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# Free vibration analysis for micro-structures used in MEMS considering surface effects

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ABSTRACT

Based on state space method, a three-dimensional (3-D) approach is proposed to study the size-dependent dynamical properties of micro-structures considering surface effects. The structure is modeled as a laminate composed of a bulk bounded with upper and bottom surface layers, which are allowed to have different material properties from the bulk layer. On the basis of 3-D fundamental elasticity, the state equations, including the surface properties of the structure, can be established to analyze the size-dependent dynamic responses of the plate-like thin film structures used in MEMS. To show the feasibility of the proposed approach, a simply supported plate-like thin structure, including the surface layers, is considered. An algorithm strategy is proposed for the calculation of the state equations obtained to ensure that the numerical results can reveal the surface effects clearly even for extremely thin surface layers. Comparing with the two-dimensional plate theories based size-dependent models for the thin film structures in literature, the present 3-D approach is exact, which can provide benchmark results to assess the accuracy of various 2-D plate theories and numerical methods. Numerical tests prepared for the prediction of natural frequency are carried out to illustrate the surface effects. Some discussions and conclusions are presented based on the results of the numerical examples.



## 1. Introduction

It has been received increasing attentions during the last few decades for the importance of surface effects on the mechanical properties of submicron or nano-scale specimens/structures due to their potential applications of ultra-thin plate or beam-like structures in micro-electro-mechanical systems (MEMS) and nano-electro-mechanical systems (NEMS) devices with high sensitivity and high frequency [1–3]. It is known that the surface of a solid is a region with thickness of several atom diameters, which has its own atom arrangement and property differing from the bulk [4,5]. For a solid with a large size, the surface effects can be ignored because the volume ratio of the surface region to the bulk is very small. However, for small-scaled solids with large surface-to-bulk ratio the significance of the surface effects is likely to be important. For this reason, a better understanding of the mechanical properties of these small elements or structures including surface effects is one of the fundamental issues in designing and predicting performance of the MEMS/NEMS devices. Hence, some experimental techniques are developed to measure the material properties for the purpose of revealing the phenomena of surface effects [6,7]. The performance of the surface can also be altered by some surface

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processing techniques, e.g. ion bombardment [8]. Miyake et al. [9] used atomic force microscopy with a spherical indenter to evaluate the surface and bulk elastic moduli of thick and thin polystyrene films. These experimental data are very useful for validating various computational models including molecular dynamics based models [10,11] and continuum elasticity based models [12]. Wong et al. [13] extended the Zener's classical work by applying the thermoelastic models to the in-plane vibration of uniform rings with rectangular cross-section for the purpose of evaluating the energy-dissipation effects in silicon MEMS resonators. By using a finite element formulation, Yi [14] investigated the effects of geometry on the energy dissipation induced by thermoelastic damping in MEMS resonators. In the study the perturbation analysis was applied to derive a linear eigenvalue equation for a silicon ring resonator under mechanical oscillation with an exponentially decaying rate. The numerical results illustrate that the quality factor (Q-factor) decreases with the radial width of the ring as a monotonic function. Even though the proposed thermoelastic models can estimate damping of the MEMS resonators efficiently by the analysis of inherent vibration and the solution of a linear eigenvalue equation, the analyses were restricted to isotropic material and some simplifications had to be made for anisotropic materials. Hence, some further improvements might be made in applying the method to analyze laminated structures or problems considering surface effects.

Although molecular dynamics based methods [10,11] have been increasingly applied to the modeling and simulation of the nano-materials and nano-structural elements, they are strongly restricted by computational capacities. For MEMS/NEMS structures and elements with at least one dimension in micro-range, such as micro/nano beam, plates, thin film etc., modeling and simulation of their overall physical and mechanical properties is likely to be left to continuum methods.

To incorporate the surface effects in continuum models, Gurtin and Murdoch [15,16] modified the theory of classical mechanics by modeling the surface as a two-dimensional membrane bonded to the underlying bulk material without slipping. The presence of surface stresses thus results in a set of non-classical boundary conditions which present the surface tractions on the bulk substrate in terms of surface stresses and inertia. The non-classical boundary conditions, the surface stress–strain relations, and the equations of classical elasticity for bulk material together form a coupled system of field equations. Based on the theory, the analysis of ultra thin beam- or plate-like structural elements have been done by many researches [12,17–23]. In these studies, the bulk layers of the structures were modeled based on two-dimensional plate theories. Since the normal stress is assumed to be zero in the classical plate theories, it is found that some of the interface boundary conditions between the bulk and the surface layers in Gurtin and Murdoch [15,16] cannot be satisfied. To improve the models, Lu et al. [22] introduced the normal stress inside and on the surface of bulk substrate to satisfy the constitutive relations on the surface. However, the variation of the normal stress along the thickness inside the bulk material cannot be determined from the governing equations, and has to be assumed in advance. Furthermore, it should be pointed out that the thickness of the surface layer was not defined in Gurtin and Murdoch theory, therefore it cannot undertake bending moment. It may introduce some errors if the thickness of studied film is reduced to its critical length scale. To account for the effects of complex shear deformations, a continuum refined model coupled with surface elasticity is proposed by Lü et al. [23] to study the size-dependent elastic behavior of FGM ultra-thin films.

To overcome some difficulties of 2-D models mentioned above, a three-dimensional approach using fundamental elasticity equations and state space method is applied here to study the thin plate-like structures with surface effects. In the approach, the structure is modeled as a laminate composed of a bulk bounded with upper and bottom surface layers, which are allowed to have different material properties from the bulk layer. This leads to a three-layered laminate model for simulating the thin plate-like structure with surface effects. Since the 3-D elasticity equations are applied to each layer and the continuity conditions between the layers are exactly satisfied, the suggested model could be considered as an exact model comparing with the Gurtin and Murdoch theory based 2-D models. In addition, both isotropic and anisotropic material properties for bulk and surface layers can be handled equally with the 3-D model, which are not convenient for the 2-D models.

