}_i=\left\{-(k_1^2-1)\left[1+\sum_{j=1}^{N}A_{ij}^{(1)}\hat{u}_j\right]G_d-\sum_{j=1,j\neq i}^{N}[(\gamma^2-k_1^2)A_{ij}^{(2)}+k_f^2A_{ij}^{(4)}]\hat{v}_j\right\}[(\gamma^2-k_1^2)A_{ii}^{(2)}+k_f^2A_{ii}^{(4)}]^{-1} \\ &\hat{u}_i=\left\{(k_1^2-1)\left[1+\sum_{j=1}^{N}A_{ij}^{(1)}\hat{u}_j\right]G_d-(\gamma^2-k_1^2)\sum_{j=1,j\neq i}^{N}B_{ij}^{(2)}\hat{u}_j\right\}[(\gamma^2-k_1^2)B_{ii}^{(2)}]^{-1},\quad (i=2,3,\ldots,N-1) \\ &\hat{v}_1=\hat{v}_N=\hat{u}_1=\hat{u}_N=0 \end{aligned} \tag{8}$$

where space sampling points

$$x_i=\frac{1}{2}\left[1-\cos\frac{(i-1)\pi}{N-1}\right]\quad (i=1,2,\ldots,N) \tag{9}$$

and the weighting coefficients are the expression [19]

$$A_{ij}^{(1)}=\prod_{k=1,k\neq i}^{N}(x_i-x_k)[(x_i-x_j)\prod_{k=1,k\neq j}^{N}(x_j-x_k)]^{-1}\quad (i,j=1,2,\cdots,N;\ j\neq i) \tag{10}$$

and the recurrence relationship

$$A_{ij}^{(r)}=r\left[A_{ii}^{(r-1)}A_{ij}^{(1)}-\frac{A_{ij}^{(r-1)}}{x_i-x_j}\right]\quad (r=2,3,4;\ i,j=1,2,\ldots,N;\ j\neq i) \tag{11}$$

and the weighting coefficient matrices $B_{ij}^{(2)}$ without modification

$$B_{ij}^{(2)} = r[A_{ii}^{(1)}A_{ij}^{(1)} - A_{ij}^{(1)}(x_i - x_j) - 1] \quad (i,j = 1,2,\ldots,N;\ j \neq i) \tag{12}$$

$$B_{ii}^{(2)} = -\sum_{k=1,k\neq i}^{N} B_{ik}^{(2)} \quad (i = 1,2,\ldots,N) \tag{13}$$

and

$$G_d = \left\{\sum_{j=1}^{N} A_{ij}^{(2)}\hat{v}_j\left[1+\sum_{j=1}^{N}A_{ij}^{(1)}\hat{u}_j\right] - \sum_{j=1}^{N}B_{ij}^{(2)}\hat{u}_j\sum_{k=1}^{N}A_{jk}^{(1)}\hat{v}_k\right\}\left\{1-\frac{3}{2}\left\{2\sum_{k=1}^{N}A_{jk}^{(1)}\hat{u}_k + \left[\sum_{k=1}^{N}A_{jk}^{(1)}\hat{u}_k\right]^2 + \left[\sum_{k=1}^{N}A_{jk}^{(1)}\hat{v}_k\right]^2\right\}\right\} \tag{14}$$

In the computations, N=17.

For motion about each bifurcated solution, the coupled equation is cast in the standard form of continuous gyroscopic systems by introducing a coordinate transform. The substitution $v(x,t)\to\hat{v}^{\pm}(x)+v(x,t)$ and $u(x,t)\to\hat{u}^{\pm}(x)+u(x,t)$ in Eq. (1) yields

$$\begin{aligned} &u_{,tt}+2\gamma u_{,xt}+(\gamma^2-k_1^2)u_{,xx}+(\gamma^2-k_1^2)\hat{u}^{\pm''} = (k_1^2-1)(\hat{v}^{\pm'}+v_{,x})G_n \\ &v_{,tt}+2\gamma v_{,xt}+(\gamma^2-k_1^2)v_{,xx}+k_{\rm f}^2 v_{,xxxx}+k_{\rm f}^2\hat{v}^{\pm(4)}+(\gamma^2-k_1^2)\hat{v}^{\pm''} = -(k_1^2-1)(G_2+u_{,x})G_n \end{aligned} \tag{15}$$

where

$$G_n = (G_3+G_2v_{,xx}-\hat{v}^{\pm'}u_{,xx}-\hat{u}^{\pm''}v_{,x}+\hat{v}^{\pm''}u_{,x}+u_{,x}v_{,xx}-v_{,x}u_{,xx})\left[G_1-\frac{3}{2}(2u_{,x}+2u_{,x}\hat{u}^{\pm'}+u_{,x}^2+2\hat{v}^{\pm'}v_{,x}+v_{,x}^2)\right] \tag{16}$$

and

$$G_1 = 1-\frac{3}{2}\left(2\hat{u}^{\pm'}+\hat{u}^{\pm'2}+\hat{v}^{\pm'2}\right), \quad G_2 = 1+\hat{u}^{\pm'}, \quad B\hat{v}^{\pm''}-\hat{v}^{\pm'}\hat{u}^{\pm''} \tag{17}$$

