



Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Validity diagrams of statistical energy analysis

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ARTICLE INFO

Article history:

Received 31 January 2009

Received in revised form

3 September 2009

Accepted 8 September 2009

Handling Editor: C.L. Morfey

Available online 6 October 2009

ABSTRACT

This paper is concerned with the validity domain of statistical energy analysis (SEA) which is defined in terms of four criteria. The mode count N and the modal overlap M must be high, the normalized attenuation factor \bar{m} and the coupling strength γ must be small. The application of dimensional analysis on the governing equations of plates gives the space of dimensionless parameters in which the validity domain of SEA must be delimited. This domain is discussed on the basis of geometry of the surfaces delimiting it. The diagrams of validity of SEA are introduced and discussed. A numerical simulation on a couple of rectangular plates coupled along one edge illustrates the theoretical approach.

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1. Introduction

Validity of statistical energy analysis (SEA) [1] was among the main concerns in the early development of the theory. The large use of SEA in engineering and, in the mean time, the difficulties encountered to meet the assumptions of SEA for practical structures, have motivated many basic studies to well understand what is SEA and what are its limitations.

The former development of SEA is largely inspired from statistical physics [2]. SEA were conceived as the theory of energy exchanges between several groups of modes [3–5]. Modes play the role of molecules that is the sites where the energy is localized. Their exact frequencies and their shapes are of no importance but their number within a frequency band and their ability to exchange energy with modes of other groups is of a great importance. SEA is a theory of collective behaviour of modes while the governing equations describe their individual behaviour. As well, SEA is the thermodynamics of structural and acoustical vibrations.

The simplicity of SEA equations largely results from the concept of thermal equilibrium. Although the foundations of SEA were laid from the beginning, the discussion rapidly turned to the question of the validity of such an approach. The assumptions of SEA have been discussed in many papers. In the early work of Woodhouse [6], the crucial question of the proportionality between exchanged power and the difference of modal energies is tackled by transforming the stiffness and mass matrices of the overall system. This rigorous approach based on linear algebra leads to several strict definitions of weak coupling. The critical overview of SEA by Fahy [7] points out several difficulties of SEA and raises several questions of a great importance for foundations of SEA. Thirty years later, some of them such as confidence, spatial distribution of energy, indirect coupling and non-conserving coupling have become fashionable subjects in SEA literature. The existence of indirect coupling between sub-systems which are not physically connected is certainly the most strange phenomenon in SEA whose discovery has shaken the base of SEA. Its systematic study by Mace [8,9] who distinguishes ‘quasi-SEA’ and ‘proper SEA’ models, largely contributes to improve the understanding of what is SEA and what are its limitations. It also

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clarifies the necessity of the assumption of large mode number [10–12]. More recently, the work of Culla and Sestieri [13] or the observations of Finnveden [14] show that the subject of the underlying assumptions of SEA is far to be closed.

Although there is some differences in the details when formulating the assumptions, the notion of weak coupling for instance is not defined in the same way by all authors, it seems that there is a large consensus on a ‘minimal’ set of assumptions. These are a large population of modes, wide-band and uncorrelated excitations, large modal overlap, diffuse field, equipartition of energy, light and conservative coupling. Some of these assumptions are redundant and therefore they are not all simultaneously required for SEA to apply [15]. But, each assumption has been discussed in the literature and for almost each of them an extension of SEA have been proposed. The strong coupling is discussed in Ref. [16], the equipartition of modal energy is relaxed in Ref. [17], non-isotropic field is considered in Ref. [18] and non-homogeneous field in Ref. [19], the dissipation in couplings is taken into account in Ref. [20], the extension to mid-frequencies in Refs. [21,22] and the transition from mid to high frequencies in Ref. [23]. Finally, a generalization of SEA valid for non-equilibrium state (non-diffuse field) and based on a transfer equation analogous to Boltzmann’s equation is proposed in Refs. [24–27].

The aim of this paper is to describe the validity domain of SEA in the space of appropriate dimensionless parameters. SEA is considered in its strict sense in the perspective of statistical physics that is without previously mentioned extensions. The assumptions are therefore large population of modes, ‘rain-on-the-roof’ excitation, large modal overlap, diffuse field, equipartition of energy, light and conservative coupling. Each assumption imposes a constraint in the space of parameters and these constraints are summarized in some diagrams where it is possible to directly visualize the position of the actual system and to check if it is located within or outside the validity domain of SEA.

This paper is organized as follows. In Section 2 are introduced the notations and the basic relationships of SEA. In Section 3, the assumptions of SEA are presented and discussed within the context of statistical physics. In Section 4 the dimensional analysis is applied to the governing equations of plates and the set of dimensionless parameters is defined. In Section 5, the validity domain of SEA is plotted in the space of dimensionless parameters and the validity diagrams are presented in the particular case of a pair of rectangular plates coupled along a common edge. A comparison of SEA results with a direct numerical simulation is presented in Section 6. Finally, a discussion for complex structures is given in Section 7.

2. Basics of SEA

SEA is a simple method to assess the vibrational energy of systems divided into n sub-systems. SEA is entirely based on some statistical considerations and the application of the energy balance. A complete derivation of SEA is available in numerous reference texts [1,28,5] and this section only summarizes the main relationships useful for the discussion.

SEA is a method intended to the prediction of vibrational levels in broadband. Therefore, the analysis is confined to a frequency band $\Delta\omega$ about the central frequency ω (rad/s). No strict definition is given for the width of the frequency band, but it is commonly admitted that octave bands are well suited. The frequency band $\Delta\omega$ is then simply related to the central frequency ω by $\Delta\omega = \omega/\sqrt{2}$.

A sub-system is defined as homogeneous set of modes whose frequencies lie in the frequency band. It may be viewed as an ensemble of N oscillators. The modal density n is defined as the number of modes per rad/s,

$$n = \frac{N}{\Delta\omega}. \quad (1)$$

The exact expression of the modal density depends on the nature of the system. In case of plates submitted to flexural vibrations, the asymptotic expression based on wave propagation analysis is [28]

$$n = \frac{S\omega}{2\pi c_g c_\phi}, \quad (2)$$

where S is the area of the plate, c_g and c_ϕ the group speed and the phase speed of the flexural wave. Several improvements of this expression can be found in the literature [1,29].

In steady-state condition, the energy balance for sub-system i reads

$$P_i^{\text{diss}} + \sum_{j \neq i} P_{ij} = P_i^{\text{inj}}. \quad (3)$$

The power being injected P_i^{inj} is assumed to be known, but the powers being dissipated P_i^{diss} and being exchanged P_{ij} must be expressed in terms of vibrational energies.

Let denote by E_i where $i = 1, \dots, n$ the vibrational energy of sub-system i . The modal energy, also called vibrational temperature, is defined as $T_i = E_i/n_i$. Both variables E_i and T_i can be adopted as primary variable of SEA. The total vibrational energy E_i is more important for a practical analysis of the system but, the modal energy T_i has a more profound physical meaning and is sometimes considered as the variable well suited for SEA. The idea that a vibrational temperature with a value different from thermal temperature can drive the system dynamics was in fact the first motivation for the early development of SEA [2].

