Journal of Sound and Vibration 329 (2010) 4335–4353

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Robust control synthesis for seat suspension systems with actuator saturation and time-varying input delay ☆

Yingbo Zhao, Weichao Sun, Huijun Gao *

Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, China

ARTICLE INFO

Article history:
Received 30 April 2009
Received in revised form
16 September 2009
Accepted 16 September 2009
Handling Editor: J. Lam
Available online 26 May 2010

ABSTRACT

The control synthesis problem is investigated in this paper for a class of semi-active seat suspension systems with norm-bounded parameter uncertainties, time-varying input delay and actuator saturation. A vertical vibration model of human body is introduced in order to make the modeling of seat suspension systems more precise. By employing a delay-range-dependent Lyapunov function and exploring the property of the saturation nonlinearity, the existence conditions of the desired state-feedback controller are derived in terms of linear matrix inequalities (LMIs). The controller is derived by solving the LMIs and the corresponding closed-loop system is asymptotically stable with a guaranteed H_∞ performance. A design example is presented to show the usefulness and advantages of the developed theoretical results.

1. Introduction

Vehicle suspensions are capable of providing a more comfortable ride by serving the basic function of isolating passengers from the roughness of the road. In other words, the most important role of suspension systems is the ride quality improvement, which is drawing increasing attention because of its impact on the drivers' fatigue, health and discomfort. Despite much work has been done on primary and secondary suspensions to improve the ride quality [1,2], seat suspensions are introduced for their simplicity and effectiveness to attenuate high-amplitude vibration in the low frequency range, in which human body is most sensitive to vibrations in the vertical direction [3–5]. To meet this demand, numerous types of seat suspension systems, including passive, semi-active and active suspensions, are currently employed and studied. Especially, the semi-active suspensions have been attracting the most attention in recent years and various approaches have been proposed to improve the performance, including optimal control [6], fuzzy logic and neural network control [7], adaptive control [8], H_∞ control [9,10], and gain-scheduling control [11], for instance.

Time delays are widely encountered in the control loops because of the electrical and electromagnetic characteristics of the actuators, which often degrade the control performances and even cause system instability. As a result, they have been widely studied during the past decades and many analytical techniques and synthesis methods have been developed using delay-dependent Lyapunov function concerning conservatism [12–18]. However, it is worth mentioning that most existing results for seat suspension systems have not taken time delays in the system into account, and ignoring them may cause deterioration in the control performance or even render the system unstable, which motivates our present study. The input

☆ This work was partially supported by National Natural Science Foundation of China (60825303), the Research Found for the Doctoral Programme of Higher Education of China (20070213084), Fok Ying Tung Education Foundation (111064).
* Corresponding author. Fax: +86 45186418091.
E-mail addresses: zhaoyingbo1986@gmail.com (Y. Zhao), 1984sunweichao@gmail.com (W. Sun), hjgao@hit.edu.cn (H. Gao).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2009.09.017

delay in this paper is assumed to be time varying in a range with upper and lower bounds, and a delay-range-dependent Lyapunov function is employed to further reduce conservatism [19,20].

Moreover, actuator saturation appears frequently in engineering systems, which is also a source of performance degradation and the closed-loop system instability. Therefore, the analysis and synthesis of control systems with actuator saturation nonlinearities have been a highlighted research topic in the research and industry domain in the past few years, and many results have been reported [21–23]. However, to the best of the authors' knowledge, this control method has not been tried for the seat suspension system in available literatures, which is another motivation of the research in this paper.

Besides, many uncertain factors such as the inaccuracies of model parameters and the errors of sensors and actuators, degrade the vibration attenuation performance and safety during the driving process. In recent years, many results have been reported to deal with the uncertainties in order to guarantee the closed-loop performance [24–27]. This robust control design method is also considered in this paper to ensure the closed-loop system asymptotical stability and H_∞ performance in spite of the parameter uncertainties.

Last but not least, most existing results concerning with seat suspensions have limited their scope to model the diver as a rigid dummy mass on the seat, which is obviously not precise enough to investigate the performances because no biodynamics are included. Therefore, the sophisticated research of ride comfort and safety improvement calls for a mathematical seated human body model. This paper utilizes a four DOF human body model to depict the essential dynamics of a seated human exposed to vertical vibration to obtain a good tradeoff between facility and accuracy as well as a better insight of the controller design.

In this paper, we are interested in the problem of robust H_∞ state-feedback controller design for a class of semi-active seat suspension systems with time-varying input delay, norm-bounded parameter uncertainties and actuator saturation. In order to obtain a better insight of the suspension system performance, a vibration model of human body is introduced and combined with the seat. By defining a Lyapunov functional more appropriate for the underlying systems and exploring the special property of the saturation nonlinearity to utilize an auxiliary feedback matrix, a less conservative condition is obtained, which turns out to be equivalent to a set of linear matrix inequalities (LMIs) [28]. And the desired controller can be obtained after solving the LMIs with standard numerical algorithms so that the corresponding closed-loop system is asymptotically stable and has a guaranteed disturbance attenuation level. Simulation results of a design example are given to show the effectiveness of the proposed controller design method.

The rest of this paper is organized as follows. Section 2 addresses the problem of multiobjective state-feedback controller design for a semi-active seat suspension system with human body model. Section 3 presents the main results, including stability and performance analysis. A design example demonstrating the effectiveness and advantages of the proposed methodology is given in Section 4 and some concluding remarks are given in Section 5.

2. Problem formulation

In this study, a three-degree-of-freedom seat suspension model shown in Fig. 1 established by Wei and Griffin in 1998 [29] is considered for controller design. In this figure, m_1 is the mass of seat frame; m_{21} and m_{22} are the masses of human thighs together with buttocks and the seat cushion, respectively, and $m_2 = m_{21} + m_{22}$; m_3 is the mass of the upper body of a seated human. The mass of lower legs and feet is neglected because of their little contribution to the biodynamic response of the seated body. c_1, c_2 and k_1, k_2 are dampings and stiffnesses of the passive suspension system, respectively; c_3 and k_3

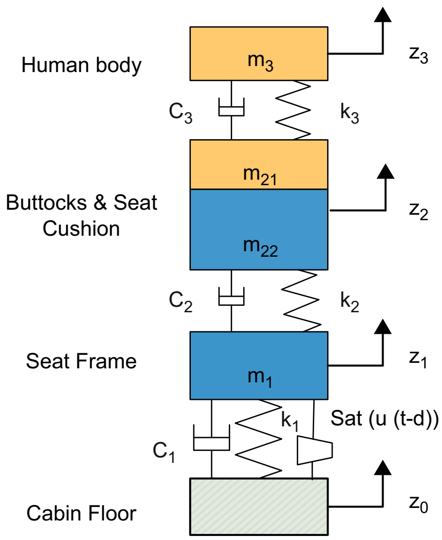

Fig. 1. Vibration model of the seat suspension system.

stand for the damping and stiffness of the components inside human body such as spines; z_1, z_2 and z_3 are the displacements of the corresponding masses; z_0 is the road displacement input, $\omega(t)=\dot{z}_0(t)$ represents the disturbance caused by road roughness; u is the active control input of the seat suspension system. To predict the biodynamic responses more reasonably, the mass of buttocks and legs is assumed to contact rigidly with the seat. The road excitation input is transmitted to the cabin floor. It is also assumed that only the vertical motion of the vehicle exists for simplification.

The governing equations of motion for the seat suspension can be expressed as

$$m_1\ddot{z}_1=-c_1(\dot{z}_1-\dot{z}_0)-k_1(z_1-z_0)+c_2(\dot{z}_2-\dot{z}_1)+k_2(z_2-z_1)-u,$$

$$m_2\ddot{z}_2=-c_2(\dot{z}_2-\dot{z}_1)-k_2(z_2-z_1)+c_3(\dot{z}_3-\dot{z}_2)+k_3(z_3-z_2),$$

$$m_3\ddot{z}_3=-c_3(\dot{z}_3-\dot{z}_2)-k_3(z_3-z_2). \tag{1}$$

By defining the state variable as

$$\mathbf{x}(t)=[x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)\ x_5(t)\ x_6(t)]^{\mathrm{T}}, \tag{2}$$

where

$$x_1(t)=z_1(t)-z_0(t),\quad x_2(t)=\dot{z}_1(t),\quad x_3(t)=z_2(t)-z_1(t),$$

$$x_4(t)=\dot{z}_2(t),\quad x_5(t)=z_3(t)-z_2(t),\quad x_6(t)=\dot{z}_3(t), \tag{3}$$

which are the deflections of the corresponding springs and velocities of the mass segments.

Then the dynamic equations in (1) can be written in the following state-space form:

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)+\mathbf{B}_w\omega(t), \tag{4}$$

where

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{k_1}{m_1} & -\dfrac{c_1+c_2}{m_1} & \dfrac{k_2}{m_1} & \dfrac{c_2}{m_1} & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & \dfrac{c_2}{m_2} & -\dfrac{k_2}{m_2} & -\dfrac{c_2+c_3}{m_2} & \dfrac{k_3}{m_2} & \dfrac{c_3}{m_2} \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & \dfrac{c_3}{m_3} & -\dfrac{k_3}{m_3} & -\dfrac{c_3}{m_3} \end{bmatrix},$$

$$\mathbf{B}=\left[0\ \ -\frac{1}{m_1}\ \ 0\ \ 0\ \ 0\ \ 0\right]^{\mathrm{T}},$$

$$\mathbf{B}_w=\left[-1\ \ \frac{c_1}{m_1}\ \ 0\ \ 0\ \ 0\ \ 0\right]^{\mathrm{T}}. \tag{5}$$

The seat suspension model becomes an uncertain model with time-varying input delay when changes in vehicle inertial properties, actuator time delays and saturation nonlinearities are taken into account, which can be expressed as

$$\dot{\mathbf{x}}(t)=\overline{\mathbf{A}}\mathbf{x}(t)+\overline{\mathbf{B}}\sigma(\mathbf{u}(t-d(t)))+\mathbf{B}_w w(t), \tag{6}$$

the actuator delay $d(t)$ is a time-varying continuous function that satisfies

$$d_1\le d(t)\le d_2,\quad 0\le\dot{d}(t)\le\mu, \tag{7}$$

where d_1 and d_2 are the lower and upper bounds of the input delay, respectively, and μ is the delay variation rate limit. The parameter uncertainties considered here are norm-bounded of the form

$$\overline{\mathbf{A}}=\mathbf{A}+\Delta\mathbf{A},\overline{\mathbf{B}}=\mathbf{B}+\Delta\mathbf{B},$$

$$[\Delta\mathbf{A}\ \ \Delta\mathbf{B}]=\mathbf{L}_1\mathbf{F}(t)[\mathbf{E_A}\ \ \mathbf{E_B}], \tag{8}$$

where $\mathbf{L}_1$, $\mathbf{E_A}$, $\mathbf{E_B}$ are known constant real matrices of appropriate dimensions, and $\mathbf{F}(t)$ is an unknown matrix function with Lebesgue-measurable elements satisfying $\mathbf{F^T(t)F(t)}\le\mathbf{I}$, and the actuator saturation nonlinearity is described by

$$\sigma(\mathbf{u}(t))=[\sigma(u_1(t))\ \ \sigma(u_2(t))\ \ \cdots\ \ \sigma(u_q(t))]^{\mathrm{T}},$$

$$\sigma(u_i(t))\triangleq\begin{cases} u_{i\max} & \text{if } u_i(t)\ge u_{i\max}, \\ u_i(t) & \text{if } -u_{i\max}\le u_i(t)\le u_{i\max}, \\ -u_{i\max} & \text{if } u_i(t)\le -u_{i\max}. \end{cases} \tag{9}$$

Before designing the state-feedback control law for a seat suspension system, we need to consider the following aspects:

(1) *Ride comfort*: Ride comfort can be generally quantified by the body acceleration in the vertical direction, thus, it is chosen as the first control output, i.e. minimizing the vertical acceleration of human body $\ddot{\mathbf{z}}_3(t)$ is one of our most concerned objectives in the controller design, that is,

$$\mathbf{z}_{o1}(t) = \ddot{\mathbf{z}}_3(t).$$

Moreover, the H_∞ norm is employed to measure the performance, whose value actually gives an upper bound of the root-mean-square gain. Hence, our goal is to minimize the H_∞ norm of the transfer function from the disturbance $w(t)$ to the control output $\mathbf{z}_{o1}(t)$ in order to improve ride comfort.

