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Flexural vibration of non-uniform beams having double-edge breathing cracks

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ABSTRACT

Flexural vibration of non-uniform Rayleigh beams having single-edge and double-edge cracks is presented in this paper. Asymmetric double-edge cracks are formed as thin transverse slots with different depths at the same location of opposite surfaces. The cracks are modelled as breathing since the bending of the beam makes the cracks open and close in accordance with the direction of external moments. The presented crack model is used for single-edge cracks and double-edge cracks having different depth combinations. The energy method is used in the vibration analysis of the cracked beams. The consumed energy caused by the cracks opening and closing is obtained along the beam's length together with the contribution of tensile and compressive stress fields that come into existence during the bending. The total energy is evaluated for the Rayleigh–Ritz approximation method in analysing the vibration of the beam. Examples are presented on simply supported beams having uniform width and cantilever beams which are tapered. Good agreements are obtained when the results from the present method are compared with the results of Chondros et al. and the results of the commercial finite element program, Ansys©. The effects of breathing in addition to crack depth's asymmetry and crack positions on the natural frequency ratios are presented in graphics.

1. Introduction

Structures can be damaged by various external or internal influences such as impacts, fatigues and corrosions. All these influences can result in flaws that lead to change of the dynamic behaviour of the structures. The most common damage type for beam shaped mechanical or structural elements under dynamic loading is the fatigue crack. Understanding the vibration effects of cracks enables their recognition in practical applications of vibration monitoring. A comprehensive review on the vibration of cracked structures was given by Dimarogonas [1].

In order to model the dynamic behaviour of cracked beams, researchers defined local flexibility changes by several methods, which included reduced cross-section or massless rotational springs. Magnitudes of the flexibility changes were estimated by fracture mechanic methods [2,3] or experimental works. Most researchers simply assumed the crack remains open and neglected the nonlinear influences of the breathing cracks for small vibration amplitudes [4–6]. Open crack assumptions were also used for the double-edge cracks. There has been some research concentrating on symmetric double-edge cracks [7–10]. However, until date there has been no work analysing asymmetric double-edge crack in the

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Nomenclature

a	crack depth
a_{cl}	depth of a closed crack
a_{op}	depth of an open crack
a_1, a_2	depths of the cracks in pair called double-edge crack
A	cross-section area
b	width of a beam
b_1	width of a beam at the root
b_2	width of a beam at the tip
CE	the energy consumed
E	modulus of elasticity
G_N	normalised Gaussian function
h	height of a beam
h_1	height of a beam at the root
h_2	height of a beam at the tip
I	second moment of inertia
k	stiffness of which type is specified by the superscripts
KE	kinetic energy
L	length of the beam
m	total number of terms of polynomial mode shape function
M	bending moment
PE	potential energy
w	coordinate axis along the beam's height
W	The transverse vibration mode shape of the beam
X	Yawing or shift of the neutral axis
y	coordinate axis along the beam's width
z	coordinate axis along the beam's length
α	Taper factor of beam's height or width
β	Additional effects of the closed crack on negative strain and compressive stress
Γ	distribution of the energy
$\Delta\theta$	angular displacement at the tip of open crack
$\Delta\phi$	angular displacement of the beam due to the positive strain at the crack location
$\Delta\psi$	angular displacement at the tip of closed crack
$\Delta\varphi$	angular displacement of the beam due to the negative strain at the crack location
γ	additional effects of the open crack on strain and tensile stress
κ	coefficient of the term of polynomial mode shape function
χ	term of polynomial mode shape function
ν	Poisson ratio
ρ	mass density
ω	circular frequency
ω_0	natural frequency of the un-cracked beam
c, dc	subscript for the words "crack" and "double-edge crack"
d	subscript for defining the direct effects of cross-section decreases
j	numerator of the mode shape terms
s	subscript for defining the effects of neutral axis shift
y	subscript for defining the effects of neutral axis yawing

literature, mainly because the definition of a crack advancement function for all different depth combinations of the double-edge crack would be quite a complicated task.

Many works contain analytical methods based on the solution of the equation set formed by means of the compatibility and continuity conditions at the crack locations. It has been possible to analyse uniform beams both numerically [11–13] and analytically [10,14]. However, there have been complications in the analytical vibration analysis of non-uniform beams that arise from the difficulty of solving nonlinear equations resulting from the geometric nonlinearities. These approaches also suffer from the lack of the crack caused stress fields decaying with the distance from the crack. The effects of decaying stress fields on the crack models utilising rotational springs were discussed by Chondros [15].

The methods, including exponentially decaying crack disturbance functions, were proposed to develop vibration equations for continuous models [7,9,16–18]. The exponential function was firstly presented by Christides and Barr [7] to model the stress/strain variation around the crack zone for one or more pairs of symmetric cracks. Shen and Pierre [16] proposed a similar approach for single cracked beams by using the many termed Galerkin's method. Another crack disturbance function was developed by Chondros et al. [9] for the vibration of simply supported beams having single-edge crack or double-edge symmetric cracks. Yang et al. [17] defined the stiffness of single and double cracked beams using strain energy variation around the crack. These approaches suffer from the overlap of the exponential functions when the multiple cracks interact with each other in close distance. An approach for defining interaction of the energy distribution functions was presented by Mazanoglu et al. [18] on the first three flexural vibration modes of multiple cracked non-uniform beams. This study also included the considerable effects of tensile stresses near the crack tip.

