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Localized bending waves in a transversely isotropic plate

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ABSTRACT

The problem of bending waves localized near the free edge of a transversely isotropic plate is investigated using the Ambartsumian higher-order plate theory which takes account of the transverse shears generated by flexural deformation. Unlike the first-order Reissner–Mindlin theory, which also takes account of transverse shears, Ambartsumian's analysis does not demand that plane normal cross-sections remain plane during bending. Within this analysis the existence of localized bending waves in transversely isotropic plates is established, and solutions of the dispersion equation obtained for different values of the elastic parameters.

The analysis of frequencies of localized bending waves shows that for thick plates the effect of anisotropy can be considerable. For the particular case of vibrations of a narrow plate, from the long wave approximation a new beam vibration equation of the Timoshenko type is obtained for a transversally isotropic plate.

1. Introduction

A recent development of sophisticated new devices using thin walled flexible waveguides has motivated a growing number of investigations concerning localized bending waves in thin walled structures. The performance of such devices can be significantly affected by the influence of localized bending waves near free edges, or at the interface between two structures composed of different materials. The analysis of localized bending waves in thin elastic plates, used as components in many modern engineering structures, can also play a significant role in the detection of imperfections, cracks or inclusions in those structures.

The study of localized bending waves propagating along the free edge of a semi-infinite isotropic elastic thin plate was originally introduced by Konenkov [1]. The analysis, carried out within the framework of the classical Kirchhoff theory of elastic plates, demonstrated the existence of waves similar to those of Rayleigh waves with an amplitude decaying exponentially with distance from the edge of the plate. Similar studies were subsequently carried out within western scientific circles by McKenna et al. [2], Thurston and McKenna [3]. In recent years the existence and propagation of such waves in anisotropic plates with cubic and orthotropic symmetries, again within the framework of Kirchhoff plate theory, has been investigated in Bagdasaryan et al. [4], Norris [5], Belubekyan and Engibaryan [6], Thompson et al. [7], Zakharov and Becker [8], Fu [9], and Mkrtchyan [10].

Similar methodologies have been used to analyze bending waves localized near the junction of two plates made of different materials [11,12], and edge and interfacial bending vibrations in elastic shells with cylindrical symmetry have been discussed in Kaplunov et al. [13], Kaplunov and Wilde [14,15] and Gulgazaryan et al.[16]. Using the exact solution for three-dimensional vibrations of a semi-infinite elastic plate [17] and [18] confirmed Konenkov's result [1] concerning the existence of localized bending edge waves within Kirchhoff's theory. The analysis of some essential properties of the edge bending waves in a thin isotropic plate based on asymptotical analysis of 3D elasticity equations is given in Zakharov [19].

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The magnetoelastic problem of localized vibrations of a conductive plate immersed in a longitudinal magnetic field parallel to the plate's middle surface have been considered in Belubekyan et al. [20], where it was shown that localized waves could be eliminated by changing the intensity and orientation of the magnetic field. Studies of the extent to which localized bending waves in thin orthotropic elastic plates can be suppressed using reinforcing ribs have been carried out in Belubekyan et al. [21]. Within classical thin plate theory the existence of bending waves confined to the free edge of a fluid loaded plate is established in Abrahams and Norris [22].

It is well known that the classical plate theory, which assumes that plane cross-sections of the plate normal to the middle plane before deformation remains plane and normal to the middle plane after deformation, underestimates deflections and overestimates natural frequencies when the plate thickness to length ratio is greater than 1/20. This is due to neglect of transverse shear and rotatory inertia that become significant for thick plates. Refined plate theories that take account of transverse shears were introduced by Reissner [23], Mindlin [24], Ambartsumian [25] and Levinson [26]. In contrast to the first order Reissner–Mindlin theory, which assumes that plane cross-sections remain plane (though not necessarily normal), Ambartsumian and later Levinson developed high order plate theories which allow some distortion of these cross-sections. For isotropic plates Norris et al. [27] demonstrated the existence of flexural edge waves in a semi-infinite elastic plate within the context of Mindlin plate theory, and similar analyses have been carried out by Belubekyan [28] within the context of Ambartsumian's thick isotropic plate.

It is known that Kirchhoff's plate theory and the first order Mindlin theory are not sensitive to the anisotropic properties of a transversally isotropic plate. The effect of anisotropy in such a plate may be revealed only in the framework of the Reissner and higher order refined plate theories. The main purpose of the present paper is to reveal the effect of anisotropy in the problem of localized bending waves in a semi-infinite transversely isotropic plate, in the framework of the higher order Ambartsumian theory. A further purpose is to obtain an analytical expression for the dispersion equation of localized wave frequencies with respect to the plate thickness and the anisotropy parameter (characterized by the transversal shear modulus). Furthermore, the Timoshenko equation for vibrations of a transversally anisotropic beam is obtained by considering the long wave approximation for a narrow plate with all edges free from loads.