For small but finite stretching problems, one can only consider the transverse motion. In this case, only the lowest order nonlinear terms need to be retained. Then the transverse motion can be decoupled so that the nonlinear the partial-differential equation

$$v_{,tt}+2\gamma v_{,xt}+(\gamma^2-1)v_{,xx}+k_f^2 v_{,xxxx} = \tfrac{3}{2}k_1^2 v_{,x}^2 v_{,xx} \tag{18}$$

and the integro-partial-differential equation

$$v_{,tt}+2\gamma v_{,xt}+(\gamma^2-1)v_{,xx}+k_f^2 v_{,xxxx} = \frac{1}{2}k_1^2 v_{,xx}\int_0^1 v_{,x}^2\,dx \tag{19}$$

are obtained [19]. The pairs of non-trivial equilibrium solutions of nonlinear equations (19) [4]

$$\hat{v}^{\pm}(x) = \pm\frac{2}{\pi k_1^2}\sqrt{\gamma^2-1-(\pi k_f)^2}\,\sin(\pi x) \tag{20}$$

bifurcate from the straight configuration and exist for $\gamma > \sqrt{1+(\pi k_f)^2}$. The substitution $v(x,\ t)\to{}^{\pm}(x)+v(x,\ t)$ in Eq. (19) yields

$$v_{,tt}+2\gamma v_{,xt}+(\gamma^2-1)v_{,xx}+k_{\rm f}^2 v_{,xxxx} = \frac{1}{2}k_1^2 v_{,xx}\left(\int_0^1 v_{,1}^2\,dx+2\int_0^1 v_{,x}\hat{v}^{\pm'}\,dx+\int_0^1\hat{v}^{\pm'2}\,dx\right)+\frac{1}{2}k_1^2\hat{v}_{,xx}^{\pm}\left(\int_0^1 v_{,x}^2\,dx+2\int_0^1 v_{,x}\hat{v}^{\pm'}\,dx\right) \tag{21}$$

The non-trivial equilibrium solutions of nonlinear equations (18) can be solved using iterative schemes

$$\begin{aligned} &\hat{v}_i = \frac{\frac{3}{2}k_1^2(\sum_{k=1}^{N}A_{jk}v_k)^2(\sum_{k=1}^{N}B_{jk}v_k)-[(\gamma^2-1)\sum_{j=1,j\neq i}^{N}A_{ij}^{(2)}+k_f^2\sum_{j=1,j\neq i}^{N}A_{ij}^{(4)}]\hat{v}_j}{(\gamma^2-1)A_{ii}^{(2)}+k_f^2A_{ii}^{(4)}}, (i=2,3,\ldots,N-1) \\ &\hat{v}_1 = \hat{v}_N = 0 \end{aligned} \tag{22}$$

The substitution $v(x,t)\to\hat{v}^{\pm}(x)+v(x,t)$ in Eq. (18) yields

$$v_{,tt}+2\gamma v_{,xt}+(\gamma^2-1)v_{,xx}+k_{\rm f}^2 v_{,xxxx} = 3k_1^2\hat{v}^{\pm'}\hat{v}^{\pm''}v_{,x}+\frac{3}{2}k_1^2\hat{v}^{\pm'2}v_{,xx}+3k_1^2\hat{v}^{\pm'}v_{,xx}v_{,x}+\frac{3}{2}k_1^2\hat{v}^{\pm''}(v_{,x})^2+\frac{3}{2}k_1^2 v_{,xx}(v_{,x})^2 \tag{23}$$

In this paper, the natural frequency of an axially moving beam in the supercritical speed ranges are focused on, the details of the non-trivial equilibria for three nonlinear models of axially moving beams have been discussed in [27], and the numerical error, the convergence and the stability of numerical schemes also have been discussed.

3. The natural frequencies of the coupled equation

The Galerkin method will be used to solve numerically the linear equations correspondingly those nonlinear equations for the natural frequencies under the boundary conditions (6) and (7). Omitting nonlinear terms of Eq. (16) yield

$$
\begin{aligned}
&u_{,tt}+2\gamma u_{,xt}+[(\gamma^2-k_1^2)+(k_1^2-1)G_1\hat{v}^{\pm\prime 2}]u_{,xx}-(k_1^2-1)G_1G_2\hat{v}^{\pm\prime}v_{,xx}\\
&-(k_1^2-1)(G_1\hat{v}^{\pm\prime}\hat{v}^{\pm\prime\prime}-3G_2G_3\hat{v}^{\pm\prime})u_{,x}-(k_1^2-1)(G_1G_3-G_1\hat{v}^{\pm\prime}\hat{u}^{\pm\prime\prime}-3G_3\hat{v}^{\pm\prime 2})v_{,x}=0\\
&v_{,tt}+2\gamma v_{,xt}+[(\gamma^2-k_1^2)+(k_1^2-1)G_1G_2^2]v_{,xx}+k_f^2v_{,xxxx}-(k_1^2-1)G_1G_2\hat{v}^{\pm\prime 2}u_{,xx}+(k_1^2-1)(G_1G_3-3G_3G_2^2+G_1G_2\hat{u}^{\pm\prime\prime})u_{,x}\\
&+(k_1^2-1)(-3G_2G_3\hat{v}^{\pm\prime}-G_1G_2\hat{u}^{\pm\prime\prime})v_{,x}=0
\end{aligned}
\tag{24}
$$