In the literature, one seldom comes across numerical and analytical solutions of vibrating non-uniform beams. An approach for determining natural frequencies and mode shapes of cracked stepped beams having varying cross-section and cracked non-uniform beams having concentrated masses was presented by Li [19,20]. However, only some specific forms of non-uniformities could be solved in these papers. A method that uses the Frobenius technique for defining transverse vibrations of tapered beams and geometrically segmented slender beams with a single crack was proposed by Chaudhari and Maiti [21,22]. Even though the beam had a single crack, their results were quite coarse. Energy based numerical approaches were also presented for non-uniform beams. Zheng and Fan [23] used the modified Fourier series for determining the approximate natural frequencies of multiple cracked non-uniform beams. A semi-analytical model for nonlinear vibrations based on an extension of the Rayleigh–Ritz method was presented by El Bikri et al. [24]. The results, which are mainly influenced by the choice of the admissible functions, were restricted with a single crack and fundamental frequency.

This paper presents a method for the flexural vibration of non-uniform Rayleigh beams having double-edge transverse cracks which are symmetric or asymmetric around the central layer of the beam's height. The breathing crack models are employed because the external moments change direction in a period of vibration. Distribution of the energy changes along the beam's length is determined together with contributing the effects of tensile and compressive stress fields that occur in the vicinity of the crack tips due to the additional angular displacement of the beam. Effects of neutral axis deviations are also included in the model. The Rayleigh–Ritz method is applied on total energy distribution for analysing the vibration of the beam. Cantilever and simply supported beams are presented as examples and good agreements are obtained when the employed method results are compared with the results of the Chondros et al. [9] and the results of the commercial finite element program (ANSYS©). The effects of crack's asymmetry and positions of cracks on the natural frequency ratios are shown graphically. Finally, the results obtained by open and breathing crack models are discussed comparatively.

2. Vibration of beams with a single-edge and double-edge crack

Fracture mechanics theory describes the change of structural strain/stress energies with crack growth [2]. The strain stored due to a crack is determined by means of the stress intensity factor for the Mode I crack and thus strain energy release rate. Clapeyron's Theorem states that only half of the work done by the external moment is stored as strain/stress energy when a crack exists on a beam. The remaining half is the energy consumed by the crack that can be formulated as follows:

$$CE = D(a)M(z_c)^2, \tag{1}$$

where $M(z_c)$ is the bending moment at the crack location of beam that can be formulated as

$$M(z_c) = E'I(z_c)\frac{\mathrm{d}^2W(z_c)}{\mathrm{d}z^2}. \tag{2}$$

E' is replaced by E for plane stress or $E/(1-v^2)$ for plane strain. $D(a)$ is the coefficient that can be defined by the following equation for a beam having a single-edge crack:

$$D(a) = \frac{18\pi F(a)^2 a^2}{Eb_c h_c^4}. \tag{3}$$

In Eq. (3), $F(a)$ is the function given for $a/h_c \leq 0.6$

$$F(a) = 1.12 - 1.4(a/h_c) + 7.33(a/h_c)^2 - 13.8(a/h_c)^3 + 14(a/h_c)^4. \tag{4}$$

The energy consumed given in Eq. (1) can also be explained by the spring model. The energy change due to crack opening can be balanced by the energy stored by a rotational spring model located at the crack tip. Since there is no spring in reality, the energy stored by the spring model is lost somewhere and is called 'the energy consumed'. The crack opening results in additional angular displacement of the beam causing also tensile stresses in the vicinity of crack tips. The energy of the tensile stress can be considered as the energy of the rotational spring model located at the unstretched side of the beam as shown in Fig. 1(a). When this effect is considered, the energy consumed is determined by taking the difference between the energy effects of the crack opening and tensile stress caused by the bending of the beam. In this case, the coefficient $D(a)$ is found as follows [18]:

$$D(a) = \frac{18\pi F(a)^2 a^2}{Eb_c h_c^4}(1 - a/h_c). \tag{5}$$

In deriving Eq. (5), minor effects of crack closing and compressive stresses caused by the bending of the beams are neglected. This open crack model can be sufficient for single-edge cracked beams vibrating in small amplitudes. When beams having double-edge cracks are bent, the crack on the stretched side opens up, and the crack on the compressed side of the beam closes. This makes it inevitable to use the breathing crack model for analysing the double-edge cracked beam. As shown in Fig. 1(b), this model covers the superposition of two cases: elongation of the beam due to crack opening and shortening of the beam due to crack closing. It is clear that, an additional crack will make the beam bend more, mainly because of the additional cross-section decrease and thus the stiffness loss of the beam. The beams will posses extra displacement near the open crack, in contrast to the displacement in negative direction near the closed crack. Tensile and compressive stresses also occur in the vicinity of the crack tips. Since some of the lost energy caused by the displacement changes is restored by the effect of the stress changes, the net energy consumed can be described by the following expression for the maximum deflection of a beam having double-edge breathing crack:

$$\begin{aligned} CE = {} & (\text{Energy of the elongation} - \text{Energy of the tensile stress}) \\ & -(\text{Energy of the shortening} - \text{Energy of the compressive stress}). \end{aligned} \tag{6}$$