2. Statement of the problem

We consider the problem of localized bending waves propagating along the free edge of a semi-infinite elastic transversely isotropic plate free of mechanical loads. Employing Cartesian coordinates (x,y,z) chosen so that the plate initially occupies the region $-\infty < x < \infty$, $0 \le y < \infty$, $-h < z < h$ (where $2h$ is the thickness of the plate), within the framework of the high order Ambartsumian and first-order Reissner–Mindlin refined plate theories the displacements u_x, u_y, u_z may be presented as follows.

1. Ambartsumian plate theory [20]:

$$u_x(x,y,z,t) = \left(z - \frac{z^3}{3h^2}\right)\varphi(x,y,t) - z\frac{\partial w}{\partial x}, \tag{1}$$

$$u_y(x,y,z,t) = \left(z - \frac{z^3}{3h^2}\right)\psi(x,y,t) - z\frac{\partial w}{\partial y}, \tag{2}$$

$$u_z(x,y,z,t) = w(x,y,t), \tag{3}$$

where $\varphi(x,y,t)$ and $\psi(x,y,t)$ are functions defining transversal shears, and $w(x,y,t)$ is the transversal displacement of the middle plane of the plate.

2. Reissner–Mindlin plate theory [23,24]:

$$u_x(x,y,z,t) = z\theta_x(x,y,t), \tag{4}$$

$$u_y(x,y,z,t) = z\theta_y(x,y,t), \tag{5}$$

$$u_z(x,y,z,t) = w(x,y,t), \tag{6}$$

where $\theta_x(x,y,t)$ and $\theta_y(x,y,t)$ are functions characterizing the rotation of the normal cross sections.

Writing

$$\theta_x = \varphi - \frac{\partial w}{\partial x} \quad \text{and} \quad \theta_y = \psi - \frac{\partial w}{\partial y}. \tag{7}$$

The Reissner–Mindlin equations can be rewritten

$$u_x(x,y,z,t) = z\varphi(x,y,t) - z\frac{\partial w}{\partial x}, \tag{8}$$

$$u_y(x,y,z,t) = z\psi(x,y,t) - z\frac{\partial w}{\partial y}. \tag{9}$$

Both models can thus be presented in the form

$$u_x(x,y,z,t) = f(z)\varphi(x,y,t) - z\frac{\partial w}{\partial x}, \tag{10}$$

$$u_y(x,y,z,t) = f(z)\psi(x,y,t) - z\frac{\partial w}{\partial y}, \tag{11}$$

$$u_z(x,y,z,t) = w(x,y,t), \tag{12}$$

where $f(z)$ is a simple linear function in the Reissner–Mindlin theory and a cubic function in the Ambartsumian theory.

Levinson [26] also proposed a high order theory in which the displacement field is given as

$$u_x(x,y,z,t) = z\left[\theta_x(x,y,t) - \frac{1}{3}\frac{z^2}{h^2}\left(\theta_x(x,y,t) + \frac{\partial w}{\partial x}\right)\right], \tag{13}$$

$$u_y(x,y,z,t) = z\left[\theta_y(x,y,t) - \frac{1}{3}\frac{z^2}{h^2}\left(\theta_y(x,y,t) + \frac{\partial w}{\partial y}\right)\right], \tag{14}$$

$$u_z(x,y,z,t) = w(x,y,t). \tag{15}$$

Using the substitution (7) Levinson's equations reduce to those of Ambartsumian.

The constitutive equations for a transversely isotropic plate have the form:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{11}-c_{12} \end{pmatrix} \begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{xz} \\ e_{xy} \end{pmatrix} \tag{16}$$

where the c_{ij} (i,j=1,…,6) are elastic constants, the $\sigma_{\alpha\beta}$ (α,β=x,y,z) are stresses, and the $e_{\alpha\beta}$ are elastic strains given by

$$e_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial \beta} + \frac{\partial u_\beta}{\partial \alpha}\right).$$