Jha and Parker [28] have found stationary beam eigenfunctions yield excellent convergence at subcritical and supercritical speeds. In the present investigation, both the trial and weight functions are chosen as eigenfunctions of a stationary beam under the boundary conditions (6) and (7), namely, suppose that the solution to Eq. (24) takes the form [15]

$$v(x,t)=\sum_{j=1}^{n}q_j^v(t)\sin(j\pi x),\quad u(x,t)=\sum_{j=1}^{n}q_j^u(t)\sin(j\pi x),\quad j=1,2,\ldots,n \tag{25}$$

where $q_j{}^v(t)$ and $q_j{}^u(t)$ are sets of generalized displacements of the beam. After substituting Eq. (25) into Eq. (24), the Galerkin procedure leads to the following set of second-order ordinary differential equations

$$
\begin{aligned}
&\sum_{j=1}^{n}M_{ij}^u\ddot{q}_j^u(t)+\sum_{j=1}^{n}C_{ij}^u\dot{q}_j^u(t)+\sum_{j=1}^{n}K_{uij}^uq_j^u(t)+\sum_{j=1}^{n}K_{vij}^uq_j^v(t)=0\\
&\sum_{j=1}^{n}M_{ij}^v\ddot{q}_j^v(t)+\sum_{j=1}^{n}C_{ij}^v\dot{q}_j^v(t)+\sum_{j=1}^{n}K_{vij}^vq_j^v(t)+\sum_{j=1}^{n}K_{uij}^vq_j^u(t)=0
\end{aligned}
\tag{26}
$$

where the dot above a variable denotes its derivative with respect to the non-dimensional time t

$$M_{ij}^u=M_{ij}^v=\int_0^1\sin(j\pi x)\sin(j\pi x)\,dx=\frac{1}{2}\delta_{ij} \tag{27}$$

$$C_{ij}^u=C_{ij}^v=2\gamma j\pi\int_0^1\cos(j\pi x)\sin(i\pi x)\,dx=\begin{cases}4\gamma ij/(i^2-j^2), & i\neq j\quad (i+j)\quad \text{even}\\ 0, & i=j,\quad \text{otherwise}\end{cases} \tag{28}$$

$$K_{uij}^u=\int_0^1\{-[(\gamma^2-k_1^2)+(k_1^2-1)G_1\hat{v}^{\pm\prime 2}](j\pi)^2\sin(j\pi x)-(k_1^2-1)\hat{v}^{\pm\prime}(G_1\hat{v}_{,xx}-3G_2G_3)j\pi\cos(j\pi x)\}\sin(i\pi x)\,dx \tag{29}$$

$$K_{vij}^u=\int_0^1(k_1^2-1)[G_1G_2\hat{v}^{\pm\prime}(j\pi)^2\sin(j\pi x)-(G_1G_3-A\hat{v}^{\pm\prime}\hat{u}^{\pm\prime\prime}-3G_3\hat{v}^{\pm\prime 2})j\pi\cos(j\pi x)]\sin(i\pi x)\,dx \tag{30}$$

$$K_{uij}^v=\int_0^1(k_1^2-1)[G_1G_2\hat{v}^{\pm\prime}(j\pi)^2\sin(j\pi x)+(G_1G_3-3B^2C+G_1G_2\hat{v}^{\pm\prime\prime})j\pi\cos(j\pi x)]\sin(i\pi x)\,dx \tag{31}$$

$$K_{vij}^v=\int_0^1\{\{k_f^2(j\pi)^2-[(\gamma^2-k_1^2)+(k_1^2-1)G_1G_2^2]\}(j\pi)^2\sin(j\pi x)+(k_1^2-1)(-3G_2G_3\hat{v}^{\pm\prime}-G_1G_2\hat{u}^{\pm\prime\prime})j\pi\cos(j\pi x)\}\sin(i\pi x)\,dx \tag{32}$$

Eq. (26) can be written in matrix-vector form as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{0} \tag{33}$$

where

$$\mathbf{q}=\begin{bmatrix}\mathbf{q}^u\\ \mathbf{q}^v\end{bmatrix},\quad \mathbf{M}=\begin{bmatrix}\mathbf{M}^u & \\ & \mathbf{M}^v\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}\mathbf{C}^u & \\ & \mathbf{C}^v\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}\mathbf{K}_u^u & \mathbf{K}_v^u\\ \mathbf{K}_u^v & \mathbf{K}_v^v\end{bmatrix} \tag{34}$$

If $\mathbf{q}(t)$ defined as

$$\mathbf{q}(t)=\mathbf{Q}\,e^{i\omega t} \tag{35}$$

Eq. (33) yield

$$[-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}]\mathbf{Q}=0 \tag{36}$$

Non-triviality of solutions to Eq. (36) requires its determinant of coefficients to be zero, therefore, the natural frequencies ω can be obtained from