The energy changes for the breathing crack can be obtained by the model including the equivalent rotational springs shown in Fig. 1(b). Additional rotational springs are located on the tip of the crack opening, for obtaining the energy of the compressive stress and are located on the tip of the closed crack for obtaining the energy change due to the displacement in negative direction. This model, which includes the extensions to the open crack model, is valid for the total depth ratio of

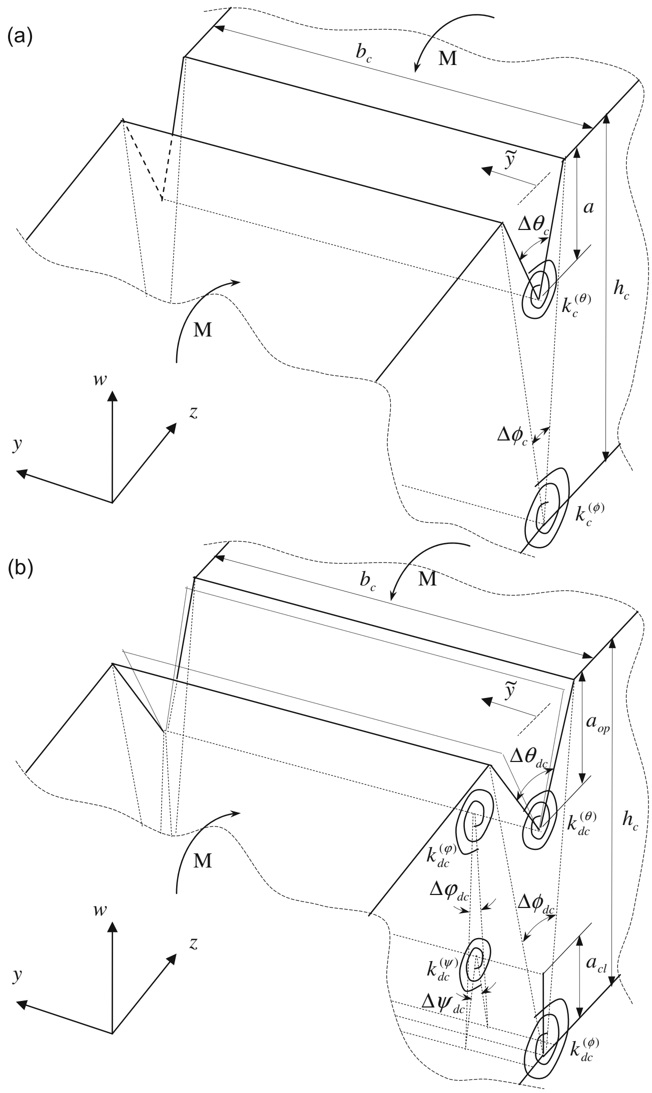

Fig. 1. Models for (a) single-edge and (b) double-edge cracks.

the cracks in pair, $(a_{\rm op}+a_{\rm cl})/h_c$, for less than 0.5. The energy consumed can be formulated by the energy of the equivalent springs as follows:

$$CE=\frac{1}{2b_c}\int_{\tilde{y}=0}^{b_c}\Big[\Big(k_{\rm dc}^{(\theta)}(\Delta\theta_{\rm dc})^2-k_{\rm dc}^{(\phi)}(\Delta\phi_{\rm dc})^2\Big)-\Big(k_{\rm dc}^{(\psi)}(\Delta\psi_{\rm dc})^2-k_{\rm dc}^{(\varphi)}(\Delta\varphi_{\rm dc})^2\Big)\Big]\,{\rm d}\tilde{y}, \tag{7}$$

where

$$\Delta\phi_{\rm dc}=\frac{a_{\rm op}}{h_c}\Delta\theta_{\rm dc},\quad \Delta\varphi_{\rm dc}=\frac{a_{\rm cl}}{h_c-a_{\rm op}}\Delta\psi_{\rm dc}. \tag{8,9}$$

The stiffness relation can also be established by providing bending moment equivalences at the stretched and compressed sides of the beam

$$k_{\rm dc}^{(\phi)}=\frac{h_c}{a_{\rm op}}k_{\rm dc}^{(\theta)},\quad k_{\rm dc}^{(\varphi)}=\frac{h_c-a_{\rm op}}{a_{\rm cl}}k_{\rm dc}^{(\psi)}. \tag{10,11}$$

If Eqs. (8)–(11) are substituted into Eq. (7), the following equation is obtained:

$$CE=\frac{1}{2b_c}\int_{\tilde{y}=0}^{b_c}\left[k_{\rm dc}^{(\theta)}(\Delta\theta_{\rm dc})^2\left(1-\frac{a_{\rm op}}{h_c}\right)-k_{\rm dc}^{(\psi)}(\Delta\psi_{\rm dc})^2\left(1-\frac{a_{\rm cl}}{h_c-a_{\rm op}}\right)\right]{\rm d}\tilde{y}. \tag{12}$$