It is assumed in the above theories that the normal stresses in the direction transverse to the plate middle plane can be disregarded so that $\sigma_{zz} = 0$, whence

$$e_{zz} = -\frac{c_{13}}{c_{33}}(e_{xx} + e_{yy}).$$

Substituting into (16) results in the following relations:

$$\sigma_{xx} = c_1\frac{\partial u_x}{\partial x} + c_2\frac{\partial u_y}{\partial y}, \quad \sigma_{yy} = c_2\frac{\partial u_x}{\partial x} + c_1\frac{\partial u_y}{\partial y}, \quad \sigma_{xy} = \frac{(c_1 - c_2)}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \tag{17}$$

where

$$c_1 = c_{11} - \frac{c_{13}^2}{c_{33}}, \quad c_2 = c_{12} - \frac{c_{13}^2}{c_{33}}. \tag{18}$$

The shear stresses σ_{xz} and σ_{yz} given by the equations

$$\sigma_{xz} = c_{44}\varphi(x,y)f'(z), \quad \sigma_{yz} = c_{44}\psi(x,y)f'(z),$$

should normally satisfy the boundary conditions

$$\sigma_{xz}(x,y,\pm h) = 0, \quad \sigma_{yz}(x,y,\pm h) = 0,$$

over the top and bottom surfaces of the plate. These conditions are satisfied by Ambartsumian's theory, in which the equations for the transverse shear stresses reduce to

$$\sigma_{xz} = c_{44}\varphi(x,y)\left(1 - \frac{z^2}{h^2}\right), \quad \sigma_{yz} = c_{44}\psi(x,y)\left(1 - \frac{z^2}{h^2}\right), \tag{19}$$

but not by the Reissner–Mindlin theory. We deduce from these equations that the stress–strain relation in the plate is determined by the three independent elastic constants. Two of these, the in-plane elastic moduli c_1 and c_2, determine the plane stresses σ_{xx}, σ_{yy} and σ_{xy}. The third, the transversal shear modulus c_{44}, determines the transverse shear stresses σ_{xz} and σ_{yz}. The in-plane moduli c_1 and c_2 are often replaced by elastic constants E and ν, where

$$c_1 = \frac{E}{1-\nu^2} \quad \text{and} \quad c_2 = \nu c_1.$$

For an isotropic plate where there are just two elastic constants

$$c_1 = \frac{E}{1-\nu^2}, \quad c_2 = \nu c_1 \quad \text{and} \quad c_{44} = \frac{E}{2(1+\nu)}. \tag{20}$$

Assuming no external mechanical loads, it follows from the equations of motion that the bending moments M_x and M_y, the twisting moment M_{xy}, and the resultant transverse shear forces Q_x and Q_y satisfy the dynamical equations

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \rho \int_{-h}^{h} z \frac{\partial^2 u_x}{\partial t^2}\,\mathrm{d}z, \tag{21}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = \rho \int_{-h}^{h} z \frac{\partial^2 u_y}{\partial t^2}\,\mathrm{d}z, \tag{22}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 2\rho h \frac{\partial^2 w}{\partial t^2}, \tag{23}$$

where

$$M_x = \int_{-h}^{+h} z\sigma_{xx}\,\mathrm{d}z, \quad M_y = \int_{-h}^{+h} z\sigma_{yy}\,\mathrm{d}z, \quad M_{xy} = \int_{-h}^{h} z\sigma_{xy}\,\mathrm{d}z, \tag{24}$$

$$Q_x = \int_{-h}^{h} \sigma_{xz}\,\mathrm{d}z, \quad Q_y = \int_{-h}^{h} \sigma_{yz}\,\mathrm{d}z, \tag{25}$$

and ρ is the mass density. Boundary conditions at the free edge $y=0$ are

$$M_y = M_{xy} = Q_y = 0. \tag{26}$$

Substituting Eqs. (10)–(13), (17) and (19) into these equations and writing $W = (\beta c_{44})^{-1} w$ yields the differential equations

$$\gamma\left[\rho \frac{\partial^2 \varphi}{\partial t^2} + \frac{c_1(1+\nu)}{2}\frac{\partial^2 \psi}{\partial x \partial y} + \frac{c_1(1-\nu)}{2}\frac{\partial^2 \varphi}{\partial y^2} + c_1 \frac{\partial^2 \varphi}{\partial x^2}\right] - \frac{2h^3}{3}\left[\rho \frac{\partial^3 W}{\partial x \partial t^2} + c_1\left(\frac{\partial^3 W}{\partial x \partial y^2} + \frac{\partial^3 W}{\partial x^3}\right)\right] = \varphi, \tag{27}$$

$$\gamma\left[\rho \frac{\partial^2 \psi}{\partial t^2} + \frac{c_1(1+\nu)}{2}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{c_1(1-\nu)}{2}\frac{\partial^2 \psi}{\partial x^2} + c_1 \frac{\partial^2 \psi}{\partial y^2}\right] - \frac{2h^3}{3}\left[\rho \frac{\partial^3 W}{\partial y \partial t^2} + c_1\left(\frac{\partial^3 W}{\partial y \partial x^2} + \frac{\partial^3 W}{\partial y^3}\right)\right] = \psi, \tag{28}$$