$$|-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}|=0 \tag{37}$$

In the computations, $n=4$, that is to say, the natural frequency is numerical calculated based on 4-term Galerkin truncation. Here only show the results of $v(x,t)\rightarrow\hat{v}^{+}(x)+v(x,t)$, as the results via $v(x,t)\rightarrow\hat{v}^{-}(x)+v(x,t)$ are the completely same. Fig. 1 illustrates the dependence of the first two natural frequencies on the axial speed for fixed nonlinear coefficient $k_1=100$ and four different flexural stiffness k_f values. In the present investigation, beams are assumed to move with a supercritical axial speed. For the given k_f, the natural frequencies increase with the growth of axial speed. Fig. 2 illustrates the effects of the nonlinear coefficient with $k_f=0.8$ on the first two natural frequencies. In Fig. 2, the dotted lines and the solid lines, respectively, stand for the natural frequencies to $k_1=100$ and 200. The comparisons indicate that nonlinear coefficient k_1 has little effects on the natural frequency. The results of 2-term Galerkin truncation, 4-term Galerkin truncation and 8-term Galerkin truncation for Eq. (24) are shown in Fig. 3, where the dash-dot lines are for 2-term Galerkin truncation results, the solid lines are for 4-term Galerkin truncation results and the dot lines are for 8-term Galerkin truncation. The 2-term Galerkin truncation results are bigger than the 4-term ones and the 8-term ones, especially for the second natural frequency, and the difference increase with the growth of axial speed. Fig. 3 demonstrates that the 2-term Galerkin method is not good enough for the first two natural frequencies for axially moving beams in the supercritical regime and the 4-term Galerkin method yields rather accurate results.

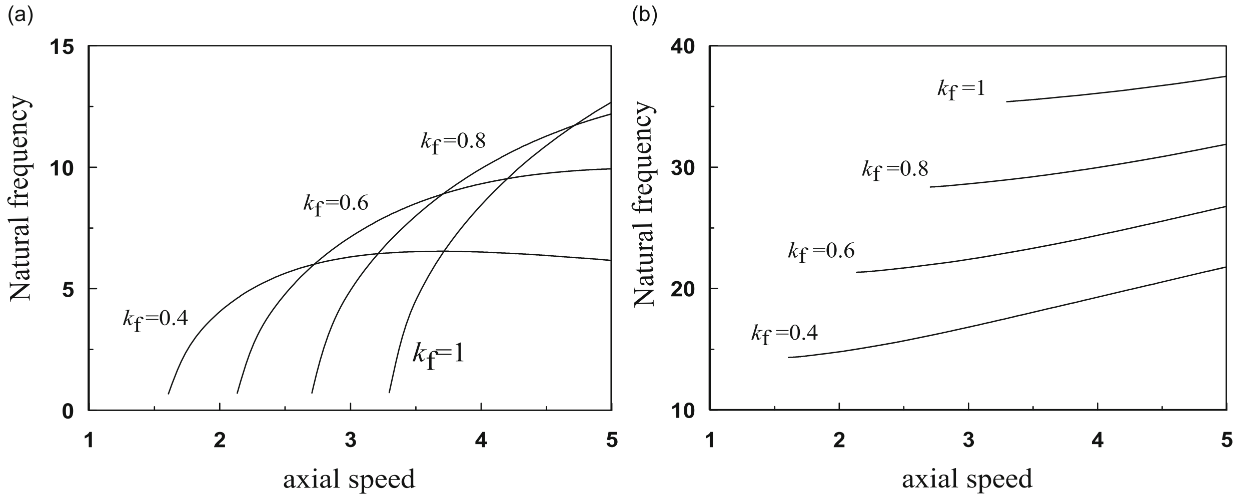

Fig. 1. Natural frequencies versus axial speed and flexural stiffness: (a) the first natural frequency and (b) the second natural frequency.

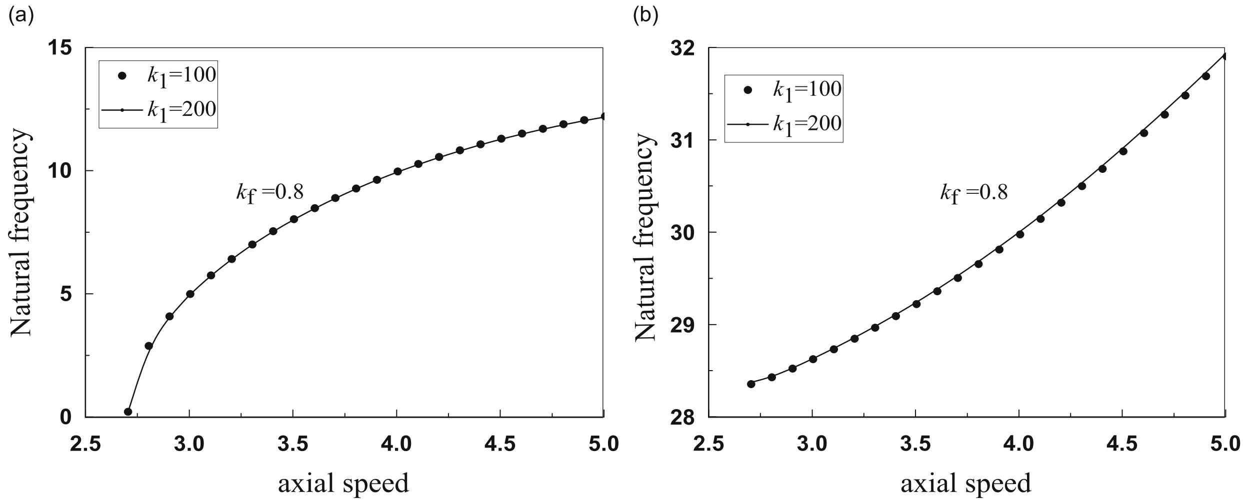

Fig. 2. The effects of the nonlinear coefficient on the natural frequencies versus axial speed: (a) the first natural frequency and (b) the second natural frequency.