The additional rotations of the open ($\Delta\theta_{ad}$) and closed ($\Delta\psi_{ad}$) cracks are influenced by three parameters which can be stated as: the cross-section decrease ($_d$), the neutral axis yawing ($_y$) due to the angular displacement difference between the cracks opening and closing, and the neutral axis shift ($_s$) due to the crack's asymmetry. Thus, the additional rotations are formulated as follows:

$$\Delta\theta_{ad} = \Delta\theta_{dc} - \Delta\theta_c = \Delta\theta_d + \Delta\theta_y + \Delta\theta_s, \tag{13}$$

$$\Delta\psi_{ad} = \Delta\psi_{dc} = \Delta\psi_d - \Delta\psi_y + \Delta\psi_s, \tag{14}$$

where $\Delta\theta_d$ and $\Delta\psi_d$ are the additional rotations caused by the direct effect of the cross-section decreases. The depth of a crack on one edge influences the opening and closing amounts of the crack on the other edge. Thus, additional rotation of the open crack due to the closed crack based cross-section drop is defined as follows:

$$\Delta\theta_d = \left(\frac{a_{cl}}{h_c - a_{op} - a_{cl}}\right)\Delta\theta_c. \tag{15}$$

Similarly, the additional rotation of the closed crack is written as follows:

$$\Delta\psi_d = \frac{a_{op}}{a_{cl}}\,\Delta\theta_d. \tag{16}$$

Neutral axis is not mentioned in the determination of the coefficient, $D(a)$, given in Eq. (5), since the crack is assumed always open and hence the beam bends with extra displacement and tensile stress only [18]. However, the breathing crack model makes it also necessary to take both the crack closing and the compressive stress effects into consideration. Nonlinear effects of the breathing cracks arise with the neutral axis modulation around the central axis during the period of vibration. The relation between the $\Delta\theta_d$ and $\Delta\psi_d$ can also be described by using yawing of the neutral axis X_y, which is caused by the difference between these additional rotations. Thus, yawing of the neutral axis is obtained as follows:

$$X_y = \frac{h_c}{2} - (h_c - a_{op})\frac{\Delta\psi_d}{\Delta\theta_d + \Delta\psi_d} \tag{17}$$

Yawing of the neutral axis leads to another additional rotation symbolised by $\Delta\theta_y$ and $\Delta\psi_y$ in Eqs. (13) and (14). The sign of the $\Delta\psi_y$ in Eq. (14) is negative, since the angular displacement due to the closed crack is always less than that caused by the crack opening. This means that the neutral axis always moves towards the closed crack during the bending. It is clear that, yawing effects reach a maximum when the cracks are located at the centre of mass of the beam and decrease as the cracks approach to the beam's ends. The yawing effects function can be represented by multiplying the maximum yawing with a normalised Gaussian function, G_N, having unit amplitude. The mean value of this function is then located on the mass centre and the standard deviation of the function is $L/6$. As a consequence, additional rotation of the beam due to the yawing of the neutral axis can be defined by the following relations:

$$\Delta\theta_y = \Delta\theta_d\left(\frac{X_y G_N}{h_c/2}\right), \quad \Delta\psi_y = \Delta\psi_d\left(\frac{X_y G_N}{h_c/2}\right). \tag{18,19}$$

The neutral axis will deviate from the central axis if there is an asymmetry in the depths of the open and closed cracks. The neutral axis shift is given by the parameter X_s

$$X_s = (a_{cl} - a_{op})/2 \tag{20}$$

and the additional rotations due to this shift is described by the following equations:

$$\Delta\theta_s = \Delta\theta_d\left(\frac{X_s}{h_c/2}\right), \quad \Delta\psi_s = \Delta\psi_d\left(\frac{X_s}{h_c/2}\right) \tag{21,22}$$

The signs of $\Delta\theta_s$ and $\Delta\psi_s$ are determined by the sign of X_s. If Eqs. (15)–(22) are considered together with Eqs. (13) and (14), the relations for additional angular displacement effects for open and closed cracks can be easily found as

$$\Delta\theta_{dc} = \Delta\theta_c\left[1 + \left(1 + \frac{X_y G_N}{h_c/2} + \frac{X_s}{h_c/2}\right)\left(\frac{a_{cl}}{h_c - a_{op} - a_{cl}}\right)\right] = \Delta\theta_c\gamma, \tag{23}$$

$$\Delta\psi_{dc} = \Delta\theta_c\left[\left(1 - \frac{X_y G_N}{h_c/2} + \frac{X_s}{h_c/2}\right)\left(\frac{a_{op}}{h_c - a_{op} - a_{cl}}\right)\right] = \Delta\theta_c\beta. \tag{24}$$

Resistances to these additional rotations, which are modelled by the rotational springs, can be determined by equating the bending moments at the stretched and compressed sides of the beam which in return gives us the following stiffness equations:

$$k_{dc}^{(\theta)} = \frac{k_c^{(\theta)}}{\gamma}, \quad k_{dc}^{(\psi)} = \frac{k_c^{(\theta)}}{\beta}. \tag{25,26}$$

Hence, the energy consumed is obtained by substituting the additional rotation and stiffness expressions into Eq. (12):

$$CE = \frac{1}{2b_c}\int_{\tilde{y}=0}^{b_c}\left[k_c^{(\theta)}(\Delta\theta_c)^2\left(1 - \frac{a_{op}}{h_c}\right)\gamma\right]\mathrm{d}\tilde{y} - \frac{1}{2b_c}\int_{\tilde{y}=0}^{b_c}\left[k_c^{(\theta)}(\Delta\theta_c)^2\left(1 - \frac{a_{cl}}{h_c - a_{op}}\right)\beta\right]\mathrm{d}\tilde{y} \tag{27}$$