$$2\rho h \frac{\partial^2 W}{\partial t^2} + \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = 0, \tag{29}$$

for φ, ψ and W, and the boundary conditions

$$\gamma\left(\frac{\partial \psi}{\partial y} + \nu \frac{\partial \varphi}{\partial x}\right) - \frac{2h^3}{3}\left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}\right) = 0, \tag{30}$$

$$\gamma\left(\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}\right) - 2\frac{\partial^2 W}{\partial x \partial y} = 0, \quad \psi = 0, \tag{31}$$

where

$$\beta = f(h) - f(-h) \quad \text{and} \quad \gamma = \frac{\int_{-h}^{h} z f(z)\,\mathrm{d}z}{c_{44}[f(h) - f(-h)]}. \tag{32}$$

Noting that for the Ambartsumian plate theory

$$f(z) = z\left(1 - \frac{z^2}{3h^2}\right), \quad \gamma = \gamma_A = \frac{2h^2}{5c_{44}},$$

whilst for the Reissner–Mindlin theory

$$f(z) = z, \quad \gamma = \gamma_{RM} = \frac{h^2}{3c_{44}},$$

since $\gamma_{RM} = 5\gamma_A/6$, the governing equations for the Reissner–Mindlin plate theory can be obtained from those for the Ambartsumian plate theory by multiplying the parameter γ by the factor 5/6. Levinson [21] noted that the equation of motion in his higher order plate theory is the same as that given by Mindlin [19] if the shear modulus is multiplied by the same factor. A more detailed study of the relationship between the Ambartsumian and Reissner–Mindlin theories of elastic plates is given in Belubekyan [28].

3. Solution of the problem

We look for edge wave solutions of the governing Eqs. (27)–(31) in the form of plane periodic waves:

$$\varphi = \mathrm{i}k\varphi_0 \exp[\mathrm{i}(\omega t - kx) - kpy], \tag{33}$$

$$\psi = k\psi_0 \exp[\mathrm{i}(\omega t - kx) - kpy], \tag{34}$$

$$w = w_0 \exp[\mathrm{i}(\omega t - kx) - kpy], \tag{35}$$

where ω is the frequency, k the wavenumber, and p is a parameter to be determined. Substituting (33)–(35) into (27)–(29) gives a system of equations for determining the unknown amplitudes φ_0,ψ_0 and w_0. From the solvability condition we obtain the following characteristic equation for the parameter p:

$$[(p^2 - 1 + \eta a)(p^2 - 1 + \eta a\theta) - \eta][2 + a\theta[(1-\nu)(1-p^2) - 2\eta a]] = 0, \tag{36}$$

$$\eta = \frac{\rho\omega^2}{ac_1k^2}, \quad a = \frac{1}{3}h^2k^2, \tag{37}$$

and $\theta = 6c_1/5c_{44}$ in the case of Ambartsumian plate theory and $\theta = c_1/c_{44}$ in the case of Reissner–Mindlin theory. Eq. (36) has 6 solutions p_j (j=1,…,6). The roots are real for any $a < 1$ if $\eta \leq \eta_0$, where

$$\eta_0 = \frac{2}{1 + a + a\theta + \sqrt{1 + a^2(1-\theta)^2 + 2a(1+\theta)}}, \tag{38}$$

and in this case can be written

$$p_1 \& p_4 = \frac{\pm\sqrt{2 - \eta a(1+\theta) - \sqrt{\eta}\sqrt{\eta a^2(1-\theta)^2 + 4}}}{\sqrt{2}}, \tag{39}$$

$$p_2 \& p_5 = \frac{\pm\sqrt{2 - \eta a(1+\theta) + \sqrt{\eta}\sqrt{\eta a^2(1-\theta)^2 + 4}}}{\sqrt{2}}, \tag{40}$$

$$p_3 \& p_6 = \pm\sqrt{1 - \frac{2\eta a}{1-\nu} + \frac{2}{\theta a(1-\nu)}}. \tag{41}$$