(2) *Suspension deflection limitation*: The controller should be capable to prevent the suspension from hitting its travel limit in order to avoid ride comfort deterioration and mechanical structural damage. The requirement is

$$\mathbf{z}_{o2}(t) = |\mathbf{z}_1(t) - \mathbf{z}_0(t)| \le \mathbf{z}_{\max}, \tag{10}$$

where $z_{\max}$ is the maximum suspension deflection limit, under all road disturbance inputs. The deflection space does not need to be minimized but its peak value needs to be limited.

Therefore, the strategy in the seat suspension system control law designing is to minimize the H_∞ norm of the transfer function from the disturbance $w(t)$ to the control output $\mathbf{z}_{o1}(t)$ and guarantee the suspension stroke requirement.

Then, the vehicle seat suspension system can be described by the following state-space equations:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \overline{\mathbf{A}}\mathbf{x}(t) + \overline{\mathbf{B}}\sigma(\mathbf{u}(t-d(t))) + \mathbf{B}_w w(t),\\ \mathbf{z}_{o1}(t) &= \overline{\mathbf{C}}_1\mathbf{x}(t),\\ \mathbf{z}_{o2}(t) &= \mathbf{C}_2\mathbf{x}(t),\end{aligned} \tag{11}$$

where $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$, $\mathbf{B}_w$ are already defined in (5), and

$$\overline{\mathbf{C}}_1 = [0\ \ 0\ \ 0\ \ c_3/m_3\ \ -k_3/m_3\ \ -c_3/m_3],$$

$$\mathbf{C}_2 = [1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0],$$

with

$$\overline{\mathbf{C}}_1 = \mathbf{C}_1 + \Delta\mathbf{C}_1, \Delta\mathbf{C}_1 = \mathbf{L}_2\mathbf{F}(t)\mathbf{E_c}.$$

In this paper, our goal is to find a state-feedback control law

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t), \tag{12}$$

such that the following requirements are satisfied:

(1) the closed-loop system is asymptotically stable;
(2) under zero initial condition, the performance $\|T_{z_{o1}w}\|_\infty < \gamma$ is minimized subject to (10) for all nonzero $w \in L_2[0,\infty)$, where $T_{z_{o1}w}$ denotes the closed-loop transfer function from the road disturbance $w(t)$ to the control output $\mathbf{z}_{o1}(t)$.

3. Robust H_∞ controller design

The sufficient conditions for the closed-loop system robust asymptotically stability and performance requirements can be derived as follows.

To begin with, for feedback gain matrix $\mathbf{K}$, we define

$$L(\mathbf{K}) \triangleq \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{k}_i\mathbf{x}| \le u_{i\max},\ i = 1, 2, \ldots, q\},$$

where $\mathbf{k}_i$ is the i th row of $\mathbf{K}$. Then $L(\mathbf{K})$ is the region in the state space where the control input is linear in $\mathbf{x}$.

Next, as shown in [28], we utilize the technique of auxiliary feedback matrices here to reduce the conservatism of dealing with the actuator saturation. Namely, for two matrices $\mathbf{K}, \mathbf{H} \in \mathbb{R}^{q\times n}$ and a vector $\mathbf{v} \in \mathbb{R}^q$, a matrix set is introduced as

$$\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H}) \triangleq \left\{ \mathbf{W} \in \mathbb{R}^{q\times n} : \mathbf{W} = \begin{bmatrix} \mathbf{v}_1\mathbf{k}_1 + (1-\mathbf{v}_1)\mathbf{h}_1 \\ \vdots \\ \mathbf{v}_q\mathbf{k}_q + (1-\mathbf{v}_q)\mathbf{h}_q \end{bmatrix} \right\},$$

where $\mathbf{v}_i = 0$ or 1, define $\psi(\mathbf{v}) \triangleq \{\mathbf{v} \in \mathbb{R}^q : \mathbf{v}_i = 0 \text{ or } 1\}$ and the auxiliary matrix $\mathbf{H}$ satisfies $|\mathbf{h}_i\mathbf{x}| \le u_{i\max}$, $i = 1, 2, \ldots, q$. And a subset of the set $L(\mathbf{K})$ will be found and chosen to be an ellipsoid of the form

$$\xi(\mathbf{P}, 1) \triangleq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^{\mathbf{T}}\mathbf{P}\mathbf{x} \le 1\},$$

where $\mathbf{P}>0$ will be determined. Combine $\xi(\mathbf{P},1)$ with

$$\begin{bmatrix} u_{i\max} & \mathbf{h}_i \\ * & u_{i\max}\mathbf{P} \end{bmatrix} \geq 0, \quad i=1,2,\ldots,q, \tag{13}$$

which means that if $\mathbf{x(t)^T Px(t)} \leq 1$, we have $2|\mathbf{h}_i\mathbf{x(t)}| \leq u_{i\max}(1+\mathbf{x(t)^T Px(t)}) \leq 2u_{i\max}$, i.e. $|\mathbf{h}_i\mathbf{x(t)}| \leq u_{i\max}$. So we can ensure that $\xi(\mathbf{P},1) \subset L(\mathbf{H})$.

Remark 1. There are 2^q elements in $\psi(\mathbf{v})$. $\mathbf{v}$ is used to choose from the rows of $\mathbf{K}$ and $\mathbf{H}$ to form a new matrix $\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H})$. If $\mathbf{v}_i=0$, then the i th row of $\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H})$ is $\mathbf{h}_i$, and if $\mathbf{v}_i=1$, then the i th row of $\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H})$ is $\mathbf{k}_i$. For example, assume $q=2$, then

$$\{\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H}) : \mathbf{v} \in \psi(\mathbf{v})\} \triangleq \left\{\mathbf{H}, \begin{bmatrix}\mathbf{k}_1\\ \mathbf{h}_2\end{bmatrix}, \begin{bmatrix}\mathbf{h}_1\\ \mathbf{k}_2\end{bmatrix}, \mathbf{K}\right\}.$$

Based on the above ideas, the following theorem gives the existence conditions of a desired state-feedback controller for system (11).

Theorem 1. *Consider system* (11) *with the input-delayed state-feedback controller in* (12), *suppose* $\gamma>0$, $\rho>0$, $0\leq d_1\leq d_2$ *and* $\mu>0$ *are given scalars. Then the closed-loop system is asymptotically stable and satisfies* $\|T_{z_{o1}w}\|_\infty<\gamma$ *for all nonzero* $w\in L_2[0,\infty)$ *under zero initial condition if there exist matrices* $\mathbf{P}>0$, $\mathbf{T}>0$, $\mathbf{Q}_i>0$, $i=1,2,3$, $\mathbf{Z}_i>0$, $i=1,2$, $\mathbf{N}_i$, $\mathbf{S}_i$, $\mathbf{M}_i$, $i=1,\ldots,5$, $\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H})$ *and* $\mathbf{W}(\mathbf{s},\mathbf{K},\mathbf{H})$ *satisfying*

$$\begin{bmatrix} \hat{\Pi} & d_2\mathbf{N} & d_{12}\mathbf{S} & d_{12}\mathbf{M} \\ * & -d_2\mathbf{Z}_1 & 0 & 0 \\ * & * & -d_{12}(\mathbf{Z}_1+\mathbf{Z}_2) & 0 \\ * & * & * & -d_{12}\mathbf{Z}_2 \end{bmatrix} < 0, \tag{14}$$

$$\begin{bmatrix} -\mathbf{I} & \sqrt{\rho}\mathbf{C}_2 \\ * & -z_{\max}^2\mathbf{P} \end{bmatrix} < 0, \tag{15}$$

where

$$\hat{\Pi} = \begin{bmatrix} \Pi_{11}+\overline{\mathbf{C}}_1^{\mathbf{T}}\overline{\mathbf{C}}_1 & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & 0 \\ * & * & \Pi_{33} & \Pi_{34} & \Pi_{35} & 0 \\ * & * & * & \Pi_{44} & \Pi_{45} & 0 \\ * & * & * & * & \Pi_{55} & \Pi_{56} \\ * & * & * & * & * & -\gamma^2 \end{bmatrix},$$

$$\begin{aligned}
&\mathbf{N^T} = [\mathbf{N}_1^{\mathbf{T}}\ \mathbf{N}_2^{\mathbf{T}}\ \mathbf{N}_3^{\mathbf{T}}\ \mathbf{N}_4^{\mathbf{T}}\ \mathbf{N}_5^{\mathbf{T}}\ 0],\\
&\mathbf{S^T} = [\mathbf{S}_1^{\mathbf{T}}\ \mathbf{S}_2^{\mathbf{T}}\ \mathbf{S}_3^{\mathbf{T}}\ \mathbf{S}_4^{\mathbf{T}}\ \mathbf{S}_5^{\mathbf{T}}\ 0],\\
&\mathbf{M^T} = [\mathbf{M}_1^{\mathbf{T}}\ \mathbf{M}_2^{\mathbf{T}}\ \mathbf{M}_3^{\mathbf{T}}\ \mathbf{M}_4^{\mathbf{T}}\ \mathbf{M}_5^{\mathbf{T}}\ \mathbf{0}],\\
&\Pi_{11} = \sum_{i=1}^{3}\mathbf{Q}_i + \mathbf{N}_1 + \mathbf{N}_1^{\mathbf{T}} + \mathbf{T}\overline{\mathbf{A}} + \overline{\mathbf{A}}^{\mathbf{T}}\mathbf{T},\\
&\Pi_{12} = \mathbf{N}_2^{\mathbf{T}} - \mathbf{N}_1 + \mathbf{S}_1 - \mathbf{M}_1 + \mathbf{T}\overline{\mathbf{B}}\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H}),\\
&\Pi_{13} = \mathbf{M}_1 + \mathbf{N}_3^{\mathbf{T}}, \Pi_{14} = -\mathbf{S}_1 + \mathbf{N}_4^{\mathbf{T}},\\
&\Pi_{15} = \mathbf{N}_5^{\mathbf{T}} - \mathbf{T} + \mathbf{P} + \overline{\mathbf{A}}^{\mathbf{T}}\mathbf{T}, \Pi_{16} = \mathbf{T}\mathbf{B}_w,\\
&\Pi_{22} = (\mu-1)\mathbf{Q}_3 + \mathbf{S}_2 + \mathbf{S}_2^{\mathbf{T}} - \mathbf{N}_2 - \mathbf{N}_2^{\mathbf{T}} - \mathbf{M}_2 - \mathbf{M}_2^{\mathbf{T}},\\
&\Pi_{23} = \mathbf{M}_2 - \mathbf{N}_3^{\mathbf{T}} + \mathbf{S}_3^{\mathbf{T}} - \mathbf{M}_3^{\mathbf{T}}, \Pi_{24} = -\mathbf{S}_2 - \mathbf{N}_4^{\mathbf{T}} + \mathbf{S}_4^{\mathbf{T}} - \mathbf{M}_4^{\mathbf{T}},\\
&\Pi_{25} = \mathbf{S}_5^{\mathbf{T}} - \mathbf{N}_5^{\mathbf{T}} - \mathbf{M}_5^{\mathbf{T}} + \mathbf{W}^{\mathbf{T}}(\mathbf{s},\mathbf{K},\mathbf{H})\overline{\mathbf{B}}^{\mathbf{T}}\mathbf{T},\\
&\Pi_{33} = -\mathbf{Q}_1 + \mathbf{M}_3 + \mathbf{M}_3^{\mathbf{T}}, \quad \Pi_{34} = -\mathbf{S}_3 + \mathbf{M}_4^{\mathbf{T}},
\end{aligned} \tag{16}$$

$$\Pi_{35} = \mathbf{M}_5^{\mathbf{T}}, \Pi_{44} = -\mathbf{Q}_2 - \mathbf{S}_4 - \mathbf{S}_4^{\mathbf{T}}, \quad \Pi_{45} = -\mathbf{S}_5^{\mathbf{T}},$$

$$\Pi_{55} = d_2\mathbf{Z}_1 + d_{12}\mathbf{Z}_2 - 2\mathbf{T}, \quad \Pi_{56} = \mathbf{T}\mathbf{B}_w, \quad d_{12} = d_2 - d_1.$$

Proof. In the first place, we define a Lyapunov–Krasovskii functional candidate for system (11) as

$$V(t) = \mathbf{x}^{\mathbf{T}}(t)\mathbf{P}\mathbf{x}(t) + \int_{t-d_1}^{t} \mathbf{x}^{\mathbf{T}}(s)\mathbf{Q}_1\mathbf{x}(s)\,\mathrm{d}s + \int_{t-d_2}^{t} \mathbf{x}^{\mathbf{T}}(s)\mathbf{Q}_2\mathbf{x}(s)\,\mathrm{d}s + \int_{t-d(t)}^{t} \mathbf{x}^{\mathbf{T}}(s)\mathbf{Q}_3\mathbf{x}(s)\,\mathrm{d}s + \int_{-d_2}^{0}\int_{t+\theta}^{t} \dot{\mathbf{x}}^{\mathbf{T}}(s)\mathbf{Z}_1\dot{\mathbf{x}}(s)\,\mathrm{d}s\,\mathrm{d}\theta + \int_{-d_2}^{-d_1}\int_{t+\theta}^{t} \dot{\mathbf{x}}^{\mathbf{T}}(s)\mathbf{Z}_2\dot{\mathbf{x}}(s)\,\mathrm{d}s\,\mathrm{d}\theta, \tag{17}$$

where $\mathbf{P}>0$, $\mathbf{Q}_i>0$, $i=1,2,3$, $\mathbf{Z}_i>0$, $i=1,2$ are matrices to be determined.