The power being dissipated by internal losses is [30]

$$P_i^{\text{diss}} = \omega \eta_i E_i, \tag{4}$$

where η_i is the damping loss factor usually determined by a direct measurement.

The power supplied by the sub-system i to the sub-system j is

$$P_{i \rightarrow j} = \omega \eta_{ij} E_i, \tag{5}$$

where η_{ij} is the coupling loss factor. The net exchanged power between sub-systems i and j is $P_{ij} = P_{i \rightarrow j} - P_{j \rightarrow i}$ and therefore,

$$P_{ij} = \omega(\eta_{ij} E_i - \eta_{ji} E_j). \tag{6}$$

The standard wave-based relationship of the coupling loss factor η_{ij} for two adjacent plates with a coupling of length L_{ij} is [31,1]

$$\eta_{ij} = \frac{L_{ij} c_{g_i}}{\pi S_i \omega} \tau_{ij}, \tag{7}$$

where τ_{ij} is the mean transmission efficiency from plates i to j .

The coupling loss factors verify the reciprocity relationship [32–34]

$$n_i \eta_{ij} = n_j \eta_{ji}. \tag{8}$$

The reciprocity relationship highlights the importance of the vibrational temperature. Substituting Eq. (8) into Eq. (6) leads to

$$P_{ij} = \omega \eta_{ij} n_i (T_i - T_j), \tag{9}$$

showing that the net exchanged power is proportional to the difference of modal energies or vibrational temperatures. This is the well-known power flow equation which has been derived in a large number of situations [4,35,3].

The SEA equation is simply obtained by substituting Eqs. (4) and (6) in Eq. (3),

$$\omega \begin{pmatrix} n_1 \sum_j \eta_{1j} & & -n_1 \eta_{1k} \\ & \ddots & \\ -n_k \eta_{kl} & & n_n \sum_j \eta_{nj} \end{pmatrix} \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} = \begin{pmatrix} P_1^{\text{inj}} \\ \vdots \\ P_n^{\text{inj}} \end{pmatrix}. \tag{10}$$

This is a linear system on vibrational temperatures whose matrix is symmetric.

3. Validity criteria of Sea

Let now turn to the discussion on the basic assumptions of SEA. SEA is a statistical method applied to the audio frequency range (macroscopic vibrations) in the same manner that thermodynamics is a statistical method applied to thermal vibrations at the molecular scale (microscopic vibrations). While in thermodynamics the statistical population is composed of a large number of molecules, atoms or any other sites which store the vibrational energy, in SEA, the energy is localized in large number N of modes. The number of molecules of thermodynamical systems is of order of Avogadro's number (10^{23}). But in SEA, the number of modes may be only of order of several thousands [7] or even several millions in the best case [36] which is always very low compared with Avogadro's number. This highlights that SEA is a statistical method applied to small populations, and indeed, this fact can cause some difficulties. Fluctuations around the mean are more important for relatively small populations. In Refs. [37,38], the relative variance (the variance divided by the square of the mean) is found to be of order of $\log N/N^2$. The size of the population of modes is therefore the first criterion for the applicability of SEA to prediction of the response of a single system (as opposed to the ensemble average response). The mode count N that is the number of modes within the frequency band $\Delta\omega$ is

$$N = n\Delta\omega, \tag{11}$$

and a large population of modes reads

$$N \gg 1. \tag{12}$$

This is the first criterion of validity of SEA.

It could be argued that a second point of view is possible. Instead of applying SEA to a single system with many modes, SEA can be applied to a population of similar systems each of them having few modes. Predictions of SEA are then compared with the ensemble average of energies of individual systems. Although this interpretation of SEA is not explicitly mentioned in early papers of SEA, it was present in the mind of these authors. For instance, Ref. [39] is a discussion on the distribution law of modes in the statistical population, and the discussion in Ref. [40] clearly presents SEA as it was. This is now a point of view widely spread in engineering especially in automotive industry where the population of systems is large [41]. The introduction of statistical population in SEA is quite similar to the so-called canonical ensemble by Gibbs in statistical mechanics where the equivalence of time average and ensemble average is ensured by the ergodic assumption.

In both version of SEA, fluctuations around the mean are present but are negligible provided that the number of modes is large in the strict version of SEA, and the size of population is large in this second interpretation of SEA. Thus, if N denotes the cumulative number of modes, i.e. the sum of all individual mode counts, the statistical method makes sense for $N \gg 1$ that is Eq. (12).

SEA is the study of incoherent vibrational energy in the same manner that thermodynamics is the study of 'degraded' mechanical energy. This state of degradation for energy only arises when the disorder prevails in the statistical population. Disorder is inherent to the statistical method. Simple laws can emerge from the behaviour of a large population provided that all 'individuals' are similar and that any of them may influence the population more than other ones. For thermal vibrations of solids, disorder means that the vibration of atoms are uncorrelated. While in the kinetic theory of gases, disorder means that the velocities of molecules before the shock are statistically independent. This is the so-called molecular chaos or *Stosszahlansatz* introduced by Boltzmann in 1872 [42]. Disorder in vibroacoustics rather means that mode amplitudes, considered as random variables, are uncorrelated. This state is reached when no mode dominates the dynamics of the system that is when the frequency response function is smooth. The modal overlap defined as

$$M = n\eta\omega, \quad (13)$$

is a measure of the overlapping of successive modes in the frequency response function. Thus, the criterion for disorder in SEA is

$$M \gg 1. \quad (14)$$

The diffuse field assumption of SEA means that the vibrational energy density is homogeneous and isotropic in each sub-system. But it has been remarked [7] that the assumption equivalent to diffuse field in the modal approach of SEA is the equipartition of modal energy (the vibrational energy is equally shared among all modes). The proportionality between the energy density and the modal energy stems from the proportionality of the modal density of two-dimensional sub-systems with the surface as in Eq. (2). But the modal energy plays the same role as the energy per molecule in thermodynamics which is exactly the definition of the temperature times Boltzmann's constant. This is why the modal energy can be called the vibrational temperature (a general demonstration of this fact is available in Refs. [43,44]). Thus, the diffuse field assumption means that the vibrational temperature is the same at any point of the sub-system or, in other words, that the sub-system is in thermal equilibrium. To reach this equilibrium state, it is necessary that rays are mixed. The general mathematical conditions under which a diffuse field can emerge are studied in billiards theory [45]. But at least, rays must cross several times the sub-system before to be attenuated. If $m = \eta\omega/c_g$ designates the attenuation factor of wave per meter,

$$\bar{m} = \frac{\eta\omega}{c_g} l, \quad (15)$$

can be called the normalized attenuation factor [46], l being the mean free path of the sub-system. Its value must be low to ensure the mixing of rays that is the thermal equilibrium,