Since the suggested 3-D model for the thin plate-like structure with surface effects is a direct extension of laminated structure models, many existed methods and technologies available in literature for the analysis of the 3-D structure models can be applied. Among these, state space analysis method is an effective approach for solving laminated structures with arbitrary thickness [24–27]. Based on a mixed representation of elasticity, the state space method converts a boundary value problem to an equivalent initial value problem in terms of the so-called state variables. Because the selected state variables are the compatible displacements and stress components at interfaces, the method is very suitable for the calculation of laminates by using the transfer matrix and recursive solving procedure. For these reasons, the method is applied in this paper to model the surface effects. To show the feasibility of the proposed model, 3-D analytical solutions for the free vibration problems of simply supported plate-like thin structures including the surface layers are derived by using the state space method. An arithmetic strategy is proposed for the calculation of the state equations obtained to ensure that the numerical results can reveal the surface effects clearly even for extremely thin surface layers. Numerical examples are carried out to exhibit the surface effects and some discussions are provided based on the results obtained.

## 2. Fundamental equations

Although the three-layered model provided by Gurtin and Murdoch [15,16] is efficient in evaluating the surface effects, the model cannot take into account the bending performance of the surface layers. In this section a new three-layered model is proposed in which the surface layers can be considered to endure bending deformation and transverse load. State space method is applied to formulate the problem and transfer matrix method is used to solve the state equation thereafter for natural frequency prediction. In order to check the effectiveness of the method and compare the numerical results with that of Ref. [22], formulae for a special case of cylindrical bending are presented.

### 2.1. General formulations for laminated structures

Consider a thin plate-like structure with thickness $h$, as shown in Fig. 1. To introduce the surface effects, the plate is modeled as a three-layered laminate composed of a bulk bounded with upper and bottom surface layers. The two surface layers are modeled to have their own material properties, and the thicknesses of the two surface layers are defined by $h_1$ and $h_3$, respectively. Therefore, the defined surface layers are allowed both in-plane and out-of-plane deformations, which may be more reasonable compared with the membrane model in Gurtin and Murdoch [15,16]. The total thickness of the plate is thus equals to $h = h_1 + h_2 + h_3$.

The coordinate parameters of the plate are denoted with $x$, $y$ and $z$, respectively, as shown in Fig. 1, while $u$, $v$ and $w$ represent the associated displacement components. The layers of the plate are assumed to have orthotropic material properties and the principal material axes coincide with the axes of the adopted Cartesian coordinate system. Hence, the basic equations for any layer of the plate (for example, the bulk layer) can be written as

(a) Stress–strain relations

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}, \tag{1}$$

(b) Dynamic equation

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = \rho \dfrac{\partial^2 u}{\partial t^2}, \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = \rho \dfrac{\partial^2 v}{\partial t^2}, \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = \rho \dfrac{\partial^2 w}{\partial t^2}, \end{cases} \tag{2}$$

(c) Strain-displacement relations

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x}, & \varepsilon_y = \dfrac{\partial v}{\partial y}, & \varepsilon_z = \dfrac{\partial w}{\partial z}, \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}, & \gamma_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}, & \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}, \end{cases} \tag{3}$$

where $C_{ij}$ are stiffness constants and $\rho$ is the volume density of the material.

The 3-D equation systems (1)–(3) for elasticity can be solved by state space method. To facilitate the derivation, define first

$$C_1 = -C_{13}/C_{33}, \quad C_2 = C_{11} - C_{13}^2/C_{33}, \quad C_3 = C_{12} - C_{13}C_{23}/C_{33}, \quad C_4 = C_{22} - C_{23}^2/C_{33}$$
$$C_5 = -C_{23}/C_{33}, \quad C_6 = C_{66}, \quad C_7 = 1/C_{33}, \quad C_8 = 1/C_{55}, \quad C_9 = 1/C_{44},$$

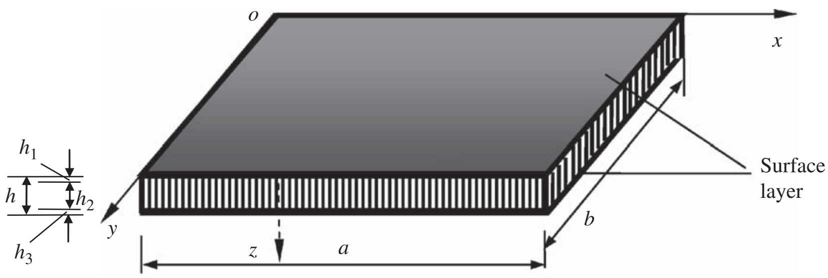


**Fig. 1.** Nomenclature of the plate considering surface effects.

and

$$\alpha = \partial/\partial x, \quad \beta = \partial/\partial y, \quad P^2 = \rho \cdot \partial^2/\partial t^2.$$

Therefore, from the third equation of Eq. (1) and considering Eq. (3), the following relation can be obtained

$$\frac{\partial w}{\partial z} = C_1 \alpha u + C_5 \beta v + C_7 \sigma_{zz}. \tag{4}$$

inserting Eqs. (3) and (4) into Eq. (1), the in-plane stresses can be calculated by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_2\alpha & C_3\beta & -C_1 \\ C_3\alpha & C_4\beta & -C_5 \\ C_6\beta & C_6\alpha & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ \sigma_z \end{Bmatrix}. \tag{5}$$