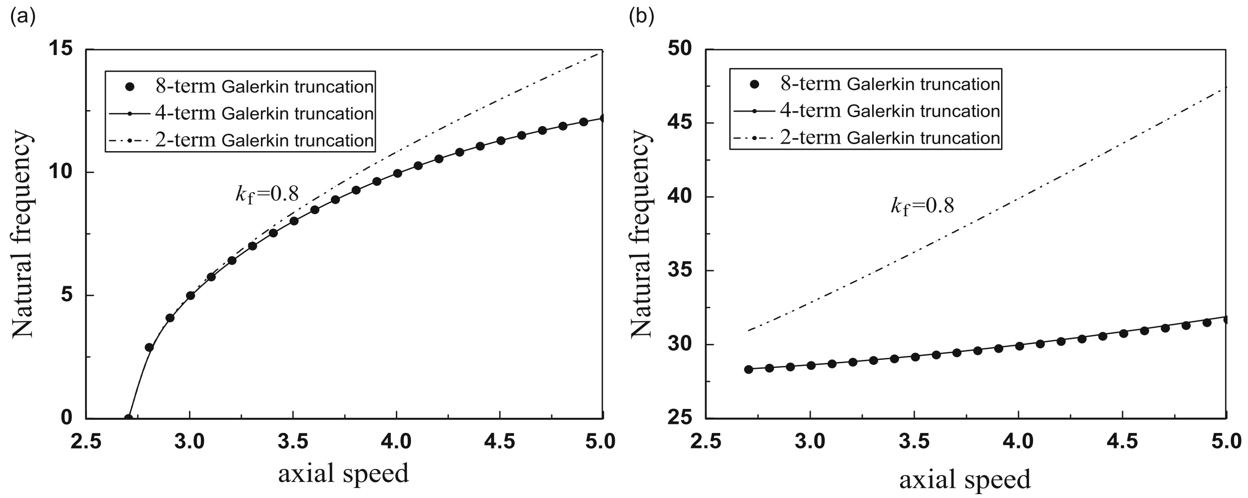

Fig. 3. Comparison between 2-term, 4-term and 8-term Galerkin truncation results: (a) the first natural frequency and (b) the second natural frequency.

4. Natural frequencies of nonlinear models of transverse vibration and comparisons

4.1. Natural frequencies of the integro-partial-differential equation

Omitting nonlinear terms of Eq. (21) yield

$$v_{,tt}+2\gamma v_{,xt}+\left(\gamma^2-1-\frac{k_1^2}{2}\int_0^1 \hat{v}^{\pm'2}\,dx\right)v_{,xx}+k_f^2 v_{,xxxx}=k_1^2\hat{v}^{\pm''}\int_0^1 v_{,x}\hat{v}^{\pm'}\,dx \tag{38}$$

In the present investigation, both the trial and weight functions are chosen as eigenfunctions of a linear non-translating beam under the boundary condition (7), namely, suppose that the solution to Eq. (24) takes the form [15]

$$v(x,t)=\sum_{j=1}^{n} q_j(t)\sin(j\pi x),\quad j=1,2,\ldots,n \tag{39}$$

where $q_j(t)$ are sets of generalized displacements of the beam. After substituting Eqs. (20) and (39) into Eq. (38), the Galerkin procedure leads to the following set of second-order ordinary differential equations:

$$\sum_{j=1}^{n} M_{ij}\ddot{q}_j(t)+\sum_{j=1}^{n} C_{ij}\dot{q}_j(t)+\sum_{j=1}^{n} K_{ij}q_j(t)=0 \tag{40}$$

where

$$M_{ij}=\int_0^1 \sin(j\pi x)\sin(i\pi x)\,dx=\frac{1}{2}\delta_{ij} \tag{41}$$

$$C_{ij}=2\gamma j\pi\int_0^1 \cos(j\pi x)\sin(i\pi x)\,dx=\begin{cases}4\gamma ij/(i^2-j^2), & i\neq j\\ 0, & i=j\end{cases} \tag{42}$$

$$K_{ij}=\int_0^1\left[(j^2-1)k_f^2\pi^2(j\pi)^2\sin(j\pi x)-(1+k_f^2\pi^2-\gamma^2)4\pi\sin(\pi x)\int_0^1 j\pi\cos(j\pi x)\cos(\pi x)\,dx\right]\sin(i\pi x)\,dx \tag{43}$$

Eq. (26) can be written in matrix-vector form as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{0} \tag{44}$$

where

$$\mathbf{q}=[q_1,q_2,\ldots,q_n]^T,\quad \mathbf{K}=\begin{bmatrix}-\pi^2(1+k_f^2\pi^2-\gamma^2) & 0 & \cdots & 0\\ 0 & 2^2(2^2-1)k_f^2\pi^4 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & j^2(j^2-1)k_f^2\pi^4\end{bmatrix} \tag{45}$$