$$\bar{m} \ll 1. \quad (16)$$

Each sub-system is in thermal equilibrium. But two adjacent sub-systems may have different vibrational temperatures. This is the assumption of local equilibrium. The same situation arises in non-equilibrium thermodynamics. The notion of local temperature makes sense providing that a local equilibrium is reached. The thermal energy flows higher temperature to lower temperature. In SEA, the linearity of the net exchanged power with vibrational temperatures, Eq. (6), is simply the expression of linearity between fluxes and forces in linear irreversible thermodynamics [47], $\omega\eta_{ij}n_i$ being the appropriate transport coefficient. It is well-known in non-equilibrium thermodynamics that the linearity of fluxes and forces is valid for systems which are not too far from equilibrium. This is the light coupling assumption. In the context of SEA, the light coupling assumption means that the flow of exchanged vibrational energy is small compared with the internal dissipation of energy. This can be enunciated as $\eta_{ij} \ll \eta_i$. Although some authors have studied the possibility to extend SEA to strong coupling [16], the usual relationships for coupling loss factors are derived under the light coupling assumption. Following Smith [48], the coupling strength is defined as $\gamma_{ij} = \eta_{ij}/\eta_i$. In case of assembled plates, η_{ij} is given by Eq. (7). Since the mean free path of two-dimensional domain is $l = \pi S/P$ where S is the area and P the perimeter of the domain, it yields

$$\gamma_{ij} = \frac{\tau_{ij}L_{ij}}{\bar{m}_iP_i}. \quad (17)$$

The light coupling condition reads

$$\gamma_{ij} \ll 1. \quad (18)$$

A complete transmission ($\tau_{ij} = 1$) over a small length in a large plate ($L_{ij} \ll P_i$) leads to a light coupling.

Indeed, the symmetric condition must also apply

$$\gamma_{ji} \ll 1. \quad (19)$$

But the couple of conditions Eqs. (18) and (19) are not independent. To check this assertion, let substitute $\eta_{ij} = \gamma_{ij}\eta_i$ in Eq. (8) and let multiply both hand-sides by ω . The modal overlap as defined in Eq. (13) then appears. It yields

$$\gamma_{ij}M_i = \gamma_{ji}M_j. \tag{20}$$

The set of dimensionless parameters γ_{ij} , γ_{ji} , M_i and M_j is therefore dependent.

Some other assumptions of SEA are not directly taken into account in these criteria. The most important of them is concerned with the nature of the excitation. It is commonly admitted that excitations must be wide-band, spatially distributed and uncorrelated in SEA. The ‘rain-on-the-roof’ excitation meets these conditions. However, SEA sometimes applies even for localized excitations and the choice of the bandwidth $\Delta\omega$ is rather important to ensure a small variance. In some texts, ‘rain-on-the-roof’ excitation is not considered as an absolutely necessary assumption but rather than a condition favourable to the establishment of diffuse field and small variance [15].

4. Space of dimensionless parameters

The purpose of this section is to apply the dimensional analysis [49] to exhibit the space of dimensionless parameters in which the validity domain of SEA must be delimited. The discussion is conducted on a vibrating system made of n rectangular plates with length a_i and a common width b . They are assumed to be coupled along their edges of length b and only the out-of-plane vibration is considered (Fig. 1).

Let first discuss the case $n = 1$. The governing equation (Love’s plate equation [50]) for the out-of-plane vibration is

$$\Delta^2 v_i - k_i^4(1 - j\eta_i)v_i = f_i, \tag{21}$$

where the imaginary part $-j\eta_i$ is the contribution of damping to the wavenumber k_i . The boundary conditions are prescribed on the deflection, the rotation angle of section, the moment and the transverse force at edges. Only two conditions are required at each edge. For instance, clamped edges at $x = 0$ and a_i ,

$$v_i(0, y) = v_i(a_i, y) = 0, \tag{22}$$

$$\frac{\partial v_i}{\partial x}(0, y) = \frac{\partial v_i}{\partial x}(a_i, y) = 0. \tag{23}$$

And simply supported edges at $y = 0, y = b$,

$$v_i(x, 0) = v_i(x, b) = 0, \tag{24}$$

$$\frac{\partial^2 v_i}{\partial y^2}(x, 0) + v_i \frac{\partial^2 v_i}{\partial x^2}(x, 0) = \frac{\partial^2 v_i}{\partial y^2}(x, b) + v_i \frac{\partial^2 v_i}{\partial x^2}(x, b) = 0. \tag{25}$$

So, the only physical parameters of this set of equations are the wavenumber k_i , the damping loss factor η_i , the length a_i , the width b , and Poisson’s coefficients ν_i that is five physical parameters. Their only physical unit is the length (the time and the mass do not appear in these parameters). The theorem of Vaschy–Buckingham [51,52] gives the number of dimensionless parameters of this problem, $5 - 1 = 4$. These dimensionless parameters can be chosen arbitrarily provided that they are independent. Among the possible choices is the dimensionless wavenumber $\kappa_i = k_i l_i / 2\pi$ where $l_i = \pi a_i b / 2(a_i + b)$ is the mean free path (this is also the number of wavelengths per mean free path), the shape ratio $\varepsilon_i = (a_i + b) / \sqrt{\pi a_i b}$ defined as the ratio between the perimeter of the plate and that of a circle of the same area, the damping loss factor η_i and Poisson’s coefficient ν_i . This set of dimensionless parameters is well-suited to rewrite the governing equation (21) and the related

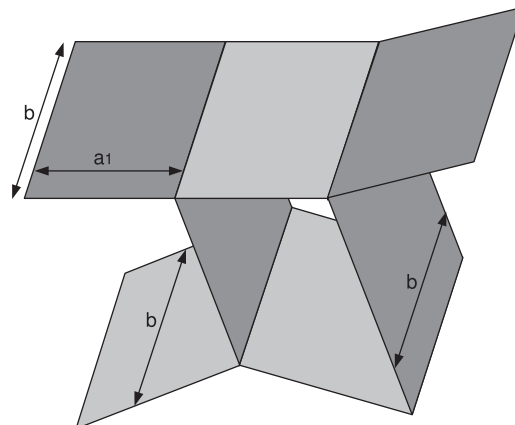


Fig. 1. Dimensional analysis on assembling of n rectangular plates of same width b and various lengths a_i and coupled along their edges of width b . Only the out-of-plane vibration is taken into account.

boundary conditions (22)–(25) in dimensionless form. As well, it will be called ‘primary’ set of dimensionless parameters. But any other choice of independent dimensionless parameters is possible. In particular, the set of dimensionless parameters of SEA introduced in the previous section is acceptable provided that a one to one map can be found:

$$\kappa_i, \eta_i, \varepsilon_i, v_i \longrightarrow N_i, M_i, \bar{m}_i, v_i. \quad (26)$$

These relationships are easily found:

$$N_i = 2\sqrt{2}\kappa_i^2 \varepsilon_i^2, \quad (27)$$

$$M_i = 4\eta_i \kappa_i^2 \varepsilon_i^2, \quad (28)$$

$$\bar{m}_i = \pi \eta_i \kappa_i, \quad (29)$$

$$v_i = v_i. \quad (30)$$

Indeed, Poisson’s coefficient remains unchanged in this transformation. The problem of a single vibrating plate is mathematically fully determined by the only four dimensionless parameters N , M , \bar{m} and v . We can admit that Poisson’s coefficient is of a low importance in SEA. This is a reasonable assumption since, excepted for non-typical materials such as rubber, Poisson’s coefficient usually ranges from 0.2 and 0.3. Since SEA is a theory included in Love’s theory of plate in the sense that equation of SEA can be derived from the governing equation of Love’s plate, the validity domain of SEA is necessarily confined into the three-dimensional space N , M , \bar{m} . The conditions (12), (14) and (16) give the boundary of the validity domain of SEA for a single plate.