By substituting Eq. (5) into Eq. (2) and considering Eq. (4) as well as the fourth and the fifth equations of Eq. (1), we can write the Eqs. (1)–(3) in the following form

$$\frac{\partial}{\partial z}\{\mathbf{R}\} = [\overline{\mathbf{G}}]\{\mathbf{R}\}, \tag{6}$$

where

$$\{\boldsymbol{R}\} = [u \;\; v \;\; w \;\; \tau_{xz} \;\; \tau_{yz} \;\; \sigma_z]^{\mathrm{T}} \tag{7}$$

is called state vector, and

$$[\overline{\boldsymbol{G}}] = \begin{bmatrix} \mathbf{0} & 0 & -\alpha & C_8 & 0 & 0 \\ \mathbf{0} & 0 & -\beta & 0 & C_9 & 0 \\ C_1\alpha & C_5\beta & 0 & 0 & 0 & C_7 \\ P^2 - C_2\alpha^2 - C_6\beta^2 & -(C_3+C_6)\alpha\beta & 0 & 0 & 0 & C_1\alpha \\ -(C_3+C_6)\alpha\beta & P^2 - C_6\alpha^2 - C_4\beta^2 & 0 & 0 & 0 & C_5\beta \\ 0 & 0 & P^2 & -\alpha & -\beta & 0 \end{bmatrix} \tag{8}$$

is the system matrix. It can be seen that Eq. (6) is a first-order partial differential equation system with respect to coordinate $z$, and the system matrix (8) is independent of $z$.

The state Eqs. (6)–(8) are written for single layered structures. They can be applied to multilayered structures by considering interface boundary conditions. The state space equation of the $j$-th layer of the structure can be written as

$$\frac{\partial}{\partial z}\{\boldsymbol{R}(z,t)\}_j = [\overline{\boldsymbol{G}}]_j\{\boldsymbol{R}(z,t)\}_j, (z_{j-1} \le z \le z_j, \quad j = 1,2,\ldots,N), \tag{9}$$

where $\{\mathbf{R}\}_j$ and $[\overline{\mathbf{G}}]_j$ are, respectively, the state vector and the system matrix for the $j$-th layer, $N$ is the number of layers of the laminates, and $z_0 = 0$, $z_j = \sum_{k=1}^{j} h_k$ with $h_k$ is the thickness of the $k$-th layer. Furthermore, the continuity conditions at interfaces require that

$$\{\mathbf{R}(z_j,t)\}_j = \{\mathbf{R}(z_j,t)\}_{j+1}, \quad (j = 1,2,\ldots,N-1). \tag{10}$$

Eqs. (9) and (10) are basic equations of the state space method used in this paper for solving the laminated structures. For specific geometry boundary conditions, the state vectors $\{\mathbf{R}(z,t)\}_j$ can be solved analytically or numerically as can be seen in Refs. [25–27], and the in-plane stress components can be determined from Eqs. (5).

### 2.2. Establishment and solution of state equation

To show effectiveness of the proposed 3-D model for studying the thin plate-like structures considering surface effects, an analytical solutions for a simply supported rectangular layered structure with length $a$ and width $b$, as shown in Fig. 1, is obtained based on the state space approach.

Considering the following simply supported boundary conditions at four edges of the plate

$$x = 0,\ a: \quad v = \sigma_x = 0; \quad y = 0,\ b: \quad u = \sigma_y = 0, \tag{11}$$

the solutions of displacements and transverse stresses for each layer are assumed to have the form as

$$\begin{Bmatrix} u \\ v \\ w \\ \tau_{xz} \\ \tau_{yz} \\ \sigma_z \end{Bmatrix}_j = \sum_m \sum_n \begin{Bmatrix} U_{mn}(z)\cos(\xi x)\sin(\eta y) \\ V_{mn}(z)\sin(\xi x)\cos(\eta y) \\ W_{mn}(z)\sin(\xi x)\sin(\eta y) \\ X_{mn}(z)\cos(\xi x)\sin(\eta y) \\ Y_{mn}(z)\sin(\xi x)\cos(\eta y) \\ Z_{mn}(z)\sin(\xi x)\sin(\eta y) \end{Bmatrix}_j \mathrm{e}^{\mathrm{i}\omega t}, \tag{12}$$

where $\xi = m\pi/a$, $\eta = n\pi/b$. By the substitution of Eq. (12) into Eq. (9), the following first-order ordinary differential equations for each pairs of $m$ and $n$ is obtained

$$\frac{\mathrm{d}}{\mathrm{d}z}\{\mathbf{R}_{mn}(z)\}_j = [\mathbf{G}]_j\{\mathbf{R}_{mn}(z)\}_j, \quad (z_{j-1} \le z \le z_j,\ j = 1, 2, \ldots, N), \tag{13}$$

in which

$$\{\mathbf{R}_{mn}(z)\}_j = [U_{mn}(z)\ V_{mn}(z)\ W_{mn}(z)\ X_{mn}(z)\ Y_{mn}(z)\ Z_{mn}(z)]_j^{\mathrm{T}}, \tag{14}$$

and

$$[\mathbf{G}]_j = \begin{bmatrix} 0 & 0 & -\xi & C_8 & 0 & 0 \\ 0 & 0 & -\eta & 0 & C_9 & 0 \\ -C_1\xi & -C_5\eta & 0 & 0 & 0 & C_7 \\ C_2\xi^2 + C_6\eta^2 - \rho\omega^2 & (C_3 + C_6)\xi\eta & 0 & 0 & 0 & C_1\xi \\ (C_3 + C_6)\xi\eta & C_6\xi^2 + C_4\eta^2 - \rho\omega^2 & 0 & 0 & 0 & C_5\eta \\ 0 & 0 & -\rho\omega^2 & \xi & \eta & 0 \end{bmatrix}_j. \tag{15}$$