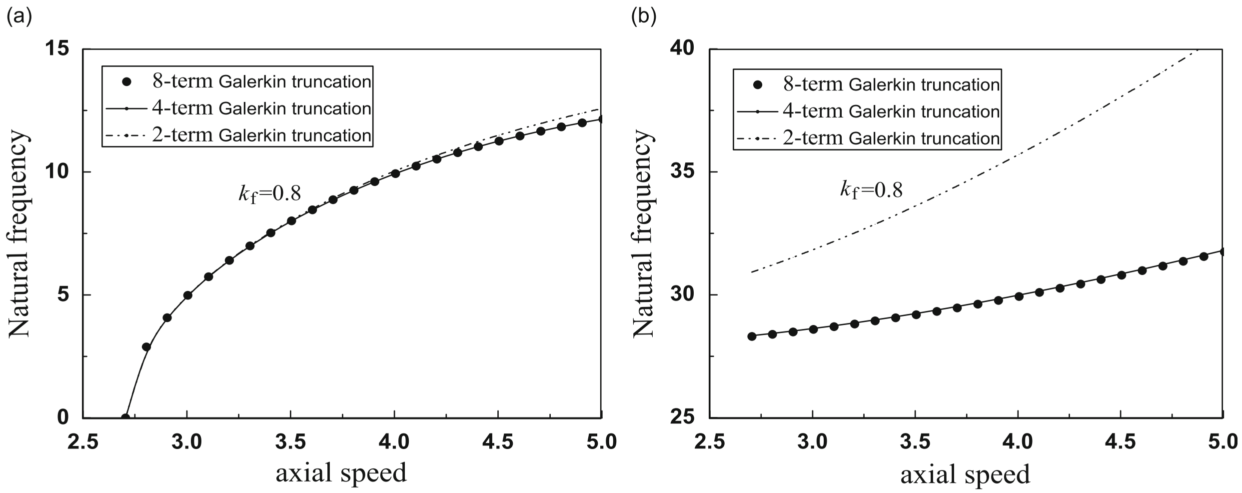

Fig. 4. Comparison between 2-term, 4-term and 8-term Galerkin truncation results: (a) the first natural frequency and (b) the second natural frequency.

Substitution of Eq. (35) into Eq. (44) yield

$$[-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}]\mathbf{Q}=0 \tag{46}$$

Non-triviality of solutions to Eq. (46) requires its determinant of coefficients to be zero, therefore, the natural frequencies ω can be obtained from

$$|-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}|=0 \tag{47}$$

Fig. 4 shows the results of the first two natural frequencies of 2-term Galerkin truncation, 4-term Galerkin truncation and 8-term Galerkin truncation for Eq. (38) dependence on the axial speed for fixed nonlinear coefficient k_1=100 and flexural stiffness k_f=0.8 values. In Fig. 4, the dash-dot lines, the solid lines and the dots, respectively, stand for 2-term, 4-term and 8-term Galerkin truncation. Fig. 4 demonstrates that 4-term Galerkin truncation predict the almost same results as 8-term Galerkin method, while the 2-term ones bigger than them, especially for the second natural frequency, and the difference increase with the growth of axial speed. That is meaning that the 4-term Galerkin method is sufficient in predicting the first two natural frequencies of axially moving beams in the supercritical regime.

4.2. Natural frequencies of the partial-differential equation

Omitting nonlinear terms of Eq. (23) yield

$$v_{,tt}+2\gamma v_{,xt}+\left(\gamma^2-1-\frac{3}{2}k_1^2\hat{v}^{\pm\prime 2}\right)v_{,xx}+k_f^2 v_{,xxxx}=k_1^2\hat{v}^{\pm\prime\prime}v_{,x}\hat{v}^{\pm\prime} \tag{48}$$

After substituting Eq. (39) into Eq. (48), the Galerkin procedure leads to the following set of second-order ordinary differential equations:

$$\sum_{j=1}^{n} M_{ij}\ddot{q}_j(t)+\sum_{j=1}^{n} C_{ij}\dot{q}_j(t)+\sum_{j=1}^{n} K_{ij}q_j(t)=0 \tag{49}$$

where M_{ij} and C_{ij}, respectively, satisfy Eqs. (41) and (42), and K_{ij} satisfy

$$K_{ij}=\int_0^1\left[\left(k_f^2 j^2\pi^2-\gamma^2+1+\frac{3}{2}k_1^2\hat{v}^{\pm\prime 2}\right)(j\pi)^2\sin(j\pi x)-3k_1^2\hat{v}^{\pm\prime}\hat{v}^{\pm\prime\prime}j\pi\cos(j\pi x)\right]\sin(i\pi x)\,dx \tag{50}$$

Eq. (49) also can be written in matrix-vector form as Eq. (44) where$\mathbf{K}=[K_{ij}]$. Therefore, the natural frequencies ω can be obtained from Eq. (47).

Fig. 5 shows the results of the first two natural frequencies of 2-term Galerkin truncation, 4-term Galerkin truncation and 8-term Galerkin truncation for Eq. (48) with k_1=100 and k_f=0.8. In Fig. 5, the dash-dot lines, the solid lines and the dots, respectively, stand for 2-term, 4-term and 8-term Galerkin truncation. Fig. 5 demonstrates the same results as Fig. 4. That is to say, the 4-term Galerkin method for the first two natural frequencies of axially moving beams yields rather accurate results in the supercritical regime.