In order to satisfy the attenuation condition that the displacement should vanish as $y \to \infty$ we take only the positive roots p_1, p_2, p_3 and now write the solution to (27)–(29) as

$$\varphi = \sum_{j=1}^{3} C_j \varphi_{00}(p_j) \exp[\mathrm{i}(\omega t - kx) - kp_jy], \tag{42}$$

$$\psi = \sum_{j=1}^{3} C_j \psi_{00}(p_j) \exp[\mathrm{i}(\omega t - kx) - kp_jy], \tag{43}$$

$$w = \sum_{j=1}^{3} C_j w_{00}(p_j) \exp[\mathrm{i}(\omega t - kx) - kp_jy], \tag{44}$$

where C_j are arbitrary constants and the functions $\varphi_{00}(p)$, $\psi_{00}(p)$, $w_{00}(p)$ have the form:

$$\varphi_{00}(p) = p[2(p^2-1) + \eta a(2 + \theta + \theta\nu)], \tag{45}$$

$$\psi_{00}(p) = 2\eta - [2(1-\eta a)(1-\eta a\theta) - p^2(2 + \eta a\theta(1-\nu))], \tag{46}$$

$$w_{00}(p) = p[2 + a\theta(1 - 2\eta a - p^2(1-\nu) - \nu)]. \tag{47}$$

Substituting these solutions into the boundary conditions (30) and (31) we obtain the following system of algebraic equations for the unknown coefficients C_1, C_2 and C_3:

$$C_1B_1(p_1)+C_2B_1(p_2)+C_3B_1(p_3)=0, \tag{48}$$

$$C_1B_2(p_1)+C_2B_2(p_2)+C_3B_2(p_3)=0, \tag{49}$$

$$C_1B_3(p_1)+C_2B_3(p_2)+C_3B_3(p_3)=0, \tag{50}$$

where

$$B_1(p)=p(2a\eta\theta+ap^4\theta(v-1)-2v-a\theta(a\eta\theta-1)(2a\eta+v+v^2-2)+p^2(2+a\theta(a\eta(\theta(v-1)-2)-(v-1)(3+v)))), \tag{51}$$

$$B_2(p)=a\theta(\eta-(a\eta-1)(a\eta\theta-1))+ap^4\theta v+p^2(2+a\theta(1-v+a\eta(\theta v-1))), \tag{52}$$

$$B_3(p)=(2\eta-2(a\eta-1)(a\eta\theta-1)+p^2(2+a\eta\theta(v-1))). \tag{53}$$

The corresponding dispersion equation defining the phase velocity η is given by the expression

$$F(\eta,v)=0, \tag{54}$$

where

$$\begin{aligned}F(\eta,v)=&-ap_2p_3\theta(v-1)(2(p_1^2-a\eta)(1+a^2\eta^2\theta-\eta(1+a+a\theta))+2p_1^2(a\eta\theta-1)v^2)-ap_1p_3\theta(v-1)(2(p_2^2-a\eta)(1+a^2\eta^2\theta\\&-\eta(1+a+a\theta))+2p_2^2(a\eta\theta-1)v^2)+2p_1p_2(2-2\eta-a^2\eta^2\theta(v-5)-2v^2+a\theta(a\eta\theta-1)(a^2\eta^2(v-3)-(v-1)^2(1+v)\\&-2a\eta(v+v^2-2))+a\eta(\theta(v+2v^2-3)-2))-2((a\eta-1)(a\eta\theta-1)-\eta)(a^3\eta^2\theta(\theta(v-1)-2)\\&-(v-1)(a\theta(v^2-1)-4)+a\eta(2-\theta(v-1)(1+2a(2+v)))).\end{aligned} \tag{55}$$

Expanding this expression as a power series in a gives a first approximation to the dispersion equation in the form:

$$F_0(\eta)=2(1-v)\sqrt{1-\eta}+1-\eta-v^2+\sqrt{a\theta(1-v)}(1-\eta-v^2)\sqrt{1+\sqrt{1-\eta}}=0, \tag{56}$$

with attenuation condition $\eta<1$, instead of $\eta<\eta_0$. When $v\neq 0$, since

$$F_0(0)=(1-v)[(3+v)+\sqrt{2a\theta(1-v)}(1+v)]>0$$

and

$$F_0(1)=-v^2(1+\sqrt{a\theta(1-v)})<0,$$

the Eq. (56) must have a root in the interval $0<\eta<1$. When $v=0$ (56) reduces to

$$\sqrt{1-\eta}[2+\sqrt{1-\eta}(1+\sqrt{a\theta}\sqrt{1+\sqrt{1-\eta}})]=0.$$

This equation has no solution satisfying the condition $\eta<1$. It follows that with no Poisson contraction effect there are no solutions corresponding to a localized wave.

Setting θ=0 (55) takes the form

$$[2(1-v)-a\eta]\sqrt{1-(1+a)\eta}-\eta-v^2+1-\eta a=0, \tag{57}$$

for edge wave propagation within Kirchhoff theory taking account of rotatory inertia.