Let now consider the coupling between two adjacent rectangular plates of size $a_i \times b$. The coupling conditions along the common edge of length b are the continuity of deflection and rotation of section and the balance of the moments and transverse forces. For instance, if the y -axis is chosen along the common edge

$$v_1(0, y) = v_2(0, y), \quad (31)$$

$$\frac{\partial v_1}{\partial y}(0, y) = \frac{\partial v_2}{\partial y}(0, y), \quad (32)$$

$$D_1 \left[\frac{\partial^2 v_1}{\partial x^2}(0, y) + v_1 \frac{\partial^2 v_1}{\partial y^2}(0, y) \right] = D_2 \left[\frac{\partial^2 v_2}{\partial x^2}(0, y) + v_2 \frac{\partial^2 v_2}{\partial y^2}(0, y) \right], \quad (33)$$

$$D_1 \left[\frac{\partial^3 v_1}{\partial x^3}(0, y) + v_1 \frac{\partial^3 v_1}{\partial y^2 \partial x}(0, y) \right] = D_2 \left[\frac{\partial^3 v_2}{\partial x^3}(0, y) + v_2 \frac{\partial^3 v_2}{\partial y^2 \partial x}(0, y) \right]. \quad (34)$$

Therefore, the bending stiffness D_1 and D_2 of plates are also relevant physical parameters. They introduce the new unit Newton. The number of independent physical parameters is now 11 including k_1 , k_2 , η_1 , η_2 , a_1 , a_2 , b , v_1 , v_2 , D_1 and D_2 . The number of dimensionless parameters is therefore $11 - 2 = 9$. Eight parameters are the previous ones, κ_i , ε_i , η_i and v_i for $i = 1, 2$. One additional dimensionless parameters is expected. One can choose the ratio of bending stiffness D_1/D_2 or, equivalently, the transmission efficiency τ_{12} of the interface (ratio of transmitted power and incident power of a plane wave with diffuse incidence). Thus, the set of ‘primary’ dimensionless parameters is, κ_1 , η_1 , ε_1 , v_1 , κ_2 , η_2 , ε_2 , v_2 , τ_{12} . The space of dimensionless parameters of SEA must include the dimensionless parameters of isolated plates, N_1 , M_1 , \bar{m}_1 , v_1 , N_2 , M_2 , \bar{m}_2 , v_2 . One additional dimensionless parameters must be prescribed for the coupling. Since γ_{12} and γ_{21} are not independent (Eq. (20)), a single coupling strength, say γ_{12} , can be retained. The one-to-one map to seek is thus

$$\kappa_1, \eta_1, \varepsilon_1, v_1, \kappa_2, \eta_2, \varepsilon_2, v_2, \tau_{12}, \longrightarrow N_1, M_1, \bar{m}_1, v_1, N_2, M_2, \bar{m}_2, v_2, \gamma_{12}. \quad (35)$$

Indeed, Eqs. (27)–(30) remain valid for $i = 1, 2$. The last relationship is

$$\gamma_{12} = \frac{\tau_{12}}{\pi \eta_1 \kappa_1} \mu, \quad (36)$$

where $\mu = b/2(a_1 + b)$. Since $\varepsilon_1 = (a_1 + b)/\sqrt{\pi a_1 b}$, μ is a function of ε_1 easily determined as

$$\frac{1}{\mu} = \begin{cases} \pi \varepsilon^2 - \sqrt{\pi \varepsilon^2 (\pi \varepsilon^2 - 4)} & \text{if } b > a, \\ \pi \varepsilon^2 + \sqrt{\pi \varepsilon^2 (\pi \varepsilon^2 - 4)} & \text{if } b < a. \end{cases} \quad (37)$$

Omitting Poisson’s coefficients, the validity domain of SEA is a subset of the seven-dimensional space N_1 , M_1 , \bar{m}_1 , N_2 , M_2 , \bar{m}_2 , γ_{12} . The limits of the SEA domain are given in Eqs. (12), (14) and (16) for each plate and Eqs. (18) and (19) for the coupling. The first seven conditions are hyper-planes in the seven-dimensional space, but the last condition ($\gamma_{21} \ll 1$) which can be expressed as

$$\gamma_{12} \frac{M_1}{M_2} \ll 1 \quad (38)$$

is a family of curves in the seven-dimensional space.

Finally, this result can be generalized to the case of n plates. The physical parameters are then, k_i , η_i , a_i , v_i and D_i for $i = 1, \dots, n$ that is five parameters per plate plus the width (the coupling length) b that is $5n + 1$ physical parameters. The physical units are always length and force, the number of dimensionless parameters is therefore $5n + 1 - 2$. The ‘primary’ dimensionless parameters are κ_i , η_i , ε_i , v_i for each plate (this provides $4n$ dimensionless parameters) and the remaining ones must be found among the transmission efficiencies τ_{ij} . But the ratios D_i/D_j are not independent since $D_i/D_j = D_1/D_j \times D_i/D_1$. The number of independent ratios D_i/D_j is $n - 1$. Therefore, a same number of τ_{ij} must be retained as independent dimensionless parameters. For instance, the sequence $\tau_{i,i+1}$, $i = 1, \dots, n - 1$ can be chosen. This provides the additional $n - 1$ ‘primary’ dimensionless parameters. The SEA dimensionless parameters indeed include those of isolated plates, N_i , M_i , \bar{m}_i and v_i , for $i = 1, \dots, n$. The other parameters stemming from the couplings are $\gamma_{i,i+1}$ for $i = 1, \dots, n - 1$. The transformation law

$$\kappa_i, \eta_i, \varepsilon_i, v_i, \tau_{i,i+1} \longrightarrow N_i, M_i, \bar{m}_i, v_i, \gamma_{i,i+1} \tag{39}$$

of the ‘primary’ set of dimensionless parameters to the SEA set of parameters is constituted by Eqs. (27)–(30) for $i = 1, \dots, n$ and,

$$\gamma_{i,i+1} = \frac{\tau_{i,i+1}}{\pi \eta_i \kappa_i} \mu(\varepsilon_i), \tag{40}$$

respectively, for $i = 1, \dots, n - 1$. The validity domain of SEA is confined into this $(5n - 1)$ -dimensional space by Eqs. (12), (14) and (16) for each plate and Eqs. (18) and (19) for each coupling.

5. Diagrams of validity

In this section, the validity conditions of SEA are clarified in the $\kappa, \eta, \varepsilon$ -space of ‘primary’ dimensionless parameters.