Then, the derivative of $V(t)$ along the solution of system (11) is given by

$$\begin{aligned}\dot{V}(t) &= \dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{P}\mathbf{x}(t) + \mathbf{x}^{\mathbf{T}}(t)\mathbf{P}\dot{\mathbf{x}}(t) + \mathbf{x}^{\mathbf{T}}(t)\mathbf{Q}_1\mathbf{x}(t) - \mathbf{x}^{\mathbf{T}}(t-d_1)\mathbf{Q}_1\mathbf{x}(t-d_1) - \mathbf{x}^{\mathbf{T}}(t-d_2)\mathbf{Q}_2\mathbf{x}(t-d_2) + \mathbf{x}^{\mathbf{T}}(t)\mathbf{Q}_3\mathbf{x}(t)\\ &\quad - \int_{t-d_2}^{t-d_1} \dot{\mathbf{x}}^{\mathbf{T}}(s)\mathbf{Z}_2\dot{\mathbf{x}}(s)\,\mathrm{d}s + d_2\dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{Z}_1\dot{\mathbf{x}}(t) - (1-\dot{d}(t))\mathbf{x}^{\mathbf{T}}(t-d(t))\mathbf{Q}_3\mathbf{x}(t-d(t)) + \mathbf{x}^{\mathbf{T}}(t)\mathbf{Q}_2\mathbf{x}(t)\\ &\quad - \int_{t-d_2}^{t} \dot{\mathbf{x}}^{\mathbf{T}}(s)\mathbf{Z}_1\dot{\mathbf{x}}(s)\,\mathrm{d}s + d_{12}\dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{Z}_2\dot{\mathbf{x}}(t).\end{aligned} \tag{18}$$

Then, for any appropriately dimensioned matrices $\mathbf{T}>0$ and $\mathbf{N}_i$, $\mathbf{S}_i$, $\mathbf{M}_i$, $i=1,\ldots,5$ we have

$$2\Omega_1\left[\mathbf{x}(t) - \mathbf{x}(t-d(t)) - \int_{t-d(t)}^{t} \dot{\mathbf{x}}(s)\,\mathrm{d}s\right] = 0,$$

$$2\Omega_2\left[\mathbf{x}(t-d(t)) - \mathbf{x}(t-d_2) - \int_{t-d_2}^{t-d(t)} \dot{\mathbf{x}}(s)\,\mathrm{d}s\right] = 0,$$

$$2\Omega_3\left[\mathbf{x}(t-d_1) - \mathbf{x}(t-d(t)) - \int_{t-d(t)}^{t-d_1} \dot{\mathbf{x}}(s)\,\mathrm{d}s\right] = 0, \tag{19}$$

$$2[\mathbf{x}^{\mathbf{T}}(t)\mathbf{T} + \dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{T}][-\dot{\mathbf{x}}(t) + \overline{\mathbf{A}}\mathbf{x}(t) + \overline{\mathbf{B}}\sigma(\mathbf{K}\mathbf{x}(t-d(t))) + \mathbf{B}_w w] = 0, \tag{20}$$

where

$$\Omega_1 = \mathbf{x}^{\mathbf{T}}(t)\mathbf{N}_1 + \mathbf{x}^{\mathbf{T}}(t-d(t))\mathbf{N}_2 + \mathbf{x}^{\mathbf{T}}(t-d_1)\mathbf{N}_3 + \mathbf{x}^{\mathbf{T}}(t-d_2)\mathbf{N}_4 + \dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{N}_5,$$

$$\Omega_2 = \mathbf{x}^{\mathbf{T}}(t)\mathbf{S}_1 + \mathbf{x}^{\mathbf{T}}(t-d(t))\mathbf{S}_2 + \mathbf{x}^{\mathbf{T}}(t-d_1)\mathbf{S}_3 + \mathbf{x}^{\mathbf{T}}(t-d_2)\mathbf{S}_4 + \dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{S}_5,$$

$$\Omega_3 = \mathbf{x}^{\mathbf{T}}(t)\mathbf{M}_1 + \mathbf{x}^{\mathbf{T}}(t-d(t))\mathbf{M}_2 + \mathbf{x}^{\mathbf{T}}(t-d_1)\mathbf{M}_3 + \mathbf{x}^{\mathbf{T}}(t-d_2)\mathbf{M}_4 + \dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{M}_5.$$

Noticing that the following equations hold

$$2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{B}}\sigma(\mathbf{K}\mathbf{x}(t-d(t))) = 2\sum_{i=1}^{q}\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))),$$

$$2\dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{B}}\sigma(\mathbf{K}\mathbf{x}(t-d(t))) = 2\sum_{i=1}^{q}\dot{\mathbf{x}}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))).$$

Then, according to (9), for each term $2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t)))$,

1. If $\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i \ge 0$ and $\mathbf{k}_i\mathbf{x}(t-d(t)) \le -u_{i\max}$, then for $-u_{i\max} \le \mathbf{h}_i\mathbf{x}(t-d(t))$ we have
$$2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) = -2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i u_{i\max} \le 2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\mathbf{h}_i\mathbf{x}(t-d(t)).$$

2. If $\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i \ge 0$ and $\mathbf{k}_i\mathbf{x}(t-d(t)) \ge -u_{i\max}$, then $\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) \le \mathbf{k}_i\mathbf{x}(t-d(t))$ and
$$2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) \le 2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\mathbf{k}_i\mathbf{x}(t-d(t)).$$

3. If $\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i \le 0$ and $\mathbf{k}_i\mathbf{x}(t-d(t)) \ge u_{i\max}$, then for $u_{i\max} \ge \mathbf{h}_i\mathbf{x}(t-d(t))$ we have
$$2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) = 2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i u_{i\max} \le 2\mathbf{x}^{\mathbf{T}}(t)\mathbf{T}\overline{\mathbf{b}}_i\mathbf{h}_i\mathbf{x}(t-d(t)).$$

4. If $\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i \leq 0$ and $\mathbf{k}_i\mathbf{x}(t-d(t)) \leq u_{i\,\max}$, then $\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) \geq \mathbf{k}_i\mathbf{x}(t-d(t))$ and

$$2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) \leq 2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i\mathbf{k}_i\mathbf{x}(t-d(t)).$$

By combining all the above four cases, we have

$$2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) \leq \max\{2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i\mathbf{h}_i\mathbf{x}(t-d(t)),\ 2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i\mathbf{k}_i\mathbf{x}(t-d(t))\}$$

for any $\mathbf{x} \in \xi(\mathbf{P},1)$ and each $i \in [1,q]$.

Now if $2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i\sigma(\mathbf{k}_i\mathbf{x}(t-d(t))) < 2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{b}}_i\mathbf{h}_i\mathbf{x}(t-d(t))$, we set $\mathbf{v}_i = 1$, otherwise we set $\mathbf{v}_i = 0$. Then it is obvious that

$$2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{B}}\sigma(\mathbf{K}\mathbf{x}(t-d(t))) \leq 2\mathbf{x}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{B}}\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H})\mathbf{x}(t-d(t)),$$

where $\mathbf{v}(\mathbf{x}) \in \psi(\mathbf{v})$. Similarly, it also follows that $2\dot{\mathbf{x}}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{B}}\sigma(\mathbf{K}\mathbf{x}(t-d(t))) \leq 2\dot{\mathbf{x}}^\mathbf{T}(t)\mathbf{T}\overline{\mathbf{B}}\mathbf{W}(\mathbf{s},\mathbf{K},\mathbf{H})\mathbf{x}(t-d(t))$ with $\mathbf{s}(\mathbf{x}) \in \psi(\mathbf{s})$.

Hence, we can see from (20) that for every $\mathbf{x} \in \xi(\mathbf{P},1)$ it holds that

$$\begin{aligned} 0 &= 2[\mathbf{x}^\mathbf{T}(t)\mathbf{T} + \dot{\mathbf{x}}^\mathbf{T}(t)\mathbf{T}][-\dot{\mathbf{x}}(t) + \overline{\mathbf{A}}\mathbf{x}(t) + \overline{\mathbf{B}}\sigma(\mathbf{K}\mathbf{x}(t-d(t))) + \mathbf{B}_w w(t)] \leq 2\mathbf{x}^\mathbf{T}(t)\mathbf{T}[-\dot{\mathbf{x}}(t) + \overline{\mathbf{A}}\mathbf{x}(t) + \overline{\mathbf{B}}\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H})\mathbf{x}(t-d(t)) \\ &\quad + \mathbf{B}_w w(t)] + 2\dot{\mathbf{x}}^\mathbf{T}(t)\mathbf{T}[-\dot{\mathbf{x}}(t) + \overline{\mathbf{A}}x(t) + \overline{\mathbf{B}}\mathbf{W}(\mathbf{s},\mathbf{K},\mathbf{H})\mathbf{x}(t-d(t)) + \mathbf{B}_w w(t)]. \end{aligned} \tag{21}$$