When a=0 Eq. (55) reduces to Konenkov's equation [1]

$$2(1-v)\sqrt{1-\eta}+1-\eta-v^2=0, \tag{58}$$

for the propagation of edge waves within the context of Kirchhoff theory for thick plates.

As it follows from (56) in problems of localized vibrations Kirchhoff theory approximation can be obtained by neglecting the relative thickness $\sqrt{a}$ of a plate, whereas in problems of vibrations of a simply supported plate neglecting the square of the plate thickness gives Kirchhoff's plate theory approximation [28].

Calculations illustrating these results are shown in Table 1 which compares wave frequencies for a transversely isotropic plate using Ambartsumian's theory with those calculated using classical Kirchhoff theory, both for a simply supported elongated plate and for a plate with free edge. In Table 1 ω_s and ω_l are respectively the minimal frequency of a simply supported elongated plate and the frequency of localized vibration calculated in the framework of Ambartsumian's refined theory, ω_{os} and ω_{ol} are the minimal frequency of the simply supported plate and the frequency of the localized vibration respectively calculated via the classical plate theory. Ambartsumian [29] has shown that for a simply supported plate the minimal natural bending frequencies ω_s and ω_{so} are related by the formula

$$\omega_s=\frac{\omega_{os}}{\sqrt{1+a\theta/4}}.$$

Table 1
Comparison of wave frequencies for Ambartsumian's theory with those calculated using classical Kirchhoff's theory.

hk	ω_s/ω_{os}				ω_l/ω_{ol}			
	$\theta=1$	$\theta=4$	$\theta=10$	$\theta=20$	$\theta=1$	$\theta=4$	$\theta=10$	$\theta=20$
0.05	0.999	0.999	0.998	0.997	0.993	0.992	0.992	0.991
0.10	0.999	0.998	0.995	0.991	0.989	0.985	0.976	0.962
0.20	0.998	0.993	0.983	0.968	0.979	0.963	0.931	0.885
0.30	0.996	0.985	0.964	0.932	0.964	0.924	0.867	0.789
0.50	0.989	0.960	0.908	0.841	0.918	0.837	0.721	0.605

The results show that whereas the classical theory provides a good approximation to the higher order theory for natural frequencies of simply supported plates, for thick plates the classical theory may significantly overestimate the frequencies of localized edge waves.

The above analysis can be relatively easily modified to provide solutions for localized bending waves along the free edges of a long narrow plate. We represent the plate as occupying the region $-\infty < x < \infty$, $-L \le y \le L$ and $-h \le z \le h$, and suppose that the edges $y = \pm L$ are free from mechanical loads. Because of the symmetry of the boundary conditions at these edges two modes of vibrations will occur: a symmetric mode and an anti-symmetric mode. For the symmetric mode we can write the general solution as

$$\varphi(y) = \sum_{j=1}^{3} C_j \varphi_{00}(p_j)\exp[\mathrm{i}(\omega t - kx)]\cosh[kp_j y], \tag{59}$$

$$\psi(y) = \sum_{j=1}^{3} C_j \psi_{00}(p_j)\exp[\mathrm{i}(\omega t - kx)]\cosh[kp_j y], \tag{60}$$

$$w(y) = \sum_{j=1}^{3} C_j w_{00}(p_j)\exp[\mathrm{i}(\omega t - kx)]\sinh[kp_j y]. \tag{61}$$

For the anti-symmetric mode we can write

$$\varphi(y) = \sum_{j=1}^{3} C_j \varphi_{00}(p_j)\exp[\mathrm{i}(\omega t - kx)]\sinh[kp_j y], \tag{62}$$

$$\psi(y) = \sum_{j=1}^{3} C_j \psi_{00}(p_j)\exp[\mathrm{i}(\omega t - kx)]\sinh[kp_j y], \tag{63}$$

$$w(y) = \sum_{j=1}^{3} C_j w_{00}(p_j)\exp[\mathrm{i}(\omega t - kx)]\cosh[kp_j y]. \tag{64}$$

From the boundary conditions (30) and (31) at free edges we again obtain the system of Eqs. (48)–(50), where in the symmetrical case $B_1(p)$ is the same as in (51), $B_2(p)$ in (52) is replaced by $B_2(p)\tanh(pkL)$, and $B_3(p)$ in (53) is replaced by $B_3(p)\tanh(pkL)$.

In the anti-symmetrical case $B_1(p)$ in (51) is replaced by $B_1(p)\tanh(pkL)$, $B_2(p)$ in (52) is replaced by $B_2(p)\tanh(pkL)$ and $B_3(p)$ in (53) remains the same.