In the case of a single plate, the space of dimensionless parameters has dimension 3. In the N, M, \bar{m} -space of SEA parameters, the conditions (12), (14), (16) define a domain limited by three planes whose equations are $N = 1$, $M = 1$ and $\bar{m} = 1$. The validity domain of SEA delimited by these three planes is shown in Fig. 2. A section along the N, M -plane is shown in Fig. 2(a), a section along the \bar{m}, M -plane in Fig. 2(b) and a section along the \bar{m}, N -plane in Fig. 2(c). It is clear that the validity domain of SEA is an unbounded domain in the N, M, \bar{m} -space. But in the $\kappa, \eta, \varepsilon$ -space of ‘primary’ dimensionless parameters, these three planes become three surfaces. By virtue of Eqs. (27)–(30), the equations of the surfaces are

$$\kappa^2 \varepsilon^2 = \frac{1}{2\sqrt{2}}, \tag{41}$$

$$\eta \kappa^2 \varepsilon^2 = \frac{1}{4}, \tag{42}$$

$$\eta \kappa = \frac{1}{\pi}, \tag{43}$$

respectively, for $N = 1$, $M = 1$ and $\bar{m} = 1$.

In the κ, η -plane, the condition $N = 1$ reads $\kappa \propto 1$ that is a vertical line which position depends on the value of ε . The condition $M = 1$ reads $\eta \propto 1/\kappa^2$ and the condition $\bar{m} = 1$ becomes $\eta \propto 1/\kappa$ a hyperbolic line. The domain of validity is thus confined within the strip shown in Fig. 3(a). A question is to check whether the top line $\bar{m} = 1$ and the bottom line $M = 1$ cross somewhere. If the answer is yes, SEA validity would be confined below this frequency limit. But the top line decreases less rapidly than the bottom line since the ratio $(1/\kappa)/(1/\kappa^2)$ goes to infinity, and it can be concluded that it does not exist a frequency limit to SEA.

In the κ, ε -plane, the condition $N = 1$ leads to $\varepsilon \propto 1/\kappa$, a hyperbolic line, the condition $M = 1$ also gives $\varepsilon \propto 1/\kappa$ but usually at a higher level since the proportionality coefficient is like $1/\eta$, and $\bar{m} = 1$ gives $\kappa \propto 1$, a vertical line. A last condition $\varepsilon > 1$ is imposed by the definition of ε which is always greater than unity (the perimeter of a circle is lower than the one of any other shape). The domain is shown in Fig. 3(b). The vertical line ($\bar{m} = 1$) imposes an upper frequency limit. However, this does not contradict the previous observation. From Eq. (30), $\bar{m} \propto \eta \kappa$ and an increase of the frequency κ while maintaining η constant leads to an increase of \bar{m} . This process is shown in Fig. 4. The upper frequency in the κ, ε -plane then corresponds to the limit $\bar{m} = 1$.

In the η, ε -plane, the condition $N = 1$ gives $\varepsilon \propto 1/\kappa$ a horizontal line, the condition $M = 1$ gives $\varepsilon \propto 1/\sqrt{\eta}$ and the condition $\bar{m} = 1$ leads to $\eta \propto 1$ a vertical line. The physical limit $\varepsilon > 1$ also holds. The relative position of the two horizontal lines $\bar{m} = 1$ and $\varepsilon = 1$ can be inverted. The resulting domain is shown in Fig. 3(c).

The case of structures with several plates is straightforward. The space of dimensionless parameters has dimension $4n - 1$ (omitting Poisson’s coefficients) and the conditions of validity are Eqs. (12)–(16) for each plate and (18) and (19) for each coupling.

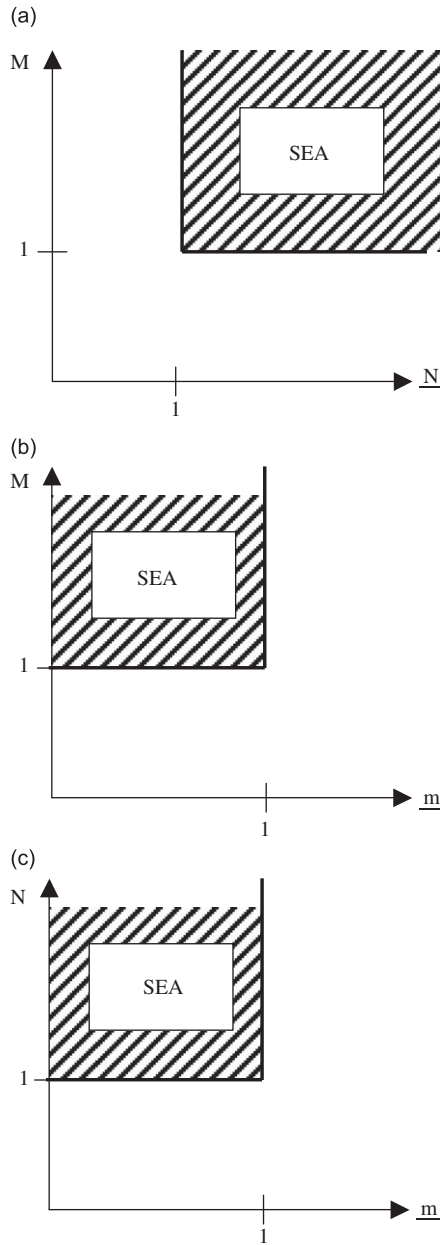


Fig. 2. Validity domain of SEA for a single plate in the N, M, \bar{m} -space of SEA dimensionless parameters. Sections of the domain on the (a) N, M -plane, (b) \bar{m}, M -plane and (c) \bar{m}, N -plane.

In the space of ‘primary’ parameters, the equations of the varieties which delimit the validity domain of SEA are

$$\kappa_i^2 e_i^2 = \frac{1}{2\sqrt{2}}, \tag{44}$$

$$\eta_i \kappa_i^2 e_i^2 = \frac{1}{4}, \tag{45}$$

$$\eta_i \kappa_i = \frac{1}{\pi}, \tag{46}$$

$$\frac{\tau_{ij} \mu(\varepsilon_i)}{\eta_i \kappa_i} = \pi, \tag{47}$$

$$\frac{\tau_{ij} \mu(\varepsilon_i)}{\eta_i \kappa_i} \times \frac{\eta_i \kappa_i^2 e_i^2}{\eta_j \kappa_j^2 e_j^2} = \pi, \tag{48}$$

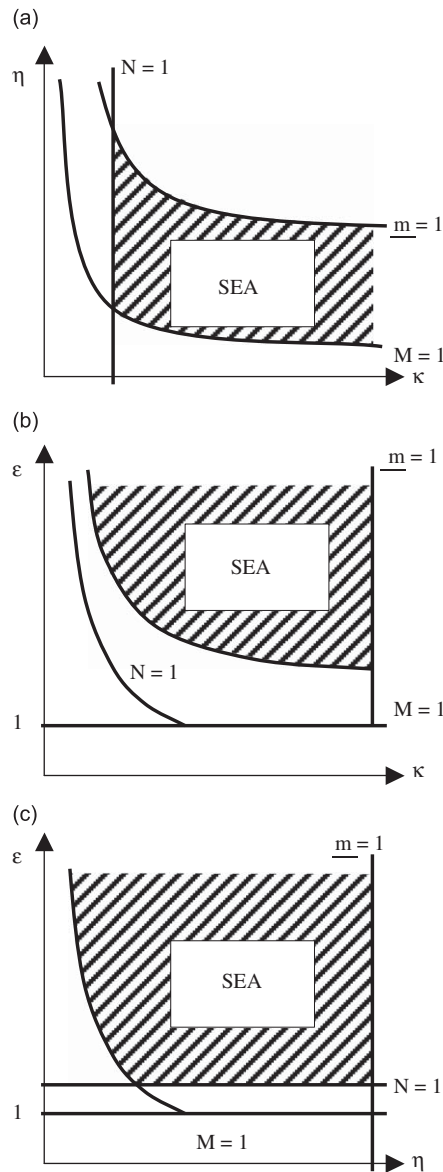


Fig. 3. Validity domain of SEA for a single plate in the $\kappa, \eta, \varepsilon$ -space of ‘primary’ dimensionless parameters. Sections of the domain on the (a) κ, η -plane, (b) κ, ε -plane and (c) η, ε -plane.