The Eq. (13) is generally called state equation and its solution can be expressed as

$$\{\mathbf{R}_{mn}(z)\}_j = \mathrm{e}^{[\mathbf{G}]_j(z - z_{j-1})}\{\mathbf{R}_{mn}(z_{j-1})\}_j = [\mathbf{D}_{mn}(z - z_{j-1})]_j\{\mathbf{R}_{mn}(z_{j-1})\}_j \quad (z_{j-1} \le z \le z_j, \quad j = 1, 2, \ldots, N) \tag{16}$$

and in particular, at $z = z_j$,

$$\{\boldsymbol{R}_{mn}(z_j)\}_j = [\boldsymbol{D}_{mn}(h_j)]_j\{\boldsymbol{R}_{mn}(z_{j-1})\}_j, \tag{17}$$

where $\{\mathbf{R}_{mn}(z_{j-1})\}_j$ and $\{\mathbf{R}_{mn}(z_j)\}_j$ are the respective values of the state vector at the top and the bottom interfaces of the $j$-th layer, and $[\boldsymbol{D}_{mn}(h_j)]_j$ is the transfer matrix of the $j$-th layer, which can be calculated either analytically or numerically. Let $\lambda_1, \lambda_2, \ldots, \lambda_6$ be the eigenvalues of the matrix $[\mathbf{G}]_j$ and $[\mathbf{P}]_j$ the matrix composed of the associated eigenvectors, the matrix $[\boldsymbol{D}_{mn}(h_j)]_j$ thus can be calculated from the following expression

$$[\mathbf{D}_{mn}(h_j)]_j = [\mathbf{P}]_j \begin{bmatrix} \mathrm{e}^{\lambda_1 h_j} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_6 h_j} \end{bmatrix} [\mathbf{P}]_j^{-1} \tag{18}$$

### 2.3. Natural frequency solution for the laminates

For a free vibration problem, a characteristic equation concerning every matrix $[\mathbf{D}_{mn}(h_j)]_j$ for the laminated plate must be established and the corresponding eigenvalue solution may be required. The problem is discussed in this section.

The continuity conditions (10) at interfaces can be expressed as

$$\{\mathbf{R}_{mn}(z_j)\}_j = \{\mathbf{R}_{mn}(z_j)\}_{j+1}, \quad (j = 1, 2, \ldots, N-1). \tag{19}$$

By applying Eqs. (17) and (19) recursively, a relationship between the state vectors of the upper and bottom surfaces of the structure is established as follows

$$\{\mathbf{R}_{mn}(z_N)\}_N = [\boldsymbol{A}]\{\boldsymbol{R}_{mn}(0)\}_1, \tag{20}$$

where

$$[\mathbf{A}] = \left(\prod_{j=N}^{1}[\mathbf{D}_{mn}(h_j)]_j\right), \tag{21}$$

and $\{\mathbf{R}_{mn}(0)\}_1$ and $\{\boldsymbol{R}_{mn}(z_N)\}_N$ are, respectively, the state vectors at the upper and bottom surfaces of the laminate, and the vector $\{\mathbf{R}_{mn}(0)\}_1$ is also called initial values.

For the free vibration problem, the traction free conditions on the upper and the bottom surfaces can be expressed as:

$$\begin{cases} [X_{mn}(z_N),\ Y_{mn}(z_N),\ Z_{mn}(z_N)]_N^{\mathrm{T}} = (0, 0, 0)^{\mathrm{T}} \\ [X_{mn}(0),\ Y_{mn}(0),\ Z_{mn}(0)]_1^{\mathrm{T}} = (0, 0, 0)^{\mathrm{T}} \end{cases} \tag{22}$$

Substituting Eq. (22) into (20) yields the following homogenous equations

$$\begin{bmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{bmatrix} \begin{Bmatrix} U_{mn}(0) \\ V_{mn}(0) \\ W_{mn}(0) \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{23}$$

where $a_{ij}$ are the relevant elements of matrix [**A**]. The condition of non-trivial solution for the above equation is equivalent to

$$\begin{vmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{vmatrix} = 0. \tag{24}$$

Eq. (24) is a characteristic equation regarding the frequency $\omega$, and is a transcendental equation as can be seen from Eq. (15). The natural frequencies for every pair of $m$ and $n$, which are corresponding to different displacement vibration modes along the thickness direction, thus can be determined by solving Eq. (24).

### 2.4. Special case: cylindrical bending

Consider a plate with infinitely wide and finite length $a$ that is simply supported at the two sides in width direction. In special case of cylindrical bending vibration, the displacements, strains and stresses depend on coordinate $x$ only and the state variables $v$ and $\sigma_y$ should be vanished. Under this condition, the Eqs. (12)–(15) become

$$\begin{Bmatrix} u \\ w \\ \tau_{xz} \\ \sigma_z \end{Bmatrix}_j = \sum_m \begin{Bmatrix} U_m(z)\cos(\xi x) \\ W_m(z)\sin(\xi x) \\ X_m(z)\cos(\xi x) \\ Z_m(z)\sin(\xi x) \end{Bmatrix}_j e^{i\omega t}, \tag{25}$$

$$\frac{d}{dz}\{\boldsymbol{R}_m(z)\}_j = [\boldsymbol{G}]_j\{\boldsymbol{R}_m(z)\}_j, (z_{j-1} \le z \le z_j,\ j = 1, 2, \ldots, N), \tag{26}$$

$$\{\boldsymbol{R}_m(z)\}_j = [U_m(z)\, W_m(z)\, Z_m(z)]_j^{\mathrm{T}}, \tag{27}$$

$$[\boldsymbol{G}]_j = \begin{bmatrix} 0 & -\xi & C_8 & 0 \\ -C_1\xi & 0 & 0 & C_7 \\ C_2\xi^2 - \rho\omega^2 & 0 & 0 & C_1\xi \\ 0 & -\rho\omega^2 & \xi & 0 \end{bmatrix}_j. \tag{28}$$

The solution process of Eq. (26) is same as that described in Sections 2.2 and 2.3.