For a narrow plate, since $kL \ll 1$, and so $\tanh(pkL) \approx pkL$, the dispersion equation for both the symmetric and anti-symmetric cases reduces to

$$1 + a^2\eta^2\theta - \eta(1 + a + a\theta) = 0. \tag{65}$$

Note that by letting $\mathrm{i}\omega \to -(\partial/\partial t)$ and $\mathrm{i}k \to \partial/\partial x$ we can rewrite this equation as

$$D\frac{\partial^4 w}{\partial x^4} + 2\rho h \frac{\partial^2}{\partial t^2}\left[w - \frac{(1+\theta)h^2}{3}\frac{\partial^2 w}{\partial x^2}\right] + \frac{4h^3(1-\nu^2)\theta}{3E}\frac{\partial^4 w}{\partial t^4} = 0. \tag{66}$$

Considering a rectangular beam of unit width and depth $2h$, with cross-sectional area S=$2h$ and second moment of area $I = 2h^2/3$, the beam vibration equation can be written as

$$E_0 I\frac{\partial^4 w}{\partial x^4} + \rho S\frac{\partial^2 w}{\partial t^2} - \rho I(1+\theta)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I\theta}{E_0}\frac{\partial^4 w}{\partial t^4} = 0, \tag{67}$$

where $E_0 = E/(1-\nu^2)$. Taking into account that for an isotropic plate $\theta = 6E_0/5G$, where G is the shear modulus, we can note that this equation coincides with Timoshenko beam equation [30] (where the numerical coefficient characterizing the form of cross-section is 5/6) and with equation of the Levinson's beam, when the Poisson ratio vanishes.

4. Analysis of the dispersion equation

The dispersion function $F(\eta,\nu)$ given by (55) has the following properties:

$$(1)\quad F(0,\nu) = 4(1-\nu)\left[3+\nu+a\theta(1-\nu^2)\left(1+\sqrt{1+\frac{2}{a\theta(1-\nu)}}\right)\right] > 0, \tag{68}$$

$$(2)\quad F(\eta_0,0) = 0, \tag{69}$$

$$(3)\quad F(\eta_0,\nu) < 0 \quad \text{for any } \nu \in [0,1/2], \tag{70}$$

$$(4)\quad F(\eta,\nu_1) < F(\eta,\nu_2) \quad \text{if } \nu_2 < \nu_1,\ \nu \in [0,1/2],\ \eta \in [0,\eta_0]. \tag{71}$$

It follows from properties (68)–(71) that for any value of the anisotropy parameter θ Eq. (55) always has a root, and hence that, since the Poisson coefficient ν is always non-zero, localized bending waves always exist. This result coincides with the corresponding results for the Kirchhoff plate [1]. Based on numerical "experimentation" of the dispersion equation the same result has been obtained in [27] for the isotropic Mindlin plate.

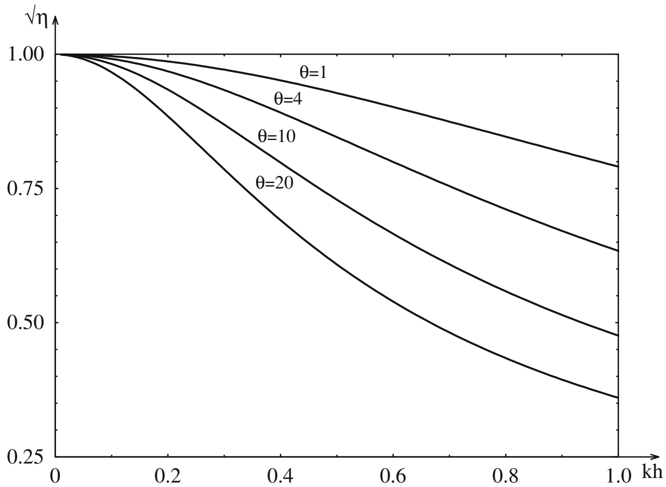

Fig. 1. Dispersion curves for different values of θ when ν=2/5.