respectively, for $N_i = 1, M_i = 1, \bar{m}_i = 1, \gamma_{ij} = 1$ and $\gamma_{ji} = 1$. Fortunately, none of these equations involves all dimensionless parameters. It is therefore not useful to represent all sections.

In the κ_i, η_i -plane, the condition $N_i = 1, M_i = 1$ and $\bar{m}_i = 1$ have ever been discussed. The condition $\gamma_{ij} = 1$ reads $\eta_i \propto 1/\kappa_i$ and the condition $\gamma_{ji} = 1$ reads $\eta_i \propto \kappa_i$. The hyperbolic line $\gamma_{ij} = 1$ confines the domain in the upper part of the κ_i, η_i -plane in the same way than the condition $M_i = 1$. The second line imposes an upper frequency limit. But this does not imply a frequency limit for SEA in the overall dimensionless space since the position of this limit in the κ_i, η_i -plane depends on some other parameters in the same way than the frequency limit observed in the κ, ε -plane of a single plate depends on the value of η . The domain is shown in Fig. 5 in the case of a single coupling.

In each diagram, the limits $\gamma_{ij} = 1$ and $\gamma_{ji} = 1$ must be applied for each coupling with the plate i . This can result in a large number of additional lines in the plane sections.

6. Numerical simulation

In order to illustrate the usefulness of validity diagrams, a numerical simulation is now presented on a pair of coupled rectangular plates. Results of SEA are compared with those of a direct numerical simulation.

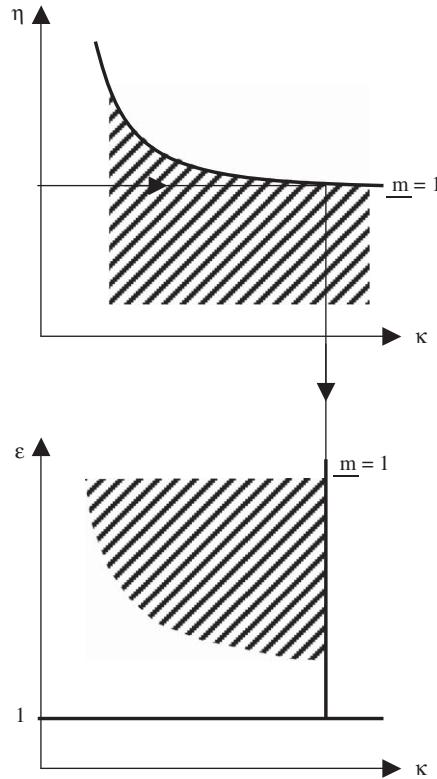


Fig. 4. The frequency limit in the κ, ε -plane stems from the fact that in the κ, η -plane, the horizontal line $\eta = \text{cste}$ encounters the limit $\bar{m} = 1$.

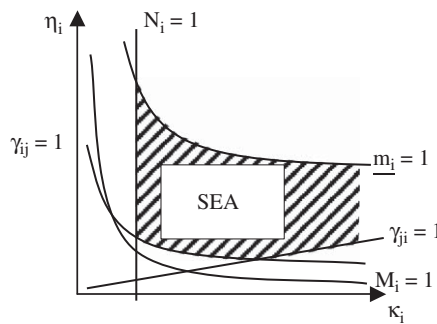


Fig. 5. Section in the κ_i, η_i -plane of the validity domain of SEA for coupled plates.

The system is shown in Fig. 6. The two plates have length a and width b . Their exterior edges are simply supported. The common edge of width b is either free or simply supported. The two plates have same flexural wave speed, damping loss factor and dimensions $a \times b$. Therefore, the dimensionless numbers $\kappa_i, \eta_i, \varepsilon_i, N_i, M_i$ and \bar{m}_i have identical values for both plates. But the plates have different values of bending stiffness and mass per unit area (but with same flexural wave speed) in order that the coupling edge is a discontinuity with transmission efficiency τ_{ij} . From reciprocity $\tau_{12} = \tau_{21}$ and since $\varepsilon_1 = \varepsilon_2, \eta_1 = \eta_2$ and $\kappa_1 = \kappa_2$ it yields, $\gamma_{12} = \gamma_{21}$. Plate 1 is excited by an out-of plane force located at $x_0/a = 0.3$ and $y_0/b = 0.25$. The frequency response functions (FRFs) between force (input) and deflection v_i but also space and time derivatives (up to order 2) of deflection, are computed on a 30×30 grid of receiver points on each plate. These FRFs are computed at 1000 or 2000 frequencies within an octave band. From these FRFs, values of energy densities for a unit force are calculated at all frequencies and receiver points. The vibrational energy of plates in the octave band is assessed by summing the contributions of all frequencies (integration over frequency) and receiver points (integration over space).

The direct numerical simulation is based on the solving of Eq. (21) for the plates. The boundary conditions are simply supported equations (24) and (25) at $y = 0$ and $y = b$, simply supported at $x = 0$ and $x = a_1 + a_2$ and either free

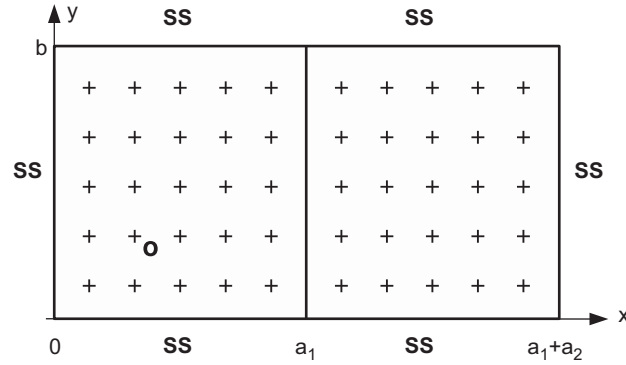


Fig. 6. Numerical simulation on a pair of rectangular plates with same width b . All external edges are simply supported and the coupling edge ensures continuity of deflection, rotation, moment and force. The left plate is excited by a driven force (o), the receiver points (+) of plates belong to a regular array centred on each plate.

Table 1
Dimensionless parameters and results for the five simulations.