## 3. An algorithm for solving state equations including surface layers

Although Eq. (20) is mathematically quite simple, it may introduce some errors when it is directly applied to the analysis of the surface effects in numerical calculations. Since the coefficient matrix [$\boldsymbol{A}$] in Eq. (21) is a product of some transfer matrixes, the solution of Eq. (20) can be non-sensitive to the contributions of the surface parameters if the thickness of the surface layers is extremely small compared to the bulk layer, which has been verified in our testing numerical calculations. Therefore, it is necessary to seek a modified approach for the purpose of providing reasonable results to reveal the surface effects effectively. It is discussed in this section.

Because the thicknesses of the surface layers are in general very small compared with the bulk layer in the proposed model, the bulk layer can be imaginarily divided into some thin plies, say $K$ plies, so that the values of the transfer matrices of these plies have reasonable orders compared to those of the transfer matrices of the surface layers. Therefore, the thickness of each thin ply for the bulk layer is given by $\Delta = h_2/K$, where the required number, $K$, can be determined according to the dimensions of the surface layers so that $\Delta \sim h_1$. Hence, the thin plate-like structure is now considered to have $N$=$K$+2 thin plies in total. It is found in numerical calculation, however, that the computational efforts and accumulated calculating errors will be increased significantly with the $K$ increases. In order to take into account the surface effects efficiently and correctly, an algorithm strategy is suggested and described below.

The state vector $\{\mathbf{R}_{mn}(z)\}_j$ of the $j$-th ply defined in (14) can be re-written in the form as

$$\{\mathbf{R}_{mn}(z)\}_j = \begin{Bmatrix} \mathbf{U}_j(z) \\ \mathbf{X}_j(z) \end{Bmatrix}, \quad \mathbf{U}_j(z) = \begin{Bmatrix} U_{mn}(z) \\ V_{mn}(z) \\ W_{mn}(z) \end{Bmatrix}_j, \quad \mathbf{X}_j(z) = \begin{Bmatrix} X_{mn}(z) \\ Y_{mn}(z) \\ Z_{mn}(z) \end{Bmatrix}_j, \tag{29}$$

where $\mathbf{U}_j(z)$ is related to the displacement components, while $\mathbf{X}_j(z)$ is related to the out-of-plane stress components. Therefore, Eq. (17) can be expressed in the following partial matrix form as

$$\begin{Bmatrix} \mathbf{U}_j(z_j) \\ \mathbf{X}_j(z_j) \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_j & \mathbf{B}_j \\ \mathbf{C}_j & \mathbf{D}_j \end{bmatrix} \begin{Bmatrix} \mathbf{U}_j(z_{j-1}) \\ \mathbf{X}_j(z_{j-1}) \end{Bmatrix}, \tag{30}$$

where $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{C}_j$ and $\mathbf{D}_j$ are all $3 \times 3$ partial matrixes relating to the transfer matrix of the $j$-th ply, or $2 \times 2$ partial matrixes for cylindrical bending problem as can be seen from Eq. (27).

For the case of free vibration, the surface stress conditions on the upper and bottom surfaces can be expressed based on Eq. (22) as

$$\{\boldsymbol{X}_1(0)\} = \{\mathbf{0}\}, \{\boldsymbol{X}_N(z_N)\} = \{\mathbf{0}\}. \tag{31}$$

Therefore, by introducing the surface conditions of the top layer to Eq (30) for $j$=1, the following result is obtained

$$\begin{bmatrix} \boldsymbol{E}_1 & -\mathbf{I} \\ \boldsymbol{F}_1 & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U}_1(0) \\ \boldsymbol{U}_1(z_1) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \boldsymbol{X}_1(z_1) \end{Bmatrix}, \tag{32}$$

where $\boldsymbol{E}_1 = \boldsymbol{A}_1$ and $\boldsymbol{F}_1 = \boldsymbol{C}_1$ for $j$=1. The solutions of the displacements on the upper and bottom surfaces of the top layer can be obtained from above equation as

$$\begin{Bmatrix} \boldsymbol{U}_1(0) \\ \boldsymbol{U}_1(z_1) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{G}_1 \\ \boldsymbol{H}_1 \end{bmatrix} \{\boldsymbol{X}_1(z_1)\}, \tag{33}$$

Similarly, by introducing the solutions (33) to Eq (30) for $j$=2, some unknown displacement and stress vectors can be expressed according to the stress vector $\mathbf{X}_2(z_2)$ as

$$\begin{bmatrix} \boldsymbol{E}_2 & -\mathbf{I} \\ \boldsymbol{F}_2 & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{X}_1(z_1) \\ \boldsymbol{U}_2(z_2) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \boldsymbol{X}_2(z_2) \end{Bmatrix}, \tag{34}$$

where the known coefficient matrixes $\boldsymbol{E}_2$ and $\boldsymbol{F}_2$ can be determined accordingly, and are not written out explicitly here. By repeating the above process recursively from $j$=1 to $j$=$N$, the following relation is obtained

$$\begin{bmatrix} \boldsymbol{E}_N & -\mathbf{I} \\ \boldsymbol{F}_N & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{X}_{N-1}(z_{N-1}) \\ \boldsymbol{U}_N(z_N) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \boldsymbol{X}_N(z_N) \end{Bmatrix}. \tag{35}$$

where the matrixes $\boldsymbol{E}_N$ and $\boldsymbol{F}_N$ are calculated according to the recursive process and their elements are related to frequency $\omega$. By introducing the bottom surface condition in Eq. (31) to Eq. (35), the following $6 \times 6$ homogenous equation is obtained

$$\begin{bmatrix} \boldsymbol{E}_N & -\mathbf{I} \\ \boldsymbol{F}_N & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{X}_{N-1}(z_{N-1}) \\ \boldsymbol{U}_N(z_N) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{36}$$

which can be used to obtain natural frequencies corresponding to the through-thickness modes for every pair of $m$ and $n$ in solution (12) or every $m$ in solution (25).