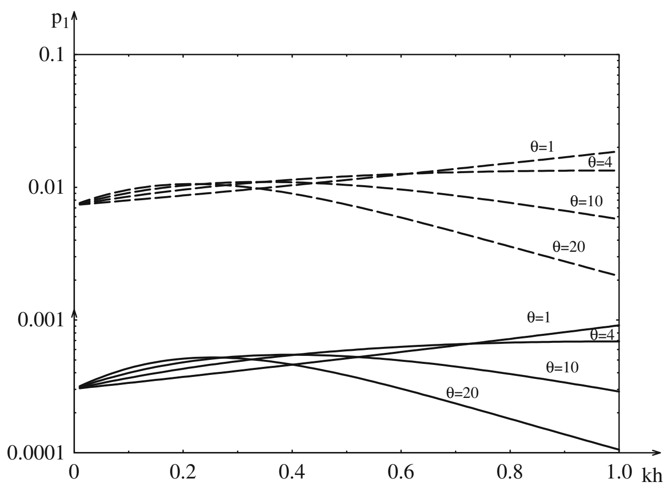

Fig. 2. Logarithmic plot of the attenuation coefficient $p_1 = \min_\eta[p_1(\eta), p_2(\eta), p_3(\eta)]$. Solid lines are for ν=1/5 and dashed lines for ν=2/5.

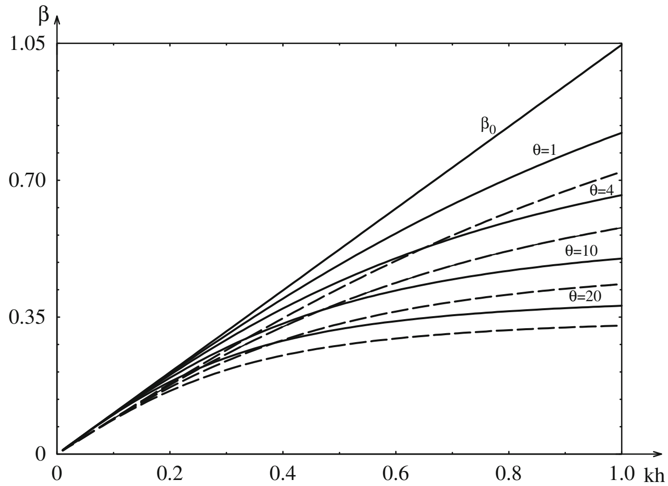

Fig. 3. Velocities for different θ, solid lines $\nu=1/5$, dash lines $\nu=2/5$. β_0 is the velocity corresponding to the classical Kirchhoff's plate.

The solutions of the dispersion Eq. (55) are shown in Figs. 1–3 which plot values of the dimensionless localized frequencies $\sqrt{\eta}$, the attenuation coefficient $\log p_1$ and the phase velocity of the localized waves normalized to the bulk shear in-plane wave velocity versus the thickness parameter kh. The ordinates of all plots in Figs. 1–3 at $kh=0$ correspond to the solutions for the Kirchhoff plate [1].

Dispersion curves for different values of the anisotropy coefficient θ are shown in Fig. 1 for $\nu=2/5$. According to formulae (20) $\theta=4$ corresponds to the isotropic plate. The numerical calculations show that for thin plates the effect of the anisotropy is unnoticeable. With the plate thickness increasing the effect of anisotropy grows and for $\theta=20$ reaches to nearly 20 percent.

Fig. 2 shows that the change of the attenuation coefficient is in the order of 10 from $\nu=1/5$ to $\nu=2/5$. For a thick plate with much stronger elastic properties in the transversal direction than elastic properties in the longitudinal direction the attenuation coefficient increases by another order. This means that localization of waves for thick plates increases i.e. the waves are attenuating faster from the free edge of the plate.

Note also that the difference between the values of the bending frequencies for the Ambartsumian plate and the Kirchhoff plate for thick plates is more than the corresponding difference between values of phase velocities β and β_0 (Fig. 3).

5. Conclusion

Localized bending waves have been studied in an elastic transversely isotropic thick plate using the high order refined theories of elastic plates. The dispersion equation has been obtained and analyzed for the existence of a localized wave at the free edge. It is noted that the dispersion equation of a simply supported plate of the Kirchhoff plate theory is obtained by neglecting the square of the plate thickness in the corresponding dispersion equation that takes account of shear stresses. In the problem of localized vibrations the Kirchhoff plate theory approximation is obtained by neglecting to first order the relative thickness of the plate. This shows that classical plate theory is more suitable for simply supported plate than for a plate with free edges. If an acceptable limit for an error is estimated to be in the order of 10 percent then Kirchhoff's plate theory will not be applicable for thick transversely isotropic plates.

It has been shown that there are no qualitative differences between results given by the first order Reissner–Mindlin and the high order Ambartsumian's theories. The correlation coefficient for shear coefficient is 5/6 the same as between the high order Levinson's and Mindlin's theories. Analysis of frequencies of localized bending waves show that for thick plates the effect of anisotropy grows up to 20 percent.

A case of a narrow plate has been also considered. It is shown that Timoshenko's beam equation can be obtained from a dispersion equation of a narrow plate with free edges.

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