Simulation	o	Δ	∇	\times	+
κ	5.0	10.0	2.9	0.91	22.4
η	0.01	0.1	0.0001	0.08	0.0005
ε	1.13	1.13	1.96	1.96	1.13
τ	0.125	0.125	0.0012	0.68	0.163
N	90	360	90	9	1800
M	1.3	51	0.013	1	1.3
\bar{m}	0.16	3.14	0.001	0.23	0.035
γ	0.2	0.01	0.62	0.14	1.16
p_1^{dns}	1.0	1.0	1.0	1.0	1.0
ωE_1^{dns}	81.5	9.72	10025	10.74	1569
ωE_2^{dns}	16.2	0.019	418	0.518	405
p_1^{sea}	1.007	1.0007	0.891	1.348	0.996
ωE_1^{sea}	86.4	9.91	6447	15.06	1297
ωE_2^{sea}	14.3	0.097	2463	1.80	695

Reference simulation (o), high attenuation (Δ), low modal overlap (∇), low mode count (\times), high coupling strength (+). Dimensionless wavenumber κ , damping loss factor η , shape ratio ε , mode count N , modal overlap M , normalized attenuation factor \bar{m} , transmission efficiency τ and coupling strength γ . Results of direct numerical simulation: injected power in plate 1 $p_1^{dns} = 1.0$, dimensionless energies ωE_i^{dns} in plate $i = 1, 2$. Results of SEA: injected power p_1^{sea} , dimensionless energies ωE_i^{sea} .

equations (31)–(34) or simply supported at the interface $x = a_1$. The solution is developed as a Fourier’s series along the y -axis:

$$v_i(x, y) = \sum_{n=1}^{\infty} v_{i,n}(x) \sin\left(\frac{n\pi y}{b}\right). \tag{49}$$

The complete description of this technique can be found in Appendix A of Ref. [46]. For the present numerical simulations, the series has been truncated to 2000 terms.

The SEA computation is realized as follows. The injected power is

$$p_1^{inj} = \frac{|F|^2}{16\sqrt{mD}}, \tag{50}$$

where $F = 1 \text{ N}$ is the excitation force, m the mass per unit area and D the bending stiffness. The modal density is given by Eq. (2) and the coupling loss factors by Eq. (7). The SEA system is Eq. (10).

Five numerical simulations have been realized. The respective dimensionless numbers are shown in Table 1. They have been chosen as follows. A first simulation (symbol o) is done with dimensionless numbers N, M and \bar{m} chosen in such a way that they are correct for SEA. They verify the criteria (12), (14), (16), (18) and (19). All subsequent simulations violate a criterion. The second simulation (symbol Δ) has a large normalized attenuation factor $\bar{m} > 1$, the third simulation (symbol ∇) has a small modal overlap ($M < 1$), the fourth simulation (symbol \times) has a small number of modes $N = 9$ and the fifth simulation (symbol +) has a strong coupling $\gamma > 1$. The positions of these five simulations are plotted in Fig. 7. In these

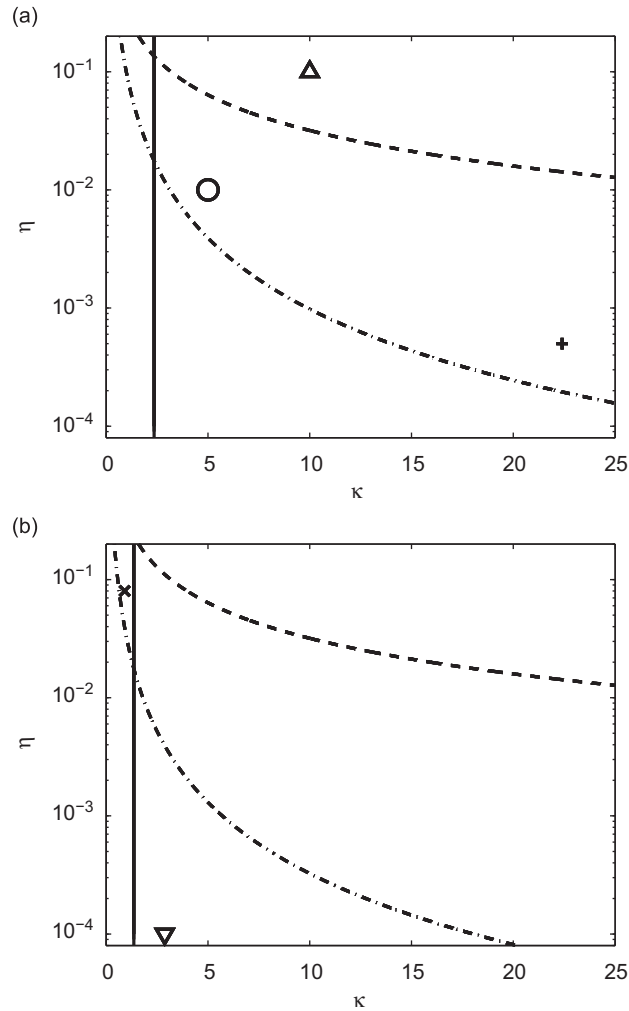


Fig. 7. Validity domain of SEA for a pair of coupled rectangular plates. The lines $\bar{m} = 1$ (---), $M = 0.5$ (-.-), $N = 20$ (-) are plotted in the κ, η -cut of 'primary' dimensionless space in cases (a) $\varepsilon = 1.13$ and (b) $\varepsilon = 1.96$. Numerical simulations are positioned in the reference case (o), for large attenuation (Δ), low modal overlap (∇), few modes (\times) and strong coupling (+).

diagrams, the validity domain of SEA is delimited by the lines $N = 20$, $M = 0.5$ and $\bar{m} = 1$. Since the equations of these lines depend on the value of ε , two diagrams have been plotted: Fig. 7(a) for $\varepsilon = 1.13$ and Fig. 7(b) $\varepsilon = 1.96$. The position of symbols relative to these lines clearly shows which assumption is violated.

At the interface, the coupling conditions are as follows: free conditions for o, Δ , ∇ , \times and simply supported conditions for +.

Numerical results are presented in Table 1. The injected power of direct numerical simulation is conventionally 1 W. It means that all results have been divided by the injected power actually computed by direct numerical simulation. For instance, the value $P_1^{sea} = 1.348$ (simulation \times) means that Eq. (50) over-estimates the actual injected power by 34.8 percent. The dimensionless energies ωE_i are also presented in Table 1. It can be checked that the power balance $P_1 = \eta(\omega E_1 + \omega E_2)$ always applies for SEA but not for direct numerical simulation. The difference is due to numerical approximations. Thus, the ratio $\eta(\omega E_1^{dns} + \omega E_2^{dns})/P_1^{dns}$ is a quality indicator of direct numerical simulation. It has been maintained below 10 percent for all simulations, which is largely smaller than the differences observable between direct numerical simulation and SEA.

Numerical results are shown in Fig. 8. The relative error between direct numerical simulation and SEA is plotted for the power being injected by the driven force and the total vibrational energies of plates 1 and 2. It is clear that the reference calculation (symbol o) shows a fine agreement between SEA and direct numerical simulation for the three results. It means not only that the injected power is well estimated by Eq. (50), but also that the power being exchanged between the two plates is well estimated by Eq. (6). When the number of modes is low (simulation \times), the SEA injected power given by Eq. (50) is a poor estimation of actual injected power. For the four simulations Δ , ∇ , \times and +, the energy of plate 2 is not well estimated. The discrepancy is large (from 2.5 to 8 dB) and cannot be only explained by the error on the injected power.