Extensions to the open crack model, seen in Fig. 1, are extracted from Eq. (27) and added into Eq. (5) which should be modified as the formulations below, for opening and closing cases of single-edge or double-edge breathing cracks

$$D(a_{\mathrm{op}})=\frac{18\pi F(a_{\mathrm{op}})^2 a_{\mathrm{op}}^2}{Eb_c h_c^4}\left(1-\frac{a_{\mathrm{op}}}{h_c}\right)\gamma, \tag{28}$$

$$D(a_{\mathrm{cl}})=\frac{18\pi F(a_{\mathrm{cl}})^2 a_{\mathrm{cl}}^2}{Eb_c h_c^4}\left(1-\frac{a_{\mathrm{cl}}}{h_c-a_{\mathrm{op}}}\right)\beta. \tag{29}$$

Of course when a beam vibrates, it will be bending in two opposite directions which will result in the exchange of the positions of the open and closed cracks. If the open and closed cracks are subscripted by the numbers also, Eq. (1), defining the energy consumed for a single-edge open crack, is modified as below for the single-edge and double-edge breathing cracks:

$$CE_{a1}=D(a_{\mathrm{op},1})[M(z_c)]^2-D(a_{\mathrm{cl},1})[M(z_c)]^2, \tag{30}$$

$$CE_{a2}=D(a_{\mathrm{op},2})[M(z_c)]^2-D(a_{\mathrm{cl},2})[M(z_c)]^2. \tag{31}$$

It should be remembered that the moment terms in Eqs. (30) and (31) include different expressions for open and closed cracks due to the difference in E'. The energy consumed is distributed along the beam length as follows [17]:

$$\Gamma^{\mathrm{CE}}=\frac{Q(a_1,z_c)}{1+[(z-z_c)/(q(a_1)a_1)]^2}+\frac{Q(a_2,z_c)}{1+[(z-z_c)/(q(a_2)a_2)]^2}, \tag{32}$$

where

$$Q(a_1,z_c)=\frac{CE_{a1}}{q(a_1)a_1\{\arctan[(L-z_c)/(q(a_1)a_1)]+\arctan[z_c/(q(a_1)a_1)]\}}, \tag{33}$$

$$q(a_1)=\frac{3\pi(F(a_1))^2(h_c-a_1-a_2)^3(a_1+a_2)}{(h_c^3-(h_c-a_1-a_2)^3)h_c}. \tag{34}$$

Eqs. (33) and (34) can be modified for the crack at the second edge.

The conservation of energy law dictates that, for a beam with no cracks, the maximum potential energy should be equal to maximum kinetic energy. If a crack exists on a beam, since the work is done by using the available maximum potential energy, the energy consumed results in a decrease of maximum potential energy with the assumption that there is no mass loss at the crack location

$$\int_{z=0}^{L}((\Gamma^{\mathrm{PE}}-\Gamma^{\mathrm{CE}})-\Gamma^{\mathrm{KE}})\,\mathrm{d}z=0, \tag{35}$$

where Γ^{PE} and Γ^{KE} represent the distributions of the maximum potential and kinetic energies as

$$\Gamma^{\mathrm{PE}}=\frac{1}{2}EI(z)\left(\frac{\mathrm{d}^2W(z)}{\mathrm{d}z^2}\right)^2, \tag{36}$$

$$\Gamma^{\mathrm{KE}}=\frac{1}{2}\rho A(z)\omega^2(W(z))^2+\frac{1}{2}\rho I(z)\omega^2\left(\frac{\mathrm{d}W(z)}{\mathrm{d}z}\right)^2. \tag{37}$$

The second term in Eq. (37) describes the effect of rotary inertia around the axis perpendicular to the bending plane. If κ_j is the coefficient of admissible mode shape function, the derivatives of Eq. (35) or those of Rayleigh quotient derived from Eq. (35) should be equal to zero

$$\partial\left(\int_{z=0}^{L}((\Gamma^{\mathrm{PE}}-\Gamma^{\mathrm{CE}})-\Gamma^{\mathrm{KE}})\,\mathrm{d}z\right)/\partial\kappa_j=0. \tag{38}$$

If $\chi_j(z)$ are a series of functions satisfying the end conditions, the mode shape function can be written as

$$W(z)=\sum_{j=1}^{m}\kappa_j\chi_j(z). \tag{39}$$

The functions, $\chi_j(z)$, are given in Table 1 for several end conditions.

Table 1
Series of functions satisfying the several end conditions.

End conditions	$\chi_j(z)$
Fixed–fixed	$(z/L)^{j+1}(1-z/L)^2$
Pinned–pinned	$(z/L)^j(1-z/L)$
Fixed–free	$(z/L)^2(1-z/L)^{j-1}$
Fixed–pinned	$(z/L)^{j+1}(1-z/L)$

3. Results and discussion

Results are presented by applying the developed method on simply supported and cantilever beams. Simply supported aluminium and steel beams having single-edge or symmetric double-edge cracks at the mid-span range are analysed and the results are compared. The aluminium beam has the following geometric properties: length L=0.235 m, width b=0.006 m, and height h=0.0254 m. The material properties of the beam are ρ=2800 kg/m^3 as density, E=72 GPa as modulus of elasticity, and ν=0.35 as Poisson ratio. A double-edge cracked steel beam of length, width, and height are given as L=0.575 m, b=0.00952 m, and h=0.03175 m, respectively. The beam has the following material properties; density ρ=7800 kg/m^3, modulus of elasticity E=206 GPa, and Poisson ratio ν=0.35. A six termed deflection function is employed in the Rayleigh–Ritz method, and a breathing crack model is used in the analysis. Frequency ratios obtained by the method agree with the results of the models presented by Chondros et al. [9] as seen in Figs. 2 and 3.