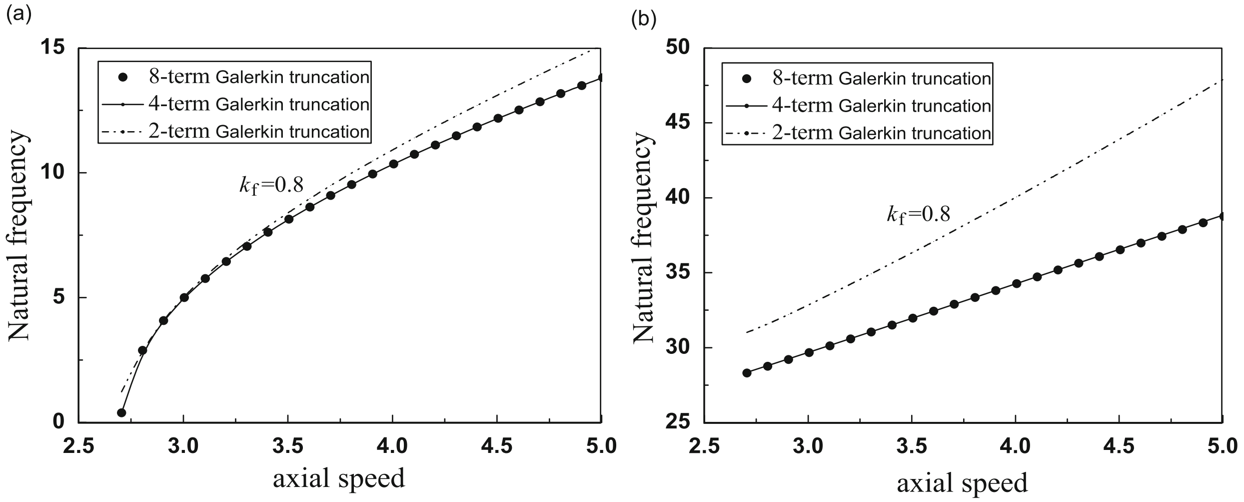

Fig. 5. Comparison between 2-term, 4-term and 8-term Galerkin truncation results: (a) the first natural frequency and (b) the second natural frequency.

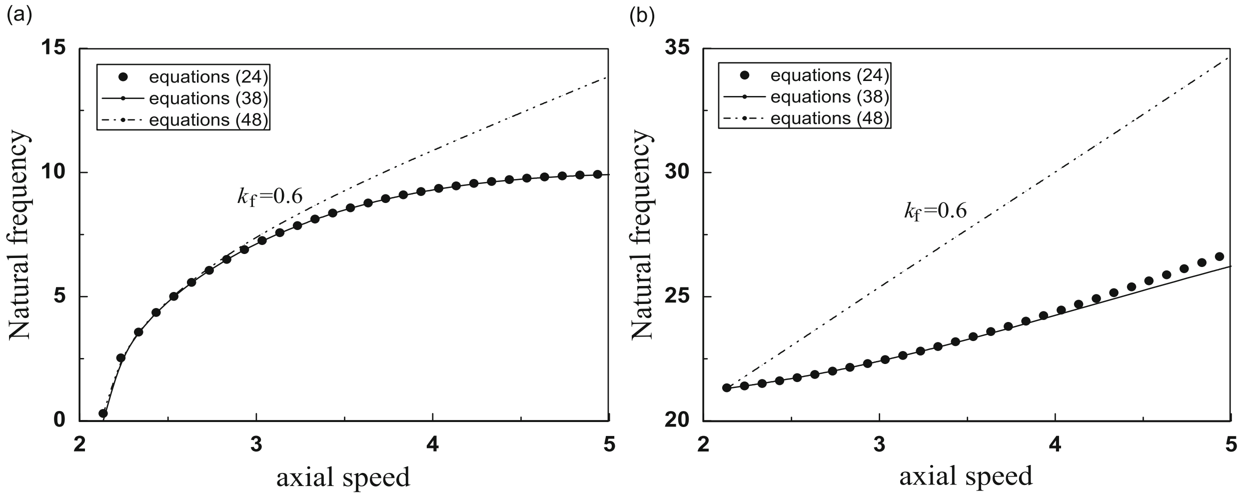

Fig. 6. The natural frequency calculated from Eqs. (24), (38) and (48) for k_f=0.6: (a) the first natural frequency and (b) the second natural frequency.

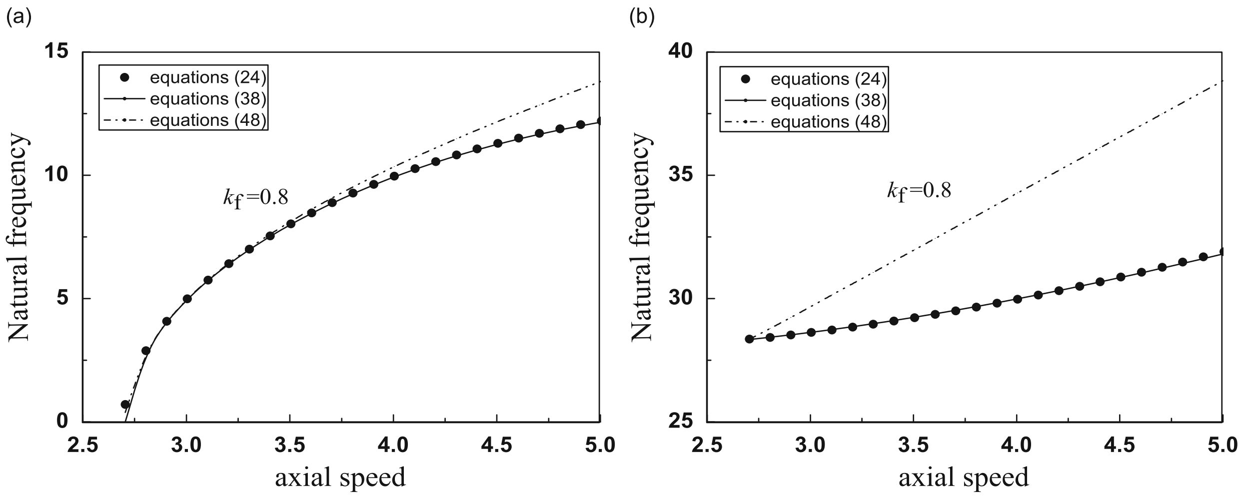

Fig. 7. The natural frequency calculated from Eqs. (24), (38) and (48) for k_f=0.8: (a) the first natural frequency and (b) the second natural frequency.