After adding Eqs. (19) and (21) to Eq. (18) and some algebraic manipulations it yields

$$\begin{aligned} \dot{V}(t) &\leq \zeta^\mathbf{T}(t)[\Pi + d_2\mathbf{N}\mathbf{Z}_1^{-1}\mathbf{N}^\mathbf{T} + d_{12}\mathbf{S}(\mathbf{Z}_1+\mathbf{Z}_2)^{-1}\mathbf{S}^\mathbf{T} + d_{12}\mathbf{M}\mathbf{Z}_2^{-1}\mathbf{M}^\mathbf{T}]\zeta(t) \\ &\quad - \int_{t-d(t)}^{t} [\zeta^\mathbf{T}(t)\mathbf{N} + \dot{\mathbf{x}}(s)\mathbf{Z}_1]\mathbf{Z}_1^{-1}[\zeta^\mathbf{T}(t)\mathbf{N} + \dot{\mathbf{x}}(s)\mathbf{Z}_1]^\mathbf{T}\,\mathrm{d}s \\ &\quad - \int_{t-d_2}^{t-d(t)} [\zeta^\mathbf{T}(t)\mathbf{S} + \dot{\mathbf{x}}^\mathbf{T}(s)(\mathbf{Z}_1+\mathbf{Z}_2)](\mathbf{Z}_1 \\ &\quad + \mathbf{Z}_2)^{-1}[\zeta^\mathbf{T}(t)\mathbf{S} + \dot{\mathbf{x}}^\mathbf{T}(s)(\mathbf{Z}_1+\mathbf{Z}_2)]^\mathbf{T}\,\mathrm{d}s - \int_{t-d(t)}^{t-d_1} [\zeta^\mathbf{T}(t)\mathbf{M} + \dot{\mathbf{x}}(s)\mathbf{Z}_2]\mathbf{Z}_2^{-1}[\zeta^\mathbf{T}(t)\mathbf{M} + \dot{\mathbf{x}}(s)\mathbf{Z}_2]^\mathbf{T}\,\mathrm{d}s \\ &\leq \zeta^\mathbf{T}(\mathbf{t})[\Pi + d_2\mathbf{N}\mathbf{Z}_1^{-1}\mathbf{N}^\mathbf{T} + d_{12}\mathbf{S}(\mathbf{Z}_1+\mathbf{Z}_2)^{-1}\mathbf{S}^\mathbf{T} + d_{12}\mathbf{M}\mathbf{Z}_2^{-1}\mathbf{M}^\mathbf{T}]\zeta(t), \end{aligned} \tag{22}$$

where

$$\zeta(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-d(t)) \\ \mathbf{x}(t-d_1) \\ \mathbf{x}(t-d_2) \\ \dot{\mathbf{x}}(t) \\ w(t) \end{bmatrix}, \quad \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & 0 \\ * & * & \Pi_{33} & \Pi_{34} & \Pi_{35} & 0 \\ * & * & * & \Pi_{44} & \Pi_{45} & 0 \\ * & * & * & * & \Pi_{55} & \Pi_{56} \\ * & * & * & * & * & 0 \end{bmatrix}.$$

By Shur complement, $\Pi + d_2\mathbf{N}\mathbf{Z}_1^{-1}\mathbf{N}^\mathbf{T} + d_{12}\mathbf{S}(\mathbf{Z}_1+\mathbf{Z}_2)^{-1}\mathbf{S}^\mathbf{T} + d_{12}\mathbf{M}\mathbf{Z}_2^{-1}\mathbf{M}^\mathbf{T} < 0$ is equivalent to

$$\begin{bmatrix} \Pi & d_2\mathbf{N} & d_{12}\mathbf{S} & d_{12}\mathbf{M} \\ * & -d_2\mathbf{Z}_1 & 0 & 0 \\ * & * & -d_{12}(\mathbf{Z}_1+\mathbf{Z}_2) & 0 \\ * & * & * & -d_{12}\mathbf{Z}_2 \end{bmatrix} < 0.$$

Next, we establish the asymptotic stability of the system in (11) with $w(t) = 0$, that is,

$$\dot{\mathbf{x}}(t) = \overline{\mathbf{A}}x(t) + \overline{\mathbf{B}}\sigma(u(t-d(t))).$$

For the above system, $\dot{V}(t)$ in (22) reduces to

$$\dot{V}(t) \leq \overline{\zeta}^\mathbf{T}(t)\tilde{\Pi}\overline{\zeta}(t),$$

where $\overline{\zeta}^\mathbf{T}(t) = [\mathbf{x}^\mathbf{T}(t)\ \mathbf{x}^\mathbf{T}(t-d(t))\ \mathbf{x}^\mathbf{T}(t-d_1)\ \mathbf{x}^\mathbf{T}(t-d_2)\ \dot{\mathbf{x}}^\mathbf{T}(t)]$ and

$$\tilde{\Pi} = \begin{bmatrix} \Pi_\mathbf{s} & d_2\mathbf{N}_\mathbf{s} & d_{12}\mathbf{S}_\mathbf{s} & d_{12}\mathbf{M}_\mathbf{s} \\ * & -d_2\mathbf{Z}_1 & 0 & 0 \\ * & * & -d_{12}(\mathbf{Z}_1+\mathbf{Z}_2) & 0 \\ * & * & * & -d_{12}\mathbf{Z}_2 \end{bmatrix},$$

$$\Pi_s = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} \\ * & * & \Pi_{33} & \Pi_{34} & \Pi_{35} \\ * & * & * & \Pi_{44} & \Pi_{45} \\ * & * & * & * & \Pi_{55} \end{bmatrix}, \quad \mathbf{N}_s = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \mathbf{N}_3 \\ \mathbf{N}_4 \\ \mathbf{N}_5 \end{bmatrix}, \quad \mathbf{S}_s = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \mathbf{S}_3 \\ \mathbf{S}_4 \\ \mathbf{S}_5 \end{bmatrix}, \quad \mathbf{M}_s = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \\ \mathbf{M}_3 \\ \mathbf{M}_4 \\ \mathbf{M}_5 \end{bmatrix}.$$

It is obvious that (14) guarantees $\tilde{\Pi} < 0$, which further leads to $\dot{V}(t) < 0$ for any $\overline{\zeta}(t) \neq 0$. Therefore, we conclude that system (11) with $w(t) = 0$, parameter uncertainty (8), actuator saturation and time delay $d(t)$ satisfying $0 \leq d_1 \leq d(t) \leq d_2$ is robust asymptotically stable.

Now, we shall establish the H_∞ performance of the system under zero initial condition. Consider the following index:

$$J \triangleq \int_0^\infty [\mathbf{z}_{o1}^{\mathbf{T}}(t)\mathbf{z}_{o1}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t)]\,\mathrm{d}t. \tag{23}$$

Then we have

$$J \leq \int_0^\infty [\mathbf{z}_{o1}^{\mathbf{T}}(t)\mathbf{z}_{o1}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(t)]\,\mathrm{d}t \tag{24}$$

for any nonzero $w(t) \in L_2[0,\infty)$.

Via some algebraic manipulations and Schur complement, it is not difficult to obtain

$$\mathbf{z}_{o1}^{\mathbf{T}}(t)\mathbf{z}_{o1}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(t) \leq \zeta^{\mathbf{T}}(t)\Pi_{\mathbf{H}}\zeta(t), \tag{25}$$

where

$$\Pi_{\mathbf{H}} = \begin{bmatrix} \hat{\Pi} & d_2\mathbf{N} & d_{12}\mathbf{S} & d_{12}\mathbf{M} \\ * & -d_2\mathbf{Z}_1 & 0 & 0 \\ * & * & -d_{12}(\mathbf{Z}_1 + \mathbf{Z}_2) & 0 \\ * & * & * & -d_{12}\mathbf{Z}_2 \end{bmatrix},$$

which is the same as (14) in Theorem 1.

Therefore, if (14) holds, i.e. $\Pi_{\mathbf{H}} < 0$, we have $\mathbf{z}_{o1}^{\mathbf{T}}(t)\mathbf{z}_{o1}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(t) < 0$, which indicates $J < 0$. Hence $\|\mathbf{z}_{o1}\|_2 < \gamma\|w(t)\|_2$ is guaranteed for any nonzero $w(t) \in L_2[0,\infty)$, and the H_∞ performance is established.

Finally, the hard constraint of suspension deflection needs to be guaranteed. From above it is ensured that $\dot{V}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) < 0$, and by integrating both sides of which we obtain

$$V(t) - V(0) < \gamma^2 \int_0^t w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t < \gamma^2\|w\|_2^2,$$

and without loss of generality we have $\|w\|_2^2 \leq w_{\max} < \infty$. Thus, it holds that

$$\mathbf{x}^{\mathbf{T}}(t)\mathbf{P}\mathbf{x}(t) < \gamma^2 w_{\max} + V(0) = \rho.$$

Moreover, it is also true that

$$\max_{t>0}|\mathbf{z}_{o2}(t)|^2 = \max_{t>0}\|\mathbf{x}^{\mathbf{T}}(t)\mathbf{C}_2^{\mathbf{T}}\mathbf{C}_2\mathbf{x}(t)\|_2 = \max_{t>0}\|\mathbf{x}^{\mathbf{T}}(t)\mathbf{P}^{1/2}\mathbf{P}^{-1/2}\mathbf{C}_2^{\mathbf{T}}\mathbf{C}_2\mathbf{P}^{-1/2}\mathbf{P}^{1/2}\mathbf{x}(t)\|_2 < \rho \cdot \lambda_{\max}(\mathbf{P}^{-1/2}\mathbf{C}_2^{\mathbf{T}}\mathbf{C}_2\mathbf{P}^{-1/2}),$$

where $\lambda_{\max}(\cdot)$ represents the maximal eigenvalue of a matrix. From above, it is easy to see that the deflection constraint (10) is guaranteed if $\rho \cdot \mathbf{P}^{-1/2}\mathbf{C}_2^{\mathbf{T}}\mathbf{C}_2\mathbf{P}^{-1/2} < z_{\max}^2\mathbf{I}$, which is equivalent to (15) according to Schur complement,the proof is completed. □

Remark 2. It is worth mentioning that the system (11) may be used to represent many important physical systems subject to inherent time-varying input delays, parameter uncertainties, exogenous disturbances and actuator saturations besides the seat suspension system. Thus, this controller designing approach has big application potentials and can be generally used.

Before proceeding further, we give the following lemma that will be used in the proof of Theorem 3.

Lemma 2. *Given appropriately dimensioned matrices* Σ_1, Σ_2, Σ_3, *with* $\Sigma_1 = \Sigma_1^{\mathbf{T}}$, *then*

$$\Sigma_1 + \Sigma_3\mathbf{W}(t)\Sigma_2 + \Sigma_2^{\mathbf{T}}\mathbf{W}^{\mathbf{T}}(t)\Sigma_3^{\mathbf{T}} < 0$$

holds for all $\mathbf{W}(t)$ *satisfying* $\mathbf{W}^{\mathbf{T}}(t)\mathbf{W}(t) \leq \mathbf{I}$ *if and only if there exists a scalar* $\varepsilon > 0$ *such that*

$$\Sigma_1 + \varepsilon\Sigma_3\Sigma_3^{\mathbf{T}} + \varepsilon^{-1}\Sigma_2^{\mathbf{T}}\Sigma_2 < 0.$$

Theorem 3. *Suppose* ρ, d_1, d_2 *and* μ *are prescribed positive scalars. Consider the semi-active suspension system in* (11), *if there exist matrices* $\overline{\mathbf{P}}>0$, $\overline{\mathbf{T}}>0$, $\overline{\mathbf{Q}}_i>0$, $i=1,2,3$, $\overline{\mathbf{Z}}_i>0$, $i=1,2$, $\overline{\mathbf{N}}_i$, $\overline{\mathbf{S}}_i$, $\overline{\mathbf{M}}_i$, $i=1,\ldots,5$, $\overline{\mathbf{W}}_\mathbf{v}$, $\overline{\mathbf{W}}_\mathbf{s}$ *and scalar* $\varepsilon>0$ *satisfying*