The method is also applied to a tapered cantilever beam having density ρ=7800 kg/m^3, modulus of elasticity E=210 GPa, and Poisson ratio ν=0.3. Variation of the height and width of the tapered beam can be expressed by the functions: $h(z)=h_2+(h_1-h_2)z/L$, $b(z)=b_2+(b_1-b_2)z/L$. The beam has also geometric properties as L=0.6 m, $h_1=b_1$=0.02 m, $\alpha_h=h_2/h_1$=0.5, and $\alpha_b=b_2/b_1$=0.75. The geometry of the tapered beam is shown in Fig. 4.

Results obtained by the present method are compared with the results of the commercial finite element program (ANSYS©) for the tapered beam in consideration. Cracks are considered as slots which are formed by subtracting thin transverse blocks from the "solid95" beam in the program. Element size is set to 0.009 m with the "esize" command, and crack widths are chosen as 0.0004 m. Smaller element size requirements in the vicinity of cracks are provided by the

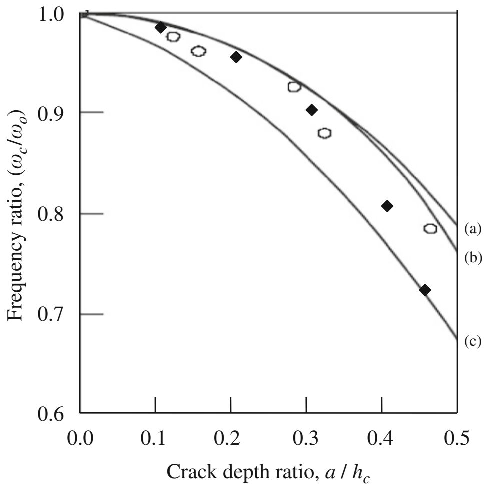

Fig. 2. First mode vibration frequency ratios of the simply supported aluminium beam with mid-span single-edge crack: (a) lumped crack flexibility model [9], (b) continuous crack model [9], (c) Christides and Barr's model [7], (○) experimental results [9], and (◆) the present model.

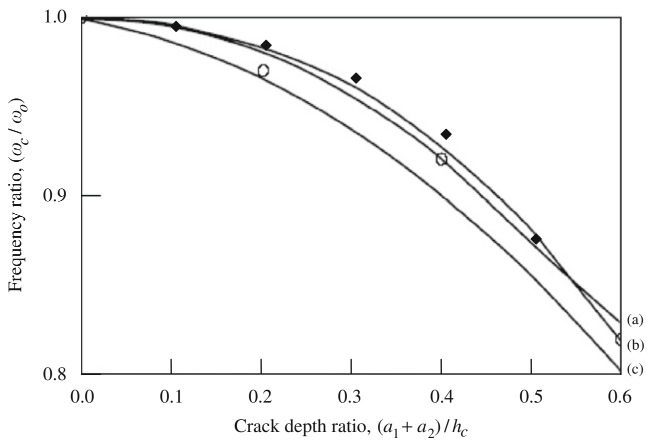

Fig. 3. First mode vibration frequency ratios of the simply supported steel beam with mid-span symmetric double-edge crack: (a) lumped crack flexibility model [9], (b) continuous crack model [9], (c) Christides and Barr's model [7], (○) experimental results [9], and (◆) the present model.

"smrtsize, 1" command, and free meshing procedures are applied. Resultantly, natural frequencies are obtained by using the analysis type called "modal analysis" in the program. It should be noted that, changes in the element number caused by the variation of crack location and crack size, have negligible effects on the results. Natural frequencies of the un-cracked beams obtained by the Rayleigh–Ritz approximations and the finite element program closely agree with each other as shown in Table 2.

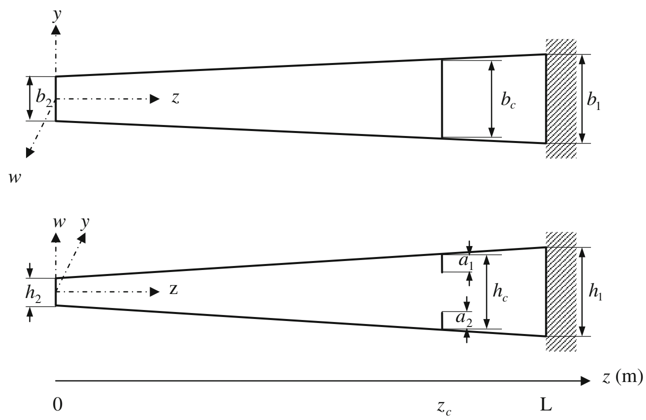

Fig. 4. Geometry of a beam.

Table 2
Natural frequencies of the uncracked beam (ω_0).