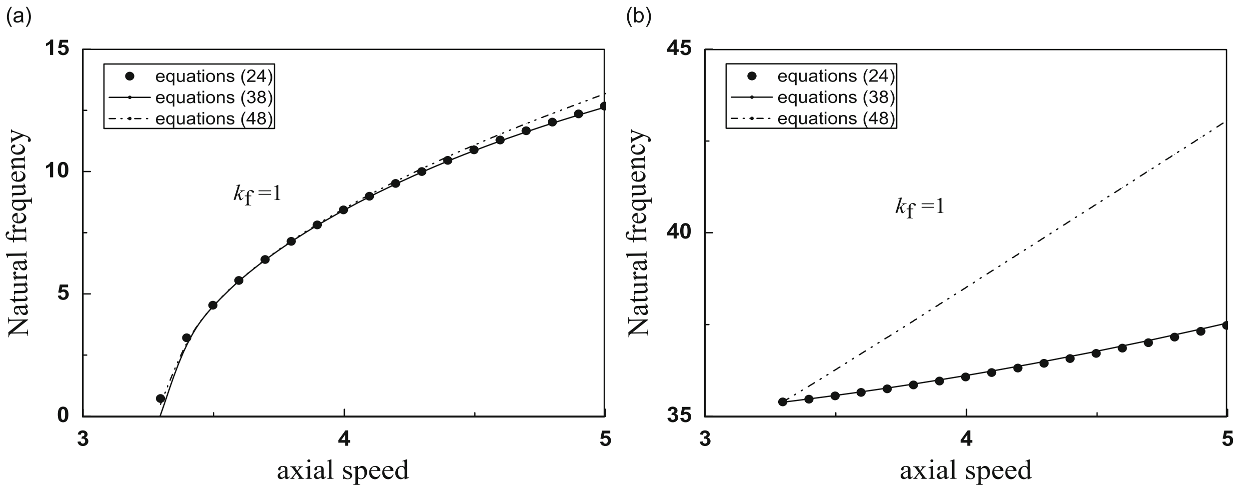

Fig. 8. The natural frequency calculated from Eqs. (24), (38) and (48) for k_f=1: (a) the first natural frequency and (b) the second natural frequency.

4.3. Comparisons

The natural frequencies are numerical calculated based on 4-trem Galerkin truncation. Based on the natural frequency of Eqs. (24), (38) and (48), the differences between the models can be investigated via the natural frequencies in the supercritical regime. Figs. 6, 7 and 8, respectively, illustrate the natural frequencies for the flexural stiffness k_f=0.6, 0.8 and 1 with fixed nonlinear coefficient k_1=100. In Figs. 6, 7 and 8, the dots, the solid lines and the dash-dot lines, respectively, stand for the numerical solutions to Eqs. (24), (38) and (48). The numerical results demonstrate that three nonlinear models qualitatively predict the same tendencies with the changing flexural stiffness k_f in the supercritical regime, while quantitatively, there are certain differences and the difference increase with the axial speed, and Eq. (48) are closer to Eq. (24), especially for the first natural frequency. Like the conclusions in the subcritical transport speed ranges, in the view of the natural frequencies in the supercritical regime, while all three models of axially moving beams compared predicted same qualitative behavior, there are differences between them quantitatively.

5. Conclusions

Axially moving systems are present in a wide class of engineering problems. The axial speed greatly affects the dynamic behavior of the system. This paper examines the natural frequency of planar vibration of axially moving elastic beams in the supercritical regime. The planar vibration is governed by a set of coupled nonlinear partial-differential equations can reduce to a nonlinear partial-differential equation and a nonlinear integro-partial-differential equation of transverse vibration. The non-trivial equilibrium equations of three nonlinear models are, respectively, solved via the differential quadrature scheme. For motion about each bifurcated solution, those equations are cast in the standard form of continuous gyroscopic systems by introducing a coordinate transform. The natural frequencies are computationally studied for the linear equations correspondingly those standard form via the Galerkin method under the simple support boundary. The investigation leads to the following conclusions:

(1) In the supercritical regime, the natural frequencies for axially moving beams increase with the growth of axial speed, and the nonlinear coefficient has little effects on the natural frequency.
(2) The 2-term Galerkin truncation for the natural frequency for axially moving beams in the supercritical range is bigger than the 4-term ones and the difference increase with the growth of axial speed, and the 4-term Galerkin method yields rather accurate results.
(3) Qualitatively, the three models predict the same tendencies of the natural frequencies with the changing flexural stiffness, especially for the first natural frequency.
(4) Quantitatively, there are certain differences. In the view of the natural frequencies, the nonlinear integro-partial-differential equation yields the results closer to those from the coupled equations. For the first natural frequency, the differences between the coupled equation and the partial-differential equation decrease with the flexural stiffness.

Acknowledgments

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