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\mathbf{T} & \Phi_{22} \end{bmatrix} < 0, \tag{26}$$

$$\begin{bmatrix} -\mathbf{I} & \sqrt{\rho}\mathbf{C}_2\overline{\mathbf{T}} \\ * & -z_{\max}^2\overline{\mathbf{P}} \end{bmatrix} < 0, \tag{27}$$

where

$$\Phi_{11} = \begin{bmatrix} \overline{\Pi}_{11}+\varepsilon\mathbf{L}_1\mathbf{L}_1^\mathbf{T} & \overline{\Pi}_{12} & \overline{\Pi}_{13} & \overline{\Pi}_{14} & \overline{\Pi}_{15} & \overline{\Pi}_{16} \\ * & \overline{\Pi}_{22} & \overline{\Pi}_{23} & \overline{\Pi}_{24} & \overline{\Pi}_{25} & \mathbf{0} \\ * & * & \overline{\Pi}_{33} & \overline{\Pi}_{34} & \overline{\Pi}_{35} & \mathbf{0} \\ * & * & * & \overline{\Pi}_{44} & \overline{\Pi}_{45} & \mathbf{0} \\ * & * & * & * & \overline{\Pi}_{55}+\varepsilon\mathbf{L}_1\mathbf{L}_1^\mathbf{T} & \overline{\Pi}_{56} \\ * & * & * & * & * & -\gamma^2 \end{bmatrix},$$

$$\Phi_{12} = \begin{bmatrix} d_2\overline{\mathbf{N}}_1 & d_{12}\overline{\mathbf{S}}_1 & d_{12}\overline{\mathbf{M}}_1 & \overline{\mathbf{T}}\mathbf{C}_1^\mathbf{T} & \overline{\mathbf{T}}\mathbf{E}_\mathbf{A}^\mathbf{T} & \overline{\mathbf{T}}\mathbf{E}_\mathbf{A}^\mathbf{T} & \overline{\mathbf{T}}\mathbf{E}_\mathbf{C}^\mathbf{T} \\ d_2\overline{\mathbf{N}}_2 & d_{12}\overline{\mathbf{S}}_2 & d_{12}\overline{\mathbf{M}}_2 & 0 & \overline{\mathbf{W}}_\mathbf{v}^\mathbf{T}\mathbf{E}_\mathbf{B}^\mathbf{T} & \overline{\mathbf{W}}_s^\mathbf{T}\mathbf{E}_\mathbf{B}^\mathbf{T} & 0 \\ d_2\overline{\mathbf{N}}_3 & d_{12}\overline{\mathbf{S}}_3 & d_{12}\overline{\mathbf{M}}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ d_2\overline{\mathbf{N}}_4 & d_{12}\overline{\mathbf{S}}_4 & d_{12}\overline{\mathbf{M}}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ d_2\overline{\mathbf{N}}_5 & d_{12}\overline{\mathbf{S}}_5 & d_{12}\overline{\mathbf{M}}_5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\Phi_{22} = \mathrm{diag}\{-d_2\mathbf{Z}_1, -d_{12}(\mathbf{Z}_1+\mathbf{Z}_2), -d_{12}\mathbf{Z}_2, -\mathbf{I}+\varepsilon\mathbf{L}_2\mathbf{L}_2^\mathbf{T}, -\varepsilon\mathbf{I}\},$$

and

$$\overline{\Pi}_{11} = \sum_{i=1}^{3}\overline{\mathbf{Q}}_i + \overline{\mathbf{N}}_1 + \overline{\mathbf{N}}_1^\mathbf{T} + \mathbf{A}\overline{\mathbf{T}} + \overline{\mathbf{T}}\mathbf{A}^\mathbf{T},$$

$$\overline{\Pi}_{12} = \overline{\mathbf{N}}_2^\mathbf{T} - \overline{\mathbf{N}}_1 + \overline{\mathbf{S}}_1 - \overline{\mathbf{M}}_1 + \mathbf{B}\overline{\mathbf{W}}_\mathbf{v}, \quad \overline{\Pi}_{13} = \overline{\mathbf{M}}_1 + \overline{\mathbf{N}}_3^\mathbf{T},$$

$$\overline{\Pi}_{14} = -\overline{\mathbf{S}}_1 + \overline{\mathbf{N}}_4^\mathbf{T}, \quad \overline{\Pi}_{15} = \overline{\mathbf{N}}_5^\mathbf{T} - \overline{\mathbf{T}} + \overline{\mathbf{P}} + \overline{\mathbf{T}}\mathbf{A}^\mathbf{T}, \quad \overline{\Pi}_{16} = \mathbf{B}_w,$$

$$\overline{\Pi}_{22} = (\mu-1)\overline{\mathbf{Q}}_3 + \overline{\mathbf{S}}_2 + \overline{\mathbf{S}}_2^\mathbf{T} - \overline{\mathbf{N}}_2 - \overline{\mathbf{N}}_2^\mathbf{T} - \overline{\mathbf{M}}_2 - \overline{\mathbf{M}}_2^\mathbf{T},$$

$$\overline{\Pi}_{23} = \overline{\mathbf{M}}_2 - \overline{\mathbf{N}}_3^\mathbf{T} + \overline{\mathbf{S}}_3^\mathbf{T} - \overline{\mathbf{M}}_3^\mathbf{T}, \overline{\Pi}_{24} = -\overline{\mathbf{S}}_2 - \overline{\mathbf{N}}_4^\mathbf{T} + \overline{\mathbf{S}}_4^\mathbf{T} - \overline{\mathbf{M}}_4^\mathbf{T},$$

$$\overline{\Pi}_{25} = \overline{\mathbf{S}}_5^\mathbf{T} - \overline{\mathbf{N}}_5^\mathbf{T} - \overline{\mathbf{M}}_5^\mathbf{T} + \overline{\mathbf{W}}_\mathbf{s}\mathbf{B}^\mathbf{T}\mathbf{T}, \quad \overline{\Pi}_{33} = -\overline{\mathbf{Q}}_1 + \overline{\mathbf{M}}_3 + \overline{\mathbf{M}}_3^\mathbf{T},$$

$$\overline{\Pi}_{34} = -\overline{\mathbf{S}}_3 + \overline{\mathbf{M}}_4^\mathbf{T}, \overline{\Pi}_{35} = \overline{\mathbf{M}}_5^\mathbf{T}, \overline{\Pi}_{44} = -\overline{\mathbf{Q}}_2 - \overline{\mathbf{S}}_4 - \overline{\mathbf{S}}_4^\mathbf{T},$$

$$\overline{\Pi}_{45} = -\overline{\mathbf{S}}_5^\mathbf{T}, \quad \overline{\Pi}_{55} = d_2\overline{\mathbf{Z}}_1 + d_{12}\overline{\mathbf{Z}}_2 - 2\overline{\mathbf{T}}, \quad \overline{\Pi}_{56} = \mathbf{B}_w,$$

then a stabilizing controller in the form of (12) *exists, such that the closed-loop system satisfies*:

(1) *asymptotically stable*;
(2) *the* H_∞ *performance* $\|T_{z_{01}w}\|_\infty < \gamma$ *is guaranteed subject to the constraint of suspension deflection for all nonzero* $w \in L_2[0,\infty)$ *under zero initial condition.*

Moreover, if inequalities (26) *and* (27) *have a feasible solution, the control gain* $\mathbf{K}$ *in* (12) *is given by*

$$\mathbf{K} = \overline{\mathbf{K}\mathbf{T}}^{-1}. \tag{28}$$

Proof. First, from Shur complement (26) is equivalent to

$$\Sigma_1 + \varepsilon\Sigma_3\Sigma_3^\mathbf{T} + \varepsilon^{-1}\Sigma_2^\mathbf{T}\Sigma_2 < 0, \tag{29}$$

where

$$\Sigma_1 = \begin{bmatrix} \overline{\Pi}_{11} + \overline{\mathbf{T}}\mathbf{C}_1^{\mathbf{T}}\mathbf{C}_1\overline{\mathbf{T}} & \overline{\Pi}_{12} & \overline{\Pi}_{13} & \overline{\Pi}_{14} & \overline{\Pi}_{15} & \overline{\Pi}_{16} \\ * & \overline{\Pi}_{22} & \overline{\Pi}_{23} & \overline{\Pi}_{24} & \overline{\Pi}_{25} & \mathbf{0} \\ * & * & \overline{\Pi}_{33} & \overline{\Pi}_{34} & \overline{\Pi}_{35} & \mathbf{0} \\ * & * & * & \overline{\Pi}_{44} & \overline{\Pi}_{45} & \mathbf{0} \\ * & * & * & * & \overline{\Pi}_{55} & \overline{\Pi}_{56} \\ * & * & * & * & * & -\gamma^2 \end{bmatrix},$$

$$\Sigma_3^{\mathbf{T}} = \begin{bmatrix} \mathbf{L}_1^{\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_1^{\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_2^{\mathbf{T}} \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} \mathbf{E_A}\overline{\mathbf{T}} & \mathbf{E_B}\overline{\mathbf{W}}_{\mathbf{v}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{E_A}\overline{\mathbf{T}} & \mathbf{E_B}\overline{\mathbf{W}}_{\mathbf{s}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{E_C}\overline{\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

By invoking Lemma 2, (29) holds if

$$\Sigma_1 + \Sigma_3\overline{\mathbf{W}}(t)\Sigma_2 + \Sigma_2^{\mathbf{T}}\overline{\mathbf{W}}^{\mathbf{T}}(t)\Sigma_3^{\mathbf{T}} < 0, \tag{30}$$

where

$$\overline{\mathbf{W}}(t) = \mathrm{diag}\{\mathbf{F}(t), \mathbf{F}(t), \mathbf{F}(t)\}.$$

From the norm-bounded parameter uncertainty defined in (8) we note that (30) is equivalent to (14) in Theorem 1 by defining

$$\overline{\mathbf{T}} = \mathbf{T}^{-1}, \quad \overline{\mathbf{N}}_i = \mathbf{T}^{-1}\mathbf{N}_i\mathbf{T}^{-1}, \quad \overline{\mathbf{S}}_i = \mathbf{T}^{-1}\mathbf{S}_i\mathbf{T}^{-1},$$

$$\overline{\mathbf{P}} = \mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-1}, \quad \overline{\mathbf{M}}_i = \mathbf{T}^{-1}\mathbf{M}_i\mathbf{T}^{-1}, \quad \overline{\mathbf{Z}}_i = \mathbf{T}^{-1}\mathbf{Z}_i\mathbf{T}^{-1},$$

$$\overline{\mathbf{Q}}_i = \mathbf{T}^{-1}\mathbf{Q}_i\mathbf{T}^{-1}, \quad \overline{\mathbf{W}}_{\mathbf{v}} = \overline{\mathbf{W}}(\mathbf{v},\mathbf{K},\mathbf{H})\mathbf{T}^{-1}, \quad \overline{\mathbf{W}}_s = \overline{\mathbf{W}}(\mathbf{s},\mathbf{K},\mathbf{H})\mathbf{T}^{-1},$$

$$\mathbf{J} = \mathrm{diag}\{\mathbf{T}^{-1},\mathbf{T}^{-1},\mathbf{T}^{-1},\mathbf{T}^{-1},\mathbf{T}^{-1},\mathbf{I},\mathbf{T}^{-1},\mathbf{T}^{-1},\mathbf{T}^{-1}\}, \tag{31}$$

and performing a congruence transformation to (14) with $\mathbf{J}$. Similarly, it also follows that (27) is equivalent to (15) in Theorem 1. Hence, the closed-loop system is asymptotically stable with an H_∞ disturbance attenuation level of γ if (26) and (27) holds and the proof is completed. □

If we assume that there is no input delay in the semi-active suspension system in (11), then we have

Corollary 4. *Suppose ρ is a prescribed positive scalar. Consider system* (11) *with the input-delayed state-feedback controller in* (12), *the closed-loop system is asymptotically stable and satisfies $\|\mathbf{z}_{o1}\|_2 < \gamma\|w\|_2$ under zero initial condition if there exist matrices $\overline{\mathbf{P}} > 0$ and $\overline{\mathbf{W}}_{\mathbf{v}}$ satisfying*

$$\begin{bmatrix} \mathrm{sym}(\mathbf{A}\overline{\mathbf{P}} + \mathbf{B}\overline{\mathbf{W}}_{\mathbf{v}}) & \mathbf{B}_w & \overline{\mathbf{P}}\mathbf{C}_1^{\mathbf{T}} \\ * & -\gamma^2 & \mathbf{0} \\ * & * & -\mathbf{I} \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\mathbf{I} & \sqrt{\rho}\mathbf{C}_2\overline{\mathbf{P}} \\ * & -z_{\max}^2\overline{\mathbf{P}} \end{bmatrix} < 0, \tag{32}$$

then a stabilizing controller in the form of (12) *exists, such that the closed-loop system satisfies*:

(1) *asymptotically stable*;
(2) *the H_∞ performance $\|T_{z_{o1}w}\|_\infty < \gamma$ is guaranteed subject to the constraint of suspension deflection under zero initial condition.*

Moreover, if (32) *has a feasible solution, then the control gain* $\mathbf{K}$ *in* (12) *is given by*

$$\mathbf{K} = \overline{\mathbf{K}}\overline{\mathbf{P}}^{-1}. \tag{33}$$

Furthermore, if we assume that the input delay is constant with an upper bound $\overline{\tau}$, then we have