Vibration modes	Frequencies (Hz) obtained by Rayleigh–Ritz (6 terms)	Frequencies (Hz) obtained by Rayleigh–Ritz (9 terms)	Frequencies (Hz) obtained by finite element program
1	54.8890	54.8890	54.935
2	249.059	249.029	248.75

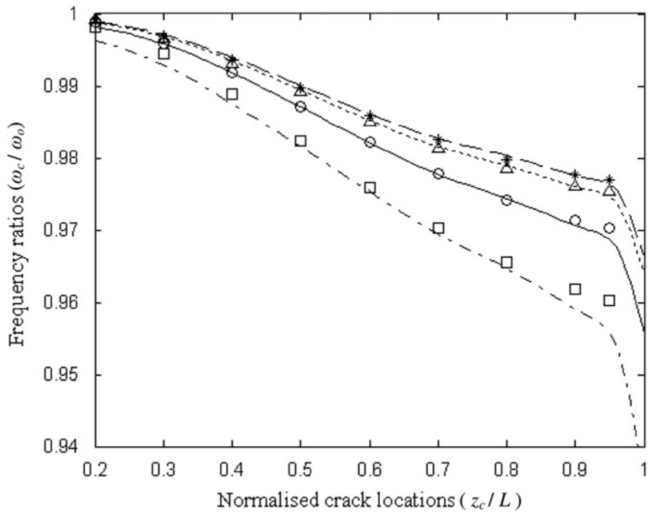

Fig. 5. First mode vibration frequency ratios of the tapered cantilever beam with several depth combinations of cracks in pair: (– –), (- - -), (—), and (– -) are the results obtained by the method with six termed deflection function in Case 1, Case 2, Case 3, and Case 4, respectively. (*), (△), (○), and (□) are the results of the Ansys© for mentioned cases.

The vibration of a beam having different combinations of symmetric and asymmetric double-edge breathing cracks with the same total depth ($a_1+a_2=0.3h_1$) is investigated as an example. The vibration of a single-edge cracked beam is also examined. The following crack cases are examined for the beam considered with variable crack locations:

Case 1: $a_1=0.15h_1$, $a_2=0.15h_1$; Case 2: $a_1=0.20h_1$, $a_2=0.10h_1$;
Case 3: $a_1=0.25h_1$, $a_2=0.05h_1$; Case 4: $a_1=0.30h_1$, $a_2=0.00h_1$.

The analyses are performed for the beams having the cracks located through $0.2L-L$ in which the total crack depth ratio remains under 0.5. The results of the present method, which uses the six termed deflection function, agree well with the results of the finite element program for the first mode of vibration as shown in Fig. 5. Second mode frequencies obtained by the method also match with the results of the finite element program for the beam having single-edge crack. However, in the cases of double-edge cracks, the matching of the second mode frequencies decreases when cracks exist through the $0.2L-0.4L$ as seen in Fig. 6. Better agreement can be observed in higher vibration modes when the deflection function used in the analysis is expanded with the larger number of terms [18]. As shown in Fig. 7, improved matching of the second mode frequencies is obtained by a nine termed approximation function for the beam

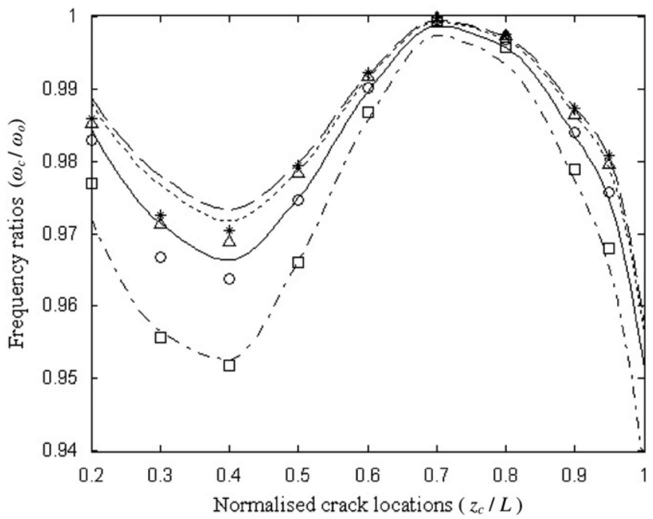

Fig. 6. Second mode vibration frequency ratios of the tapered cantilever beam with several depth combinations of cracks in pair: (– –), (- - -), (—), and (– -) are the results obtained by the method with six termed deflection function in Case 1, Case 2, Case 3, and Case 4, respectively. (*), (△), (○), and (□) are the results of the Ansys© for mentioned cases.

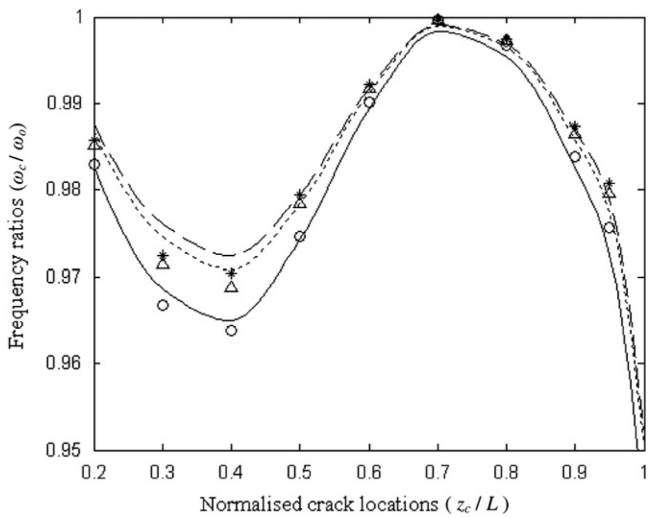

Fig. 7. Second mode vibration frequency ratios of the tapered cantilever beam with several depth combinations of cracks in pair: (– –), (- - -), and (—) are the results obtained by the method with nine termed deflection function in Case 1, Case 2, and Case 3, respectively. (*), (△), and (○) are the results of the Ansys© for mentioned cases.