## 4. Numerical examples

To validate the present method, numerical calculations are carried out and compared with the result of Ref. [22]. As an example, cylindrical bending vibration problem for a simply supported thin plate with infinitely width and finite length $a$ is considered. The original material parameters used in Ref. [22] are given in the following

$$\begin{gathered} E = 5.625 \times 10^{10}\,\mathrm{N/m^2}, \quad \nu = 0.25, \quad \rho = 3 \times 10^3\,\mathrm{kg/m^3}, \\ \lambda_0 = 7 \times 10^3\,\mathrm{N/m}, \quad \mu_0 = 8 \times 10^3\,\mathrm{N/m}, \quad \rho_0 = 7 \times 10^{-4}\,\mathrm{kg/m^2}. \end{gathered} \tag{37}$$

where $E$ and $\nu$ are Young's modulus and Poisson's ratio for the material of the bulk layer, respectively, $\lambda_0$ and $\mu_0$ are Lamé constants of the surface layer. It is known that the dimension of $\lambda_0$ or $\mu_0$ is [N/m$^2$], same as that of Young's modulus. However, the dimension of $\lambda_0$ or $\mu_0$ given by Ref. [22] is [N/m] because the thickness of the surface layers were not be taken into account. Hence, in order to compare the results with that of Ref. [22] the dimensions and values of the Lamé constants $\lambda_0$ and $\mu_0$ should be altered in the present calculation. Assume that the thicknesses of the surface layers are fixed in the numerical examples, which are defined by $h_1 = h_3 = 1.0 \times 10^{-8}$ m. Considering the thickness of the surface layer, the equivalent material constants are taken as

$$\begin{gathered} E = 5.625 \times 10^{10}\,\mathrm{N/m^2}, \quad \nu = 0.25, \rho = 3 \times 10^3\,\mathrm{kg/m^3} \\ \lambda_0 = 7 \times 10^{11}\,\mathrm{N/m^2}, \quad \mu_0 = 8 \times 10^{11}\,\mathrm{N/m^2}, \quad \rho_0 = 7 \times 10^4\,\mathrm{kg/m^3}, \end{gathered} \tag{38}$$

The span to thickness ratio $s = a/h$ is taken $s$=10 as that of Ref. [22]. By changing the thicknesses $h_2$ of the bulk layer or $h$ of the plate-like structure, the effects of the surface layers on the overall mechanical responses of the structure can be studied accordingly. Furthermore, as discussed in the preceding section, to reduce the possible errors due to numerical calculations, the bulk layer in the numerical examples is divided into thin plies so that the ratio between the thickness of each thin ply and the surface layer meets the requirement of $\boldsymbol{\Delta}/h_1 \leq 10^3$. Numerical results indicate that the results obtained under this condition are satisfied.

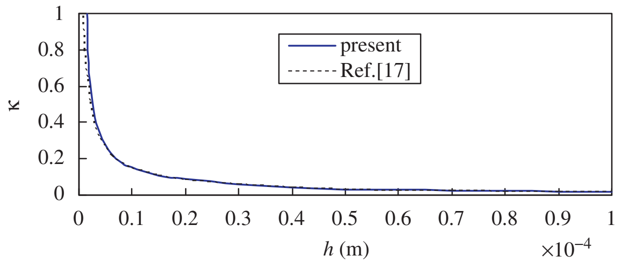


**Fig. 2.** Difference of first-order frequency predicted between the results with and without surface effects.

The non-dimensional difference between the first-order frequencies with and without the surface effects is calculated and illustrated by the following parameter

$$\kappa = \frac{\overline{\omega}_1^2 - \omega_1^2}{\omega_1^2}, \tag{39}$$

where $\overline{\omega}_1$ is the first-order frequency of the plate considering surface effects. The comparison result between present method and Ref. [22] is shown in Fig. 2. It is indicated that the surface effect revealed by the present method is more significant than that of Ref. [22] for micro-structures.

Next example is to consider a simply supported square plate-like structure. The material properties for the bulk are considered to be orthotropic, and the non-dimensional elastic constants are assumed to be

$$\begin{array}{lll} C_{12}/C_{11} = 0.246269 & C_{13}/C_{11} = 0.0831715 & C_{22}/C_{11} = 0.543103 \\ C_{23}/C_{11} = 0.115017 & C_{33}/C_{11} = 0.530172 & C_{44}/C_{11} = 0.266810 \\ C_{55}/C_{11} = 0.159914 & C_{66}/C_{11} = 0.262931 & \end{array} \tag{40}$$

Since very little data regarding the surface elastic properties can be found in literature, it is assumed in the numerical examples that the two surface layers have the same non-dimensional elastic constants as described in (40). The differences of the physical material properties between the bulk and the surfaces are distinguished by the ratio $T = C_{11}^{(S)}/C_{11}^{(B)}$, where the superscripts $S$ and $B$ denote the surface layer and the bulk layer, respectively. When $T$=1, the structure is degenerated into a single-layered classical plate model without considering the surface effects. In addition, the other geometry and material properties of the plate-like structure in the numerical examples are assumed to be

$$a = b, \quad \rho^{(S)}/\rho^{(B)} = 1, \tag{41}$$

where $\rho$ is the material density.

To assess the surface effects on the dynamic responses of the plate, free vibration of the plate-like structure with the material properties given above is analyzed with $h_1 = h_3 = 1.0 \times 10^{-8}$ m, $h/a = 0.1$ and $T = 10.0$. Fig. 3 shows the variations of the vibration frequency parameter, $\Omega = \omega h \sqrt{\rho^{(B)}/C_{11}^{(B)}}$, with the ratio $\overline{h} = h/h_1$, and Fig. 4 shows the variations of the relative frequency difference $\tilde{\omega} = (\Omega_B - \Omega_S)/\Omega_B$ with the ratio of $\overline{h}$ for some vibration modes. It is seen that the influences of the surface effects on the dynamic properties of the structure components become significant when $h/h_1 < 10^4$. It is also noted from Fig. 4 that for the relative frequency difference $\tilde{\omega}$ calculated from the same order vibration modes but with different pairs of $m$ and $n$, the influences of the surface effects are almost same.