Corollary 5. *Suppose $\overline{\tau}>0$ and $\rho>0$ are proscribed scalars. Consider system (11) with the input-delayed state-feedback controller in (12), the closed-loop system is asymptotically stable and satisfies $\|\mathbf{z}_{o1}\|_2<\gamma\|w\|_2$ under zero initial condition if there exist matrices $\overline{\mathbf{P}}>0$, $\overline{\mathbf{Q}}>0$, $\overline{\mathbf{S}}$, and $\overline{\mathbf{W}}_\mathbf{v}$ satisfying*

$$\begin{bmatrix} \overline{\Psi}_1+\overline{\Psi}_2+\overline{\Psi}_2^\mathbf{T}+\Psi_3 & \sqrt{\overline{\tau}}\overline{\Phi}_1^\mathbf{T} & \sqrt{\overline{\tau}}\overline{\mathbf{S}} & \overline{\Phi}_2^\mathbf{T} \\ * & \overline{\mathbf{Q}}-2\overline{\mathbf{P}} & \mathbf{0} & \mathbf{0} \\ * & * & -\overline{\mathbf{Q}} & \mathbf{0} \\ * & * & * & -\mathbf{I} \end{bmatrix}<0, \tag{34}$$

$$\begin{bmatrix} -\mathbf{I} & \sqrt{\rho}\mathbf{C}_2\overline{\mathbf{P}} \\ * & -\mathbf{z}_{\max}^2\overline{\mathbf{P}} \end{bmatrix}<0, \tag{35}$$

where

$$\overline{\Psi}_1=\begin{bmatrix} \overline{\mathbf{A}\mathbf{P}}+\overline{\mathbf{P}\mathbf{A}}^\mathrm{T} & \overline{\mathbf{B}\mathbf{W}}_\mathbf{v} & \mathbf{B}_w \\ * & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} \end{bmatrix}, \quad \overline{\Psi}_2=[\overline{\mathbf{S}} \quad -\overline{\mathbf{S}} \quad \mathbf{0}],$$

$$\overline{\Phi}_1=[\overline{\mathbf{A}\mathbf{P}} \;\; \overline{\mathbf{B}\mathbf{W}}_\mathbf{v} \;\; \mathbf{B}_w], \quad \overline{\Phi}_2=[\overline{\mathbf{C}}_1\overline{\mathbf{P}} \;\; \mathbf{0} \;\; \mathbf{0}],$$

$$\Psi_3=\mathrm{diag}[\mathbf{0} \;\; \mathbf{0} \;\; -\gamma^2\mathbf{I}],$$

then a stabilizing controller in the form of (12) exists, such that the closed-loop system satisfies:

(1) *asymptotically stable*;
(2) *the H_∞ performance $\|T_{z_{o1}w}\|_\infty<\gamma$ is guaranteed subject to the constraint of suspension deflection under zero initial condition.*

Moreover, if inequalities (34) and (35) have a feasible solution, the control gain $\mathbf{K}$ in (12) is given by

$$\mathbf{K}=\overline{\mathbf{K}\mathbf{P}}^{-1}. \tag{36}$$

Proof. First, we define a Lyapunov–Krasovskii functional candidate for system (11) as

$$V(t)=\mathbf{x}^\mathbf{T}(t)\mathbf{P}\mathbf{x}(t)+\int_{-\overline{\tau}}^{0}\int_{t+\beta}^{t}\mathbf{x}^\mathbf{T}(\alpha)\mathbf{Q}\mathbf{x}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{37}$$

where $P>0$ and $\mathbf{Q}>0$ are matrices to be determined. Then define

$$\overline{\mathbf{P}}=\mathbf{P}^{-1}, \quad \overline{\mathbf{Q}}=\mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1}, \quad \overline{\mathbf{W}}_\mathbf{v}=\mathbf{W}(\mathbf{v},\mathbf{K},\mathbf{H})\mathbf{P}^{-1},$$

and follow the same procedures as Theorems 1 and 3, and the proof is completed. □

Remark 3. The above results may actually be extended to deal with the analysis and synthesis of stochastic systems. Because if we model the road excitation and actuator fault occurrences stochastic, which is quite natural in practice [30]. The deduction methods and procedures are basically similar.

From Theorem 3 we can see that the conditions are LMIs not only over the matrix variables, but also over the objective scalar γ when γ_g is given, which implies that γ can be included as an optimization variable to obtain a lower bound of the guaranteed H_∞ performance. That is, the controller design problem has been transformed into a set of LMI conditions. Based on these conditions, the robust multiobjective state-feedback controller design can be accomplished by solving the following convex optimization problem:

$$\min\gamma \quad \text{s.t. (26) and (27).} \tag{38}$$

Table 1
System parameters of the proposed seat suspension.

Mass (kg)		Damping coefficient (N s/m)		Spring constant (N/m)	
m_1	15	c_1	830	k_1	31 000
m_2	1+7.8	c_2	200	k_2	18 000
m_3	43.4	c_3	1485	k_3	44 130

4. A design example

In order to evaluate the effectiveness and usefulness of the controller design method proposed in the above section, an example is introduced in this section. The schematic and biodynamical parameters for this study are borrowed from [31,32] and listed in Table 1 and the maximum suspension deflection is defined as $z_{max} = 0.08$ m, the maximum control force is assumed as $u_{max} = 1500$ N. Furthermore, assume that the input time delay lower bound $d_1 = 0$ ms, upper bound $d_2 = 25$ ms, the delay variation rate $\mu = 1.5$, and the norm-bounded parameter uncertainties are expressed as

$$\mathbf{L}_1 = \delta_1 \times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{L}_2^{\mathbf{T}} = \delta_2 \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{E_B} = \delta_{\mathbf{B}} \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$\mathbf{E_A} = \delta_{\mathbf{A}}\mathbf{I}$, $\mathbf{E_C} = \delta_{\mathbf{C}}\mathbf{I}$, where δ_1, δ_2, $\delta_{\mathbf{A}}$, $\delta_{\mathbf{B}}$, $\delta_{\mathbf{C}}$ are all set to 0.02 for simplicity.

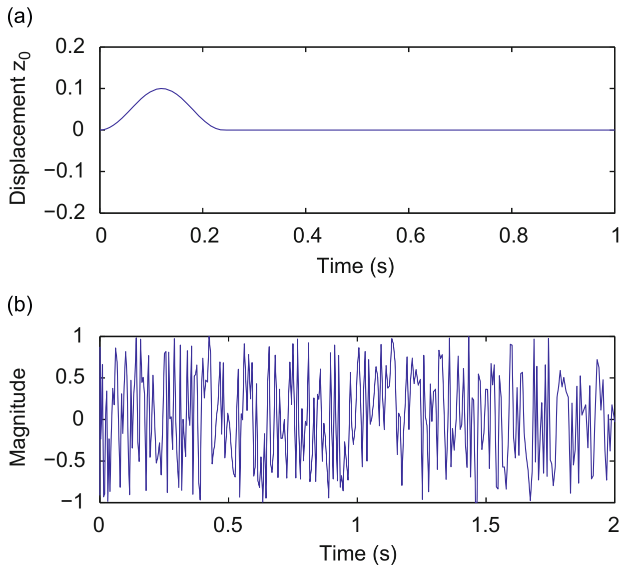

Fig. 2. Bump (a) and white-noise (b) inputs from ground.

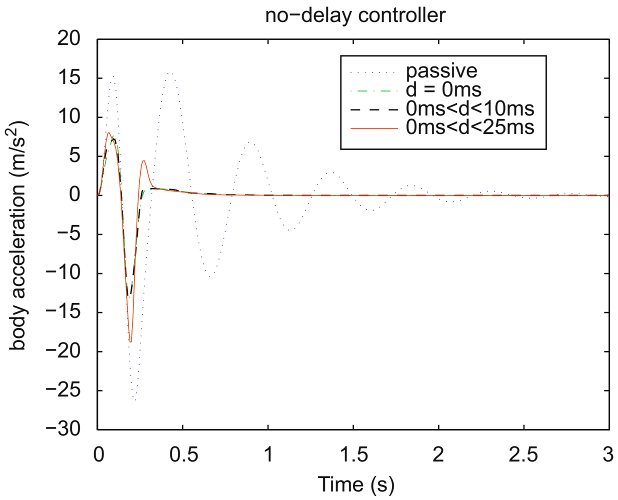

Fig. 3. Vertical accelerations of open-loop system and closed-loop system with controller $\mathbf{K_{nom}}$ under bump excitation.

First, we give a controller $\mathbf{K_{nom}}$ with a guaranteed H_∞ performance $\gamma = 8.8867$, which is designed by employing the approach in Corollary 4 without considering the time delay in the actuator to see its performance for no input delay ($d = 0$ ms), minor input delay ($0\,\text{ms} < d < 10\,\text{ms}$) and biggish input delay ($0\,\text{ms} < d < 25\,\text{ms}$) cases, which are denoted as case 1, 2 and 3. And the controller is given by

$$\mathbf{K_{nom}} = 10^4 \times [-0.2632 \quad 0.0599 \quad -2.9348 \quad 0.1195 \quad -2.2877 \quad 0.3544].$$

In the context of seat suspension performance, road disturbances can be generally assumed as shocks. Shocks are discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an otherwise smooth road surface. In this work, this case of road profile is considered first to reveal the transient response characteristic, which is given by

$$z_0(t) = \begin{cases} \dfrac{a}{2}\left(1 - \cos\left(\dfrac{2\pi V_0}{l}t\right)\right), & 0 \le t \le \dfrac{l}{V_0}, \\ 0, & t > \dfrac{l}{V_0}, \end{cases} \tag{39}$$

and illustrated in Fig. 2(a), where a is the height of the bump, and l is the length of the bump. Here we choose $a = 0.1$ m, $l = 2$ m and the vehicle forward velocity $V_0 = 30\,(\text{km/h})$. The second type of disturbance input from the ground $w(t)$ is assumed to be zero-mean white noise with identity power spectral density, which is shown in Fig. 2(b) (Fig. 3).

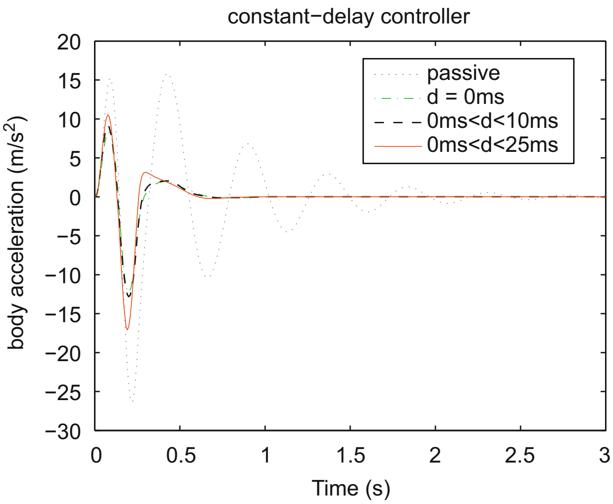

Fig. 4. Vertical accelerations of open-loop system and closed-loop system with controller $\mathbf{K_{con}}$ under bump excitation.

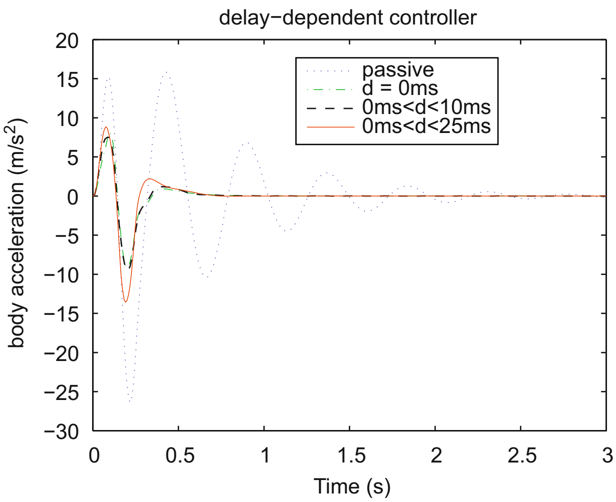

Fig. 5. Vertical accelerations of open-loop system and closed-loop system with controller $\mathbf{K_{drd}}$ under bump excitation.