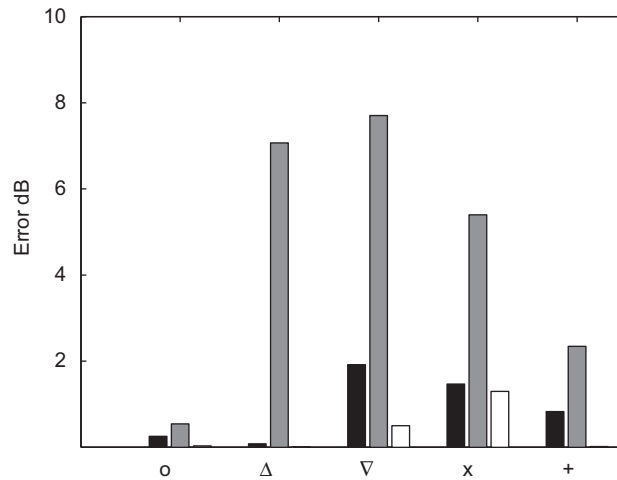


Fig. 8. Relative errors (dB) of SEA results compared with the direct numerical simulation. Errors on energy of plate 1 (black), energy of plate 2 (grey), injected power (white). The reference situation (o) and strong coupling (+) lead to good results on injected power as well as vibrational energies. But, large attenuation (Δ), low modal overlap (∇) and low number of modes (\times) show discrepancies between SEA and direct numerical simulation.

In these cases, the exchanged power given in SEA by Eq. (6) is a poor estimation of the power actually exchanged. Also interesting is the last simulation (+) with strong coupling ($\gamma = 1.16$). The assumption $\gamma \ll 1$ is violated which results in a bad estimation of energy 2. But, the discrepancy is not so high compared with other simulations. The assumption of light coupling seems to be robust in this case. This phenomenon, which is something well-known by ‘practitioners’ of SEA, can be explained by the fact that when the coupling is strong, the equilibrium between sub-systems is rapidly reached. Therefore, the vibrational energies are very little sensitive to the quality of Eq. (9) since they are rather governed by the equality $T_i = T_j$ which is roughly sufficient to obtain a good result.

7. Complex structures

One would wonder whether the previous analysis applies for complex structures. The previous analysis is limited to rectangular plates with a common width b . To relax the assumption of common width, it is necessary to introduce several additional parameters, the width b_i of plate i and the coupling length L_{ij} between plates i and j . The physical parameters are now k_i , η_i , a_i , b_i , v_i and D_i for $i = 1, \dots, n$ and L_{ij} for all couplings. Their number $6n + n(n-1)/2$ and the number of physical units is always 2. The number of dimensionless parameters is therefore $6n + n(n-1)/2 - 2$ to completely describe the structure.

In the same way, if the plates are no longer assumed to be rectangular but, for instance, polygonal, further lengths and angles are necessary to fully describe the geometry. This results in additional physical parameters, which, in turn, will increase the number of dimensionless parameters.

But the complexity not only appear in the geometry, but also in the physical properties of the structure. Stiffeners, ribs, joints, attached equipments, varying thickness and so on, are some examples of what are actual structures. Each time, complexity results in a greater number of dimensionless parameters necessary to fully describe the structure from the mathematical point of view.

But the question is now to know if these additional parameters are necessary from the practical point of view. In the case of complex geometry, the additional parameters carry information only on geometry. But, it is well-known that the state of diffuse field does not depend on the detail of geometry. The theory of mixing billiards [45] establishes the conditions under which a geometry leads to diffuse field and, in the same time, gives some counter-examples of geometries which never lead to diffuse field. But, generally speaking, geometrical complexity is a property favourable for diffuse field. The more complex is the shape of billiards, the greater is the chance to have diffuse field. Thus, one can expect that these additional dimensionless parameters are not so important in SEA.

The case of structural complexity is different. Stiffeners, ribs and other structural elements modify frequency and shape of eigenmodes. As well, they can have a strong influence on the modal density and/or the modal overlap. This can drastically modify SEA results. But it is well-known that friction in joints can be the most important cause of dissipation of vibrational energy. Once again, this can modify SEA results. More important, a strong structural modification can also affect the homogeneity of mode groups. This would question the splitting of the structures into sub-systems. And the SEA model itself could become irrelevant.

So, in principle, the dimensional analysis performed in this paper only applies for the case of assembled rectangular plates. Its application to more complex geometries seems to be reasonable. If this is the case, the numbers N_i , M_i and \bar{m}_i for

$i = 1, \dots, n$ and $\gamma_{i,i+1}$ for $i = 1, \dots, n - 1$ constitute a set of $4n - 1$ independent dimensionless parameters in which SEA validity domain is included. But for structural complexity, the analysis must be re-done and each particular case could give a different set of dimensionless parameters. The only certainty is that the numbers N_i , M_i , \bar{m}_i , $\gamma_{i,i+1}$ are necessary but they are not sufficient.

8. Conclusion

In this paper, the validity of SEA has been examined from the dimensional analysis point of view. Four conditions of validity have been clarified in the space of dimensionless parameters given by the dimensional analysis. The equivalence between two sets of dimensional parameters has been established. The 'primary' set of parameters is frequency κ , damping loss factor η , shape ε and the transmission efficiency τ while the set of SEA parameters is the mode count N , the modal overlap M , the normalized attenuation factor \bar{m} and the coupling strength γ .

The conditions of validity drawn in the plane sections of the 'primary' space give the diagrams of validity. Although theoretically an infinite number of sections is necessary to entirely define the domain, the section along the κ_i, η_i -plane is the most interesting one for at least two reasons. First of all, when analysing and actual structures, engineers who want to control the energy flow usually add damping on some components. And to know the low frequency limit of SEA is indeed the most important concern to apply SEA with success. Secondly, the five surfaces associated with the conditions of validity cut the κ_i, η_i -plane. Therefore, all the information is available on this diagram.

It has not been proved that these four conditions are the only ones to entirely define the validity domain of SEA. Some other assumptions such as the nature of excitation, the problem of variance in SEA have not been discussed in this text. For instance, to have a small variance on vibrational energies can be considered as an additional assumption of SEA. But a direct numerical simulation of variance by Monte Carlo's method would require to increase the number of physical parameters (for instance by considering $\langle k \rangle$ and $\text{var}(k)$ instead of only k) and therefore to add a dimensionless parameter (that could be the statistical overlap [15] in this case). It is also possible that further assumptions of SEA will be discovered in future. But anyway, these additional assumptions will lead to additional lines in the diagrams presented in this paper.

Acknowledgements

The authors gratefully acknowledge R.H. Lyon to give precise details about the former works on SEA, the Gaus Laboratory (Sherbrooke, Quebec, Canada) for the invitation to a seminar and the fruitful discussion on SEA, and ANR (France) for the financial support under Contract ANR06-BLAN-0115.

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