In summary, the numerical examples show that the surface effects increase with the decrease of the structure size. It indicates that the 3-D model suggested in the paper can also predict the surface effects for the low-dimensional structures effectively. Based on the numerical results obtained with the assumed material data, the following observations are summarized:

(a) Surface effects can be ignored reasonably when $h/h_1 \geq 10^5$, whereas the effects become significant for micro structures as the ratio of $h/h_1$ decreases.
(b) The surface effects on the relative differences of frequencies for different vibration modes are generally non-sensitive to thickness-span ratio $h/a$.
(c) The solution system based on the state space method and the algorithm strategy proposed can provide an ideal mathematical tool for the analysis of minute elements that widely used in MEMS/NEMS structures. The present method can also be used as benchmarks for assessing some other numerical and simplified analytical solutions.

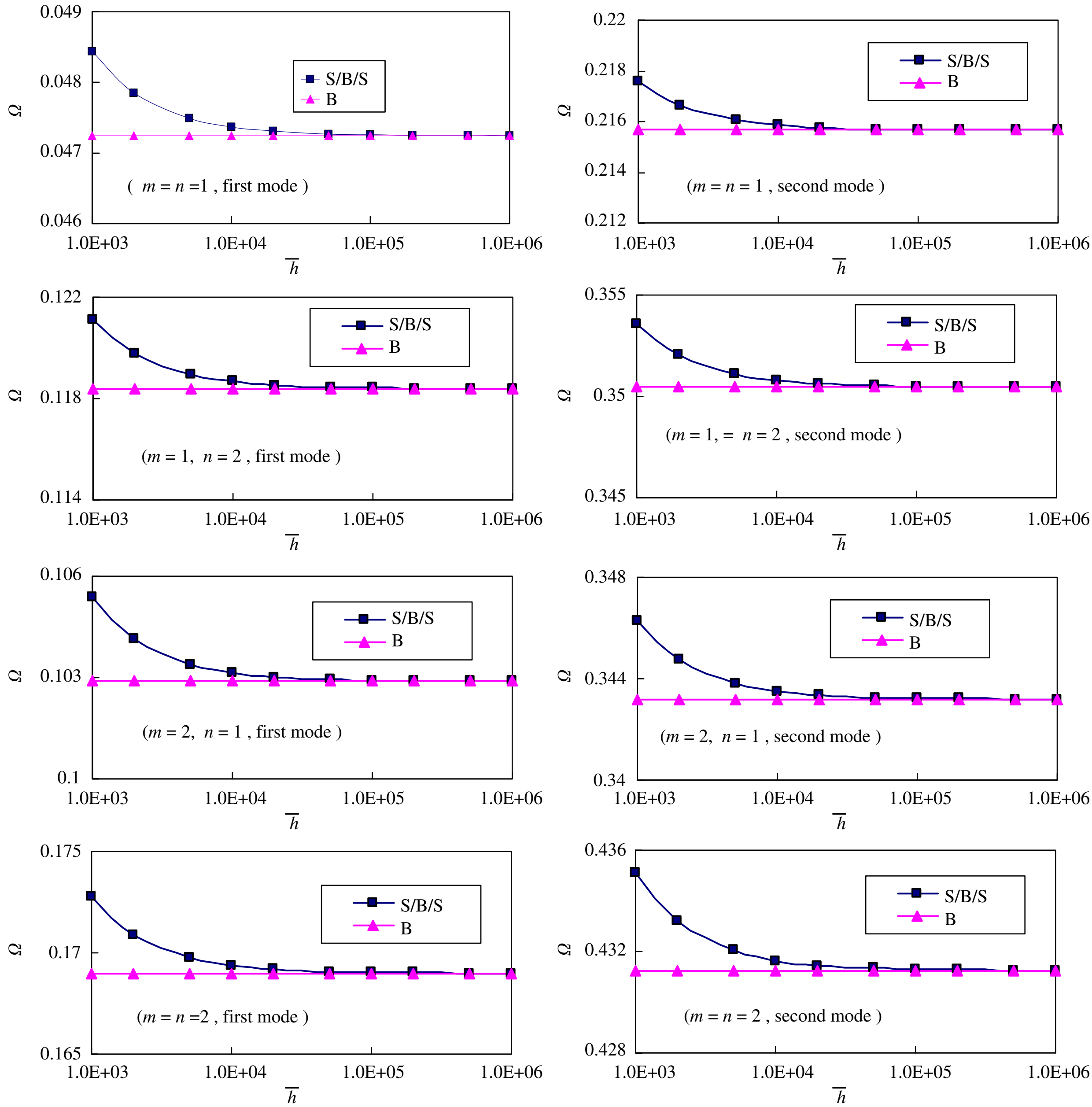


**Fig. 3.** Variations of frequency differences for some modes with ratio $\bar{h} = h/h_1$ for $h/a = 0.1$.

## 5. Discussions and concluding remarks

In the present work, a three-dimensional approach based on continuum elasticity is suggested to study the response of ultra-thin elastic film structures including surface effects, in which the plate-like element is modeled as a laminate composed of a bulk bounded with upper and bottom surface layers. The state space method is applied to solve the laminated structures with arbitrary thickness. An algorithm is presented for the calculation of the state equation to ensure that the solutions can reveal the surface effects clearly when the thickness of the surface layer is tremendously small. In the algorithm, every step in the recursive process only needs solving an $6 \times 6$ algebra equations for the bending problem or an $4 \times 4$ algebra equations for the cylindrical bending problem. The coefficients of the algebra equations are determined automatically step-by-step and the solutions are obtained in a recursive procedure. Meanwhile, the solutions given satisfy the boundary conditions along all edges of the plate and the continuity conditions at the interfaces of the laminate. Numerical examples show that the algorithm is effective.