The bump responses of the passive suspension and the active suspension with controller $\boldsymbol{K}_{\mathbf{nom}}$ for cases 1, 2 and 3 are compared in Fig. 4. It demonstrates that the closed-loop system is asymptotically stable and has a better performance with or without actuator time delays. However, the closed-loop performance degrades significantly when the actuator delay bound gets larger.

Next, we give another controller $\boldsymbol{K}_{\mathbf{con}}$ with a guaranteed H_∞ performance $\gamma = 14.0432$, which is designed with consideration for constant input time delay using the approach in Corollary 5 with $\overline{\tau} = 25\,\text{ms}$ to see its performance for the three cases, which is given by

$$\mathbf{K_{con}} = 10^4 \times [-1.3144 \;\; 0.0266 \;\; -1.1652 \;\; 0.0625 \;\; -2.1263 \;\; 0.0614].$$

In Fig. 5, the bump responses of the passive suspension and the active suspension with controller $\mathbf{K_{con}}$ for cases 1, 2 and 3 are illustrated. From Fig. 5 we can see that the closed-loop performance level with controller $\mathbf{K_{con}}$ is a bit more stable than controller $\mathbf{K_{nom}}$ when the actuator delay happens, which is still not a satisfaction though.

Finally, we give the controller $\mathbf{K_{drd}}$ with a guaranteed H_∞ performance $\gamma = 16.4012$, which is designed with delay-range-dependent method proposed in this paper by solving the convex optimization problem (38) in the MATLAB environment, and the associated matrices are as follows (for brevity, here we only list the matrices that are necessary for the construction of the desired controller):

$$\mathbf{K_{drd}} = [-27.6826 \quad 859.9575 \quad 12.2584 \quad 109.4245 \quad 3.0885 \quad 78.1620],$$

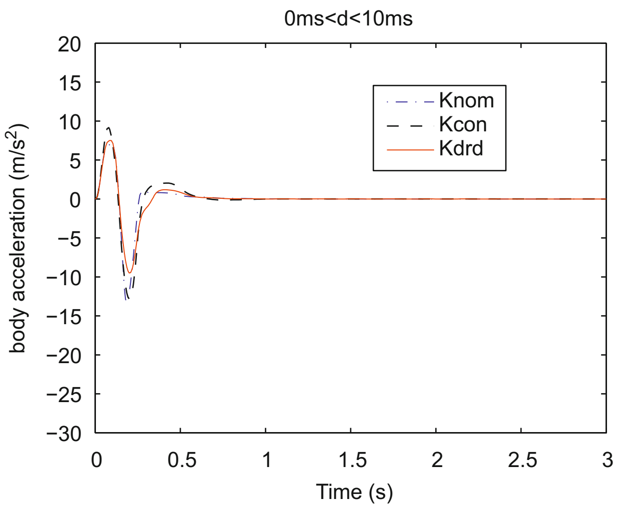

Fig. 6. Vertical accelerations of closed-loop systems with different controllers under bump excitation when $0\,\text{ms} < d < 10\,\text{ms}$.

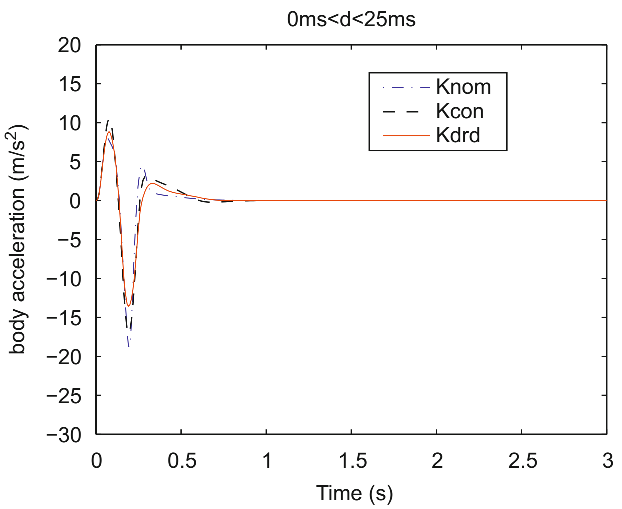

Fig. 7. Vertical accelerations of closed-loop systems with different controllers under bump excitation when $0\,\text{ms} < d < 25\,\text{ms}$.

$$\overline{\mathbf{T}} = \begin{bmatrix} 0.0009 & -0.0191 & -0.0003 & -0.0078 & -0.0002 & -0.0024 \\ -0.0191 & 1.1318 & 0.0136 & 0.2072 & 0.0075 & 0.0011 \\ -0.0003 & 0.0136 & 0.0003 & -0.0025 & -0.0001 & 0.0022 \\ -0.0078 & 0.2072 & -0.0025 & 0.4762 & 0.0126 & -0.0770 \\ -0.0002 & 0.0075 & -0.0001 & 0.0126 & 0.0004 & -0.0025 \\ -0.0024 & 0.0011 & 0.0022 & -0.0770 & -0.0025 & 0.0537 \end{bmatrix},$$

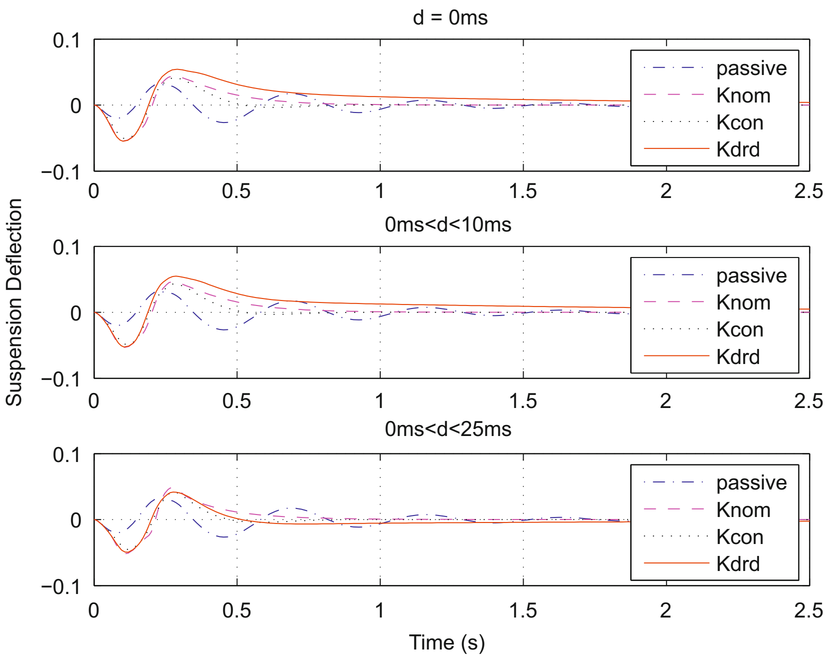

Fig. 8. Suspension deflections of closed-loop systems with different controllers.

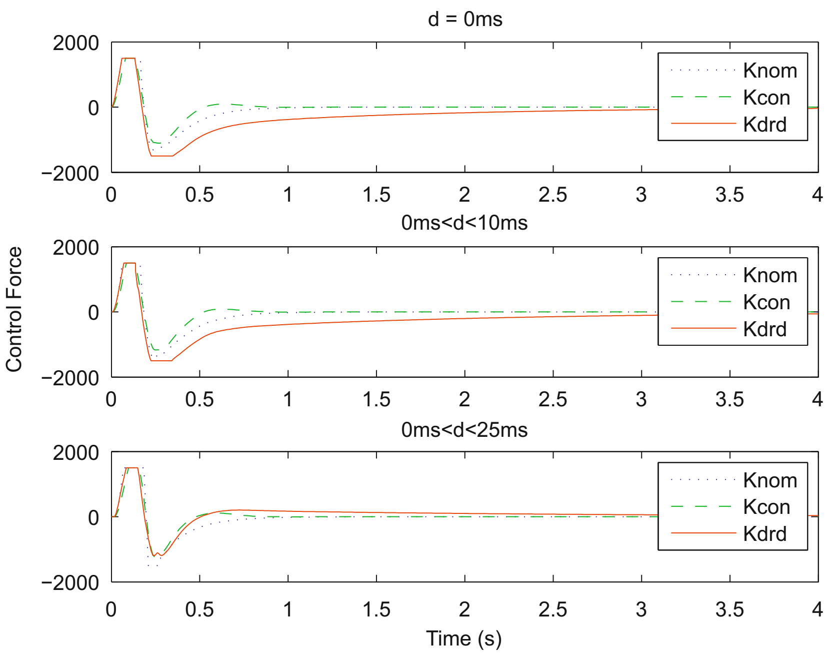

Fig. 9. Control forces of closed-loop systems with different controllers.

and therefore we can obtain the delay-range-dependent state-feedback controller $\mathbf{K_{drd}}$ as

$$\mathbf{K_{drd}} = 10^4 \times [-3.0358 \;\; 0.0629 \;\; -1.7496 \;\; -0.0103 \;\; -2.2889 \;\; -0.0425].$$

The bump responses of controller $\mathbf{K_{drd}}$ for the three cases are shown in Fig. 5. It is obvious that this delay-range-dependent state-feedback controller is capable to provide the best time domain performance among the three controllers under time-varying delay condition. To make this point clear, the performances of the three closed-loop systems with controllers $\mathbf{K_{nom}}$, $\mathbf{K_{con}}$ and $\mathbf{K_{drd}}$ are compared in Figs. 6 and 7 for cases 2 and 3, respectively.

The bump response of suspension deflection is plotted in Fig. 8, from which it can be seen that this time domain constraint is guaranteed to be less than its prescribed limit in spite of the large bump energy by all of the three designed state-feedback controllers.

Fig. 9 depicts the active control forces of the closed-loop systems, which are confined within a reasonable range which can be generated by hydraulic or electrorheological actuators in practice. It is confirmed that the designed robust active seat suspension system is able to guarantee a better performance under a pronounced bump disturbance and limited actuator control force in spite of the parameter uncertainty and time-varying input delay.

At last, vertical human body accelerations of open- and closed-loop systems under white-noise disturbance are illustrated in Figs. 10–12, and compared in Figs. 13 and 14, from which it can be seen that the closed-loop system performance with controller $\mathbf{K_{nom}}$ degrades the fastest among the three controllers, while controller $\mathbf{K_{drd}}$ maintains the most satisfying performance when the actuator delay bound is large. Here, the root mean square (RMS) value, which is a statistical measure of the magnitude of a varying quantity, is employed to investigate the seat suspension performances.

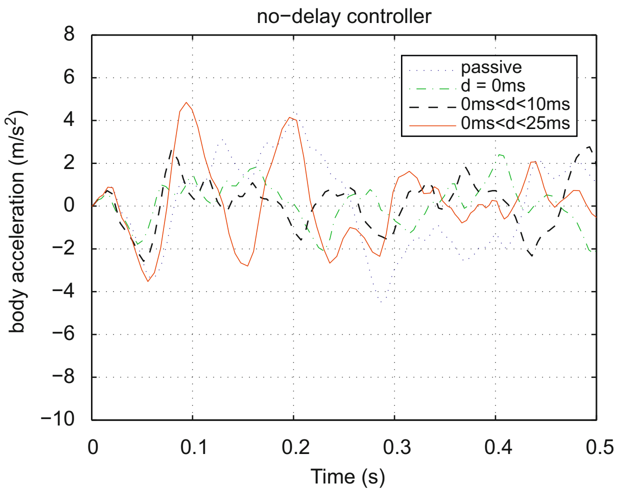

Fig. 10. Vertical accelerations of open-loop system and closed-loop system with controller $\mathbf{K_{nom}}$ under white noise disturbance for cases 1, 2 and 3.

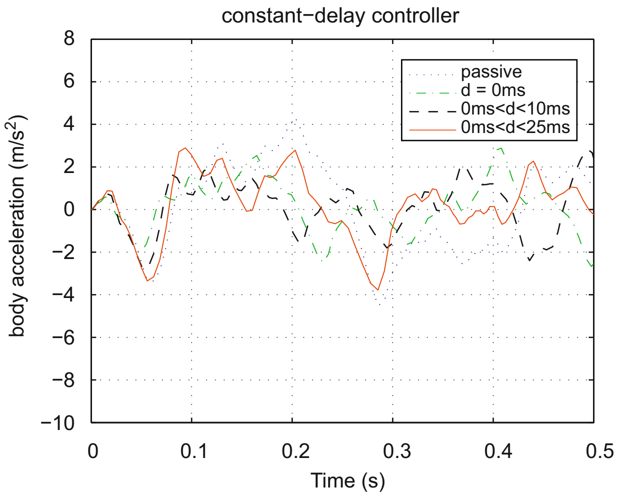

Fig. 11. Vertical accelerations of open-loop system and closed-loop system with controller $\mathbf{K_{con}}$ under white noise disturbance for cases 1, 2 and 3.