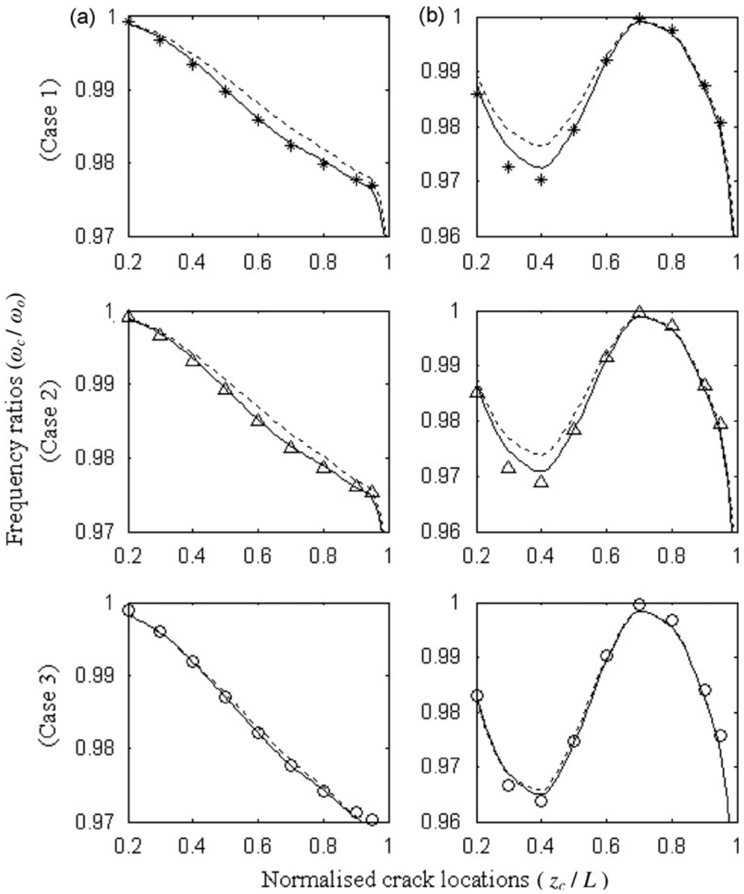

Fig. 8. (a) First mode and (b) second mode natural frequency ratios obtained by the method using (a) six termed and (b) nine termed deflection functions. The method including breathing (—) and open (- - -) crack models are compared with the results of the Ansys© figured by (*), (△), and (○) representing Case 1, Case 2, and Case 3, respectively.

with a double-edge crack. This result shows that in the analysis of the double-edge cracked beam, the number of terms used in the deflection function should be more than the size of the function used in the analysis of the single-edge cracked beam. It is also seen from the figures that natural frequency ratios decrease with increasing asymmetry of the cracks in pair.

The differences between the results of open and breathing crack models are shown in Fig. 8. The effects of crack closing, compressive stresses, additional rotations, and neutral axis changes are not included in the open crack model. Results show that better matching with the finite element program can be obtained when the breathing crack model is used in the analysis of a double-edge cracked beam. It is also seen from the figure that the differences between the results of open and breathing crack models become smaller when larger asymmetry exists between the cracks.

4. Conclusion

A method is presented to obtain the vibration of non-uniform beams having symmetric and asymmetric double-edge breathing cracks. The open crack model presented by Mazanoglu et al. [18] is modified by taking into account the effects of crack closing and compressive stress in addition to crack opening and tensile stress for modelling the breathing cracks. In addition to the direct effect of extra cross-section decrease, the effects of neutral axis yawing due to the difference between the opening and closing amounts and neutral axis shift due to the depth difference of the cracks in pair are also included in the model. The energy effects of the rotary inertia are also taken into consideration. Overall energy is analysed by the Rayleigh–Ritz approximation method.

This paper presents the first application of the vibration analysis of non-uniform beams having double-edge cracks. Up until now, there has been no work in existing literature for analysing the vibration of beams with asymmetric double-edge crack although symmetric double-edge crack models have been presented for uniform beams. The model presented in this paper is valid for both single-edge and symmetric double-edge cracks. The model has also the capability of analysing the vibration of beams with different depth combinations of asymmetric double-edge cracks.

Results of the method including open and breathing crack models are compared and examined in this paper. When the results we obtain for the double-edge cracked beams are compared with the results of the finite element program, we see that the results of the breathing crack model are more accurate than that of the open crack model. The differences between the results of open and breathing crack models become negligible for the single-edge cracks.

In the paper, it is shown that higher modes of vibration frequencies require larger number of terms to use in the deflection function. It is also observed that an extended number of terms are required for analysing the vibration of double-edge cracked beams. This means vibration analysis of the double-edge cracked beams needs more time than that of the single-edge cracked beams. However, a significant advantage of the method is that the computing time is shorter compared with the time it takes to solve this using a finite element program. Thus, natural frequencies required for the frequency based inverse methods can be easily obtained for different beams.

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