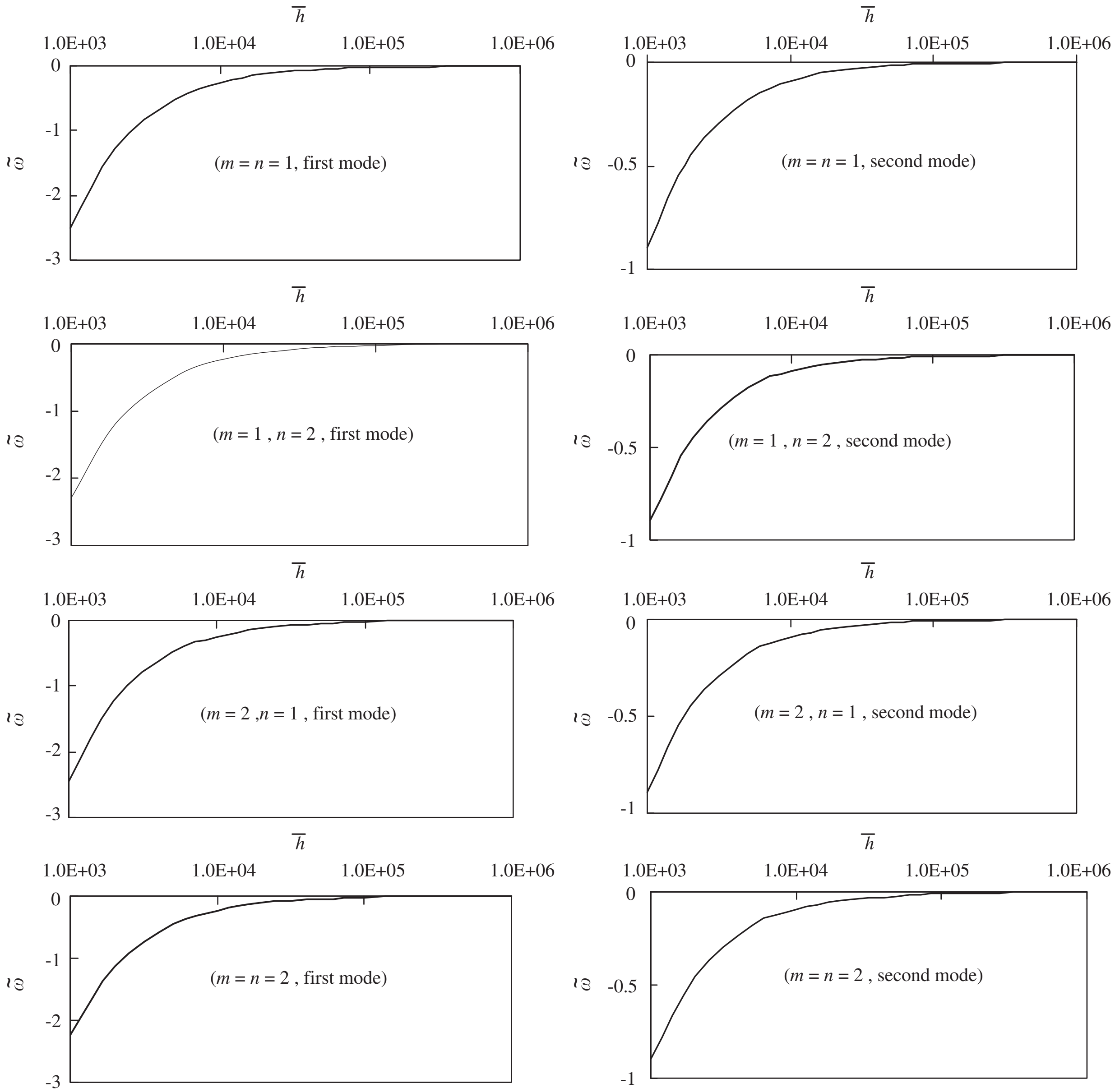


**Fig. 4.** Variations of relative frequency differences $\tilde{\omega}$ (percent) for some modes with $\overline{h}$ for $h/a = 0.1$.

Compared with the 2-D plate theory based models for the thin film structures considering surface effects in literature, the 3-D model suggested in the paper may show some advantages. Since the model is based on 3-D elasticity equations, it is exact in the extent of continuum model range, and can be used to model the structures with any thickness-span ratios. The 2-D theory based models can be considered to be the simplifications of the 3-D model, and can only be applied to the structures with restricted thickness-span ratios. Therefore, the results obtained with the 3-D model can be used to assess the accuracies and application ranges of various 2-D based theories and methods. In addition, the surface anisotropy properties and the interface boundary conditions between the surface and the bulk layers can be handled naturally with the 3-D model, which may encounter some difficulties in the 2-D models.

Although the surface effect included model suggested in the paper is mathematically just a normal extension of conventional laminate models, it deals with different physical problems. A classical laminate is a structure with many laminas physically bounded together, while the thin plate-like structure considered here is a single solid but considering the effects induced by different atom arrangements on the surface region. The influences of the effects on the overall mechanical responses of large-scale structures still exist but are too small to be considered. In the analysis of small material structures, however, the surface effects can significantly influence their mechanical properties due to large surface-to-bulk ratio as shown in the numerical examples, and thus need to be considered in modeling process. Since the methodology of modeling structures with surface effects based on continuum theories has received increasing interest in just a few years as the rapid development of MEMS/NEMS, and various theories and methods are still under exploration, this study attempts to show that some conventional models can still be extended to this kind of problems effectively if the physics behind the problems are understood. In addition, the present method can also be extended to solve the surface effects problems of FGM as that of Ref. [23] with some improvements in the solution process are to be made.

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