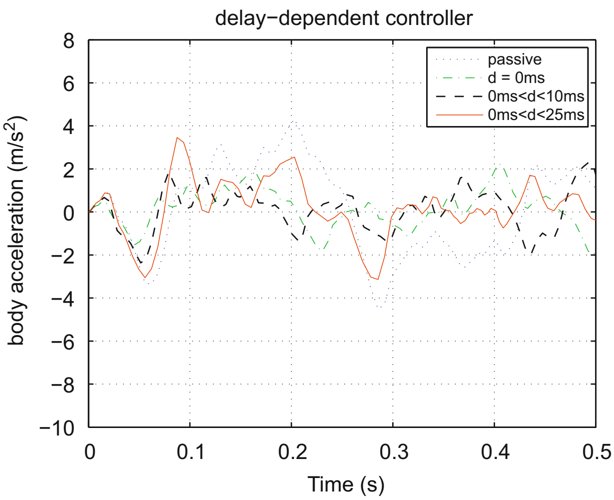

Fig. 12. Vertical accelerations of open-loop system and closed-loop system with controller $\mathbf{K}_{\mathbf{drd}}$ under white noise disturbance for cases 1, 2 and 3.

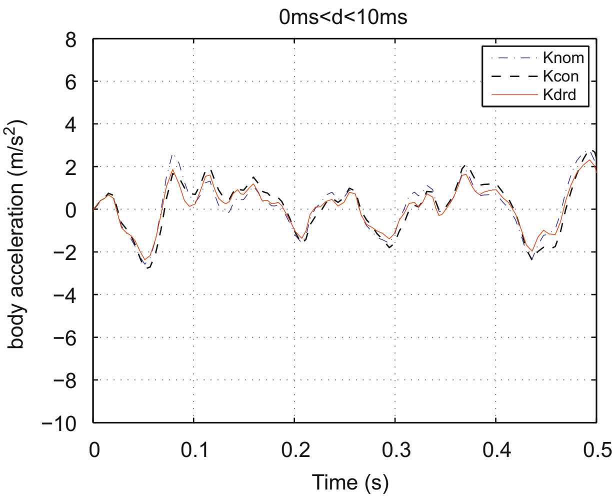

Fig. 13. Vertical accelerations of closed-loop systems with different controllers under white noise disturbance when $0\,\text{ms} < d < 10\,\text{ms}$.

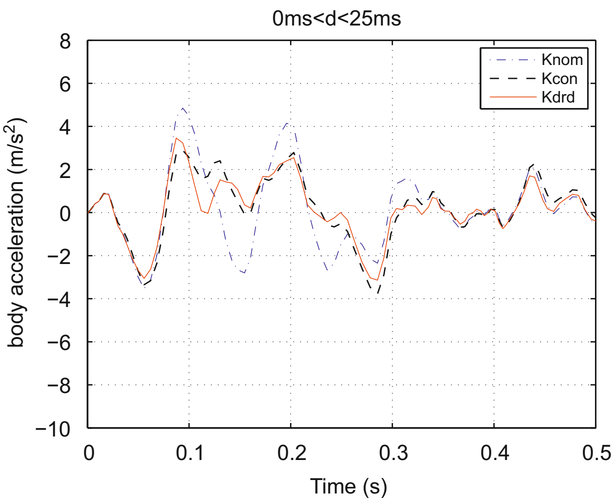

Fig. 14. Vertical accelerations of closed-loop systems with different controllers under white noise disturbance when $0\,\text{ms} < d < 25\,\text{ms}$.

Table 2
Comparison of RMS values of vertical human body acceleration (m/s^2) for passive system and closed-loop systems with different controllers.

	$d=0$ ms	$0\,\text{ms}<d<10\,\text{ms}$	$0\,\text{ms}<d<25\,\text{ms}$
Passive	3.6826	3.6826	3.6826
$\mathbf{K_{nom}}$	1.1803	1.2904	2.1289
$\mathbf{K_{con}}$	1.5797	1.6309	1.9773
$\mathbf{K_{drd}}$	1.1952	1.2691	1.5806

Because it is especially useful when variants are positive and negative. It can be calculated for a series of discrete values or for a continuously varying function. The name comes from the fact that it is the square root of the mean of the squares of the values. It is a special case of the power mean with the exponent is 2. The corresponding RMS values of vertical human body accelerations of open- and closed-loop systems for the three cases are listed and compared in Table 2, which indicates that controller $\mathbf{K_{drd}}$ has the best vibration attenuation ability than the other two controllers especially when time-varying delay happens.

5. Conclusions

The problem of robust multiobjective H_∞ control synthesis for a class of uncertain seat suspension systems with actuator saturation and time-varying input delay has been dealt with in this paper by proposing a state-feedback controller. A delay-range-dependent Lyapunov function and an auxiliary feedback matrix have been considered to reduce the conservatism so that a better closed-loop performance can be guaranteed. The H_∞ performance has been established via a Lyapunov approach and the controller design has been cast into a convex optimization problem with LMI constraints via some algebraic manipulations. Then the desired controller can be achieved by solving the corresponding LMIs. Finally, a design example has been given to demonstrate the effectiveness and advantages of the proposed controller design approach.

References

[1] H. Gao, J. Lam, C. Wang, Multi-objective control of vehicle active suspension systems via load-dependent controllers, *Journal of Sound and Vibration* 290 (3–5) (2006) 654–675.
[2] R. Guclu, K. Gulez, Neural network control of seat vibrations of a non-linear full vehicle model using PMSM, *Mathematical and Computer Modelling* 47 (11–12) (2008) 1356–1371.
[3] S. Choi, Y. Han, MR seat suspension for vibration control of a commercial vehicle, *International Journal of Vehicle Design* 31 (2) (2003) 202–215.
[4] S. Rakheja, Y. Afework, S. Sankar, An analytical and experimental investigation of the driver-seat-suspension system, *Vehicle System Dynamics* 23 (1) (1994) 501–524.
[5] D. Wilder, M. Magnusson, J. Fenwick, M. Pope, The effect of posture and seat suspension design on discomfort and back muscle fatigue during simulated truck driving, *Applied Ergonomics* 25 (2) (1994) 66–76.
[6] Z. Ying, Y. Ni, J. Ko, Semi-active optimal control of linearized systems with multi-degree of freedom and application, *Journal of Sound and Vibration* 279 (1–2) (2005) 373–388.
[7] Ş. Yildirim, Vibration control of suspension systems using a proposed neural network, *Journal of Sound and Vibration* 277 (4–5) (2004) 1059–1069.
[8] R. Guclu, Fuzzy logic control of seat vibrations of a non-linear full vehicle model, *Nonlinear Dynamics* 40 (1) (2005) 21–34.
[9] H. Du, J. Lam, K. Sze, Non-fragile output feedback H_∞ vehicle suspension control using genetic algorithm, *Engineering Applications of Artificial Intelligence* 16 (7–8) (2003) 667–680.
[10] H. Du, K. Yim Sze, J. Lam, Semi-active H_∞ control of vehicle suspension with magneto-rheological dampers, *Journal of Sound and Vibration* 283 (3–5) (2005) 981–996.
[11] I. Fialho, G. Balas, Road adaptive active suspension design using linear parameter-varying gain-scheduling, *IEEE Transactions on Control Systems Technology* 10 (1) (2002) 43–54.
[12] H. Gao, C. Wang, Delay-dependent robust H_∞ and L_2-L_∞ filtering for a class of uncertain nonlinear time-delay systems, *IEEE Transactions on Automatic Control* 48 (9) (2003) 1661–1666.
[13] Y. He, M. Wu, J. She, G. Liu, Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays, *Systems & Control Letters* 51 (1) (2004) 57–65.
[14] P. Park, A delay-dependent stability criterion for systems with uncertain time-invariant delays, *IEEE Transactions on Automatic Control* 44 (4) (1999) 876–877.
[15] P. Shi, E. Boukas, R. Agarwal, Control of Markovian jump discrete-time systems with norm bounded uncertainty and unknown delay, *IEEE Transactions on Automatic Control* 44 (11) (1999) 2139–2144.
[16] Z. Wang, Y. Liu, X. Liu, On global asymptotic stability of neural networks with discrete and distributed delays, *Physics Letters A* 345 (4–6) (2005) 299–308.
[17] Z. Wang, F. Yang, D. Ho, X. Liu, Robust H_∞ filtering for stochastic time-delay systems with missing measurements, *IEEE Transactions on Signal Processing* 54 (7) (2006) 2579–2587.
[18] L. Wu, P. Shi, C. Wang, H. Gao, Delay-dependent robust H_∞ and L_2-L_∞ filtering for LPV systems with both discrete and distributed delays, *IEE Proceedings Control Theory and Applications* 153 (4) (2006) 483–492.
[19] Y. He, M. Wu, J. She, G. Liu, Parameter-dependent Lyapunov functional for stability of time-delay systems with polytopic-type uncertainties, *IEEE Transactions on Automatic Control* 49 (5) (2004) 828–832.
[20] Y. He, Q. Wang, C. Lin, M. Wu, Delay-range-dependent stability for systems with time-varying delay, *Automatica* 43 (2) (2007) 371–376.
[21] Y. Cao, Z. Lin, D. Ward, An anti-windup approach to enlarging domain of attraction for linear systems subject to actuator saturation, *IEEE Transactions on Automatic Control* 47 (1) (2002) 140–145.
[22] T. Hu, Z. Lin, Exact characterization of invariant ellipsoids for single inputlinear systems subject to actuator saturation, *IEEE Transactions on Automatic Control* 47 (1) (2002) 164–169.

[23] L. Zhang, E. Boukas, A. Haidar, Delay-range-dependent control synthesis for time-delay systems with actuator saturation, *Automatica* 44 (10) (2008) 2691–2695.
[24] Y. Cao, Y. Sun, J. Lam, Delay-dependent robust H_∞ control for uncertain systems with time-varying delays, *IEE Proceedings—Control Theory and Applications* 145 (3) (1998) 338–344.
[25] Z. Wang, B. Huang, H. Unbehauen, Robust H_∞ observer design of linear state delayed systems with parametric uncertainty: the discrete-time case, *Automatica* 35 (6) (1999) 1161–1167.
[26] Z. Wang, H. Zeng, D. Ho, H. Unbehauen, Multiobjective control of a four-link flexible manipulator: a robust H_∞ approach, *IEEE Transactions on Control Systems Technology* 10 (6) (2002) 866–875.
[27] S. Xu, P. Van Dooren, R. Stefan, J. Lam, Robust stability and stabilization for singular systems with state delay and parameter uncertainty, *IEEE Transactions on Automatic Control* 47 (7) (2002) 1122–1128.
[28] T. Hu, Z. Lin, B. Chen, An analysis and design method for linear systems subject to actuator saturation and disturbance, *Automatica* 38 (2) (2002) 351–359.
[29] L. Wei, J. Griffin, The prediction of seat transmissibility from measures of seat impedance, *Journal of Sound and Vibration* 214 (1) (1998) 121–137.
[30] Z. Wang, B. Huang, K. Burnham, Stochastic reliable control of a class of uncertain time-delay systems with unknown nonlinearities, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* 48 (5) (2001) 646–650.
[31] S. Choi, Y. Han, Vibration control of electrorheological seat suspension with human-body model using sliding mode control, *Journal of Sound and Vibration* 303 (1–2) (2007) 391–404.
[32] C. Liang, C. Chiang, A study on biodynamic models of seated human subjects exposed to vertical vibration, *International Journal of Industrial Ergonomics* 36 (10) (2006) 869–890.