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Asymptotic stabilization of a nonlinear axially moving string by adaptive boundary control

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ABSTRACT

In this paper, a robust adaptive boundary control for an axially moving string that shows nonlinear behavior resulting from spatially varying tension is investigated. A hydraulic actuator equipped with a damper is used as the control actuator at the right boundary of the string. The Lyapunov redesign method is employed to derive a robust control algorithm employing adaptation laws that estimate three unknown system parameters (mass per unit length of string, lumped mass of hydraulic actuator, and damping coefficient of damper) and an unknown boundary disturbance. The uniform asymptotic stability (when the three parameters are all unknown), the exponential stability (when they are known), and the uniform ultimate boundedness (with a bounded boundary disturbance) of the closed loop system are investigated. The convergence of the parameter estimates to the true values is shown. Numerical simulations are performed to demonstrate the effectiveness of the proposed robust adaptive boundary control.

1. Introduction

In mechanical engineering, many operations of components such as power-transmission belts, chain drives, high-speed magnetic tapes, plastic-films, and paper-sheets under processing as well as steel-strips are modeled as axially moving systems. For these systems, the mechanical vibrations (particularly in the transverse direction) of the moving part become the main quality- and productivity-limiting factor, especially for high-speed precision machine systems. Therefore, reduction of transverse vibrations in axially moving systems has become an important research area. In efforts to solve the vibration problem, many researchers have investigated the application of control actions at a system's left or right boundary (a measure known as boundary control), the provision of control inputs through a supporting roller being more cost-effective than the addition of an extra actuator in the middle of a system.

In recent years, there have been many papers published on dynamic analysis [1–6] and control [7–32] of axially moving systems as well as flexible structure systems. Notably, a number of creative control techniques by which axially moving systems are stabilized at their boundaries have been addressed [8–32]. Such approaches primarily pursue point(s) control (not distributed control). Control laws to reduce the total mechanical energy to zero can be derived using a Lyapunov function candidate, which is equivalent to the total mechanical energy of a moving system. These control laws use

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measured signals of the transverse displacement and the time-rate of the slope of the moving material at the boundary, obtainable by the addition of laser sensors at the boundary point. Therefore, in so far as actuators and sensors can be easily assembled at the boundary, the boundary control method can provide a practical control solution for axially moving systems. Boundary control can be implemented in one of two ways: active control [8–29] or, with the proper damping mechanism, passive control [30–32] (or semi-active control). Alternatively, distributed control [7] might yield better control performance, though it is difficult to implement the algorithm because distributed control forces and feedback signals are required. Essentially, the boundary control method is more advantageous than the distributed control method.

A number of researchers have investigated adaptive and intelligent boundary controls for axially moving systems as well as flexible structure systems. Queiroz et al. [9] proposed an adaptive control law for an axially moving string system in which two control inputs are located in the middle of the moving string to reduce the vibration of the controlled span. Fung et al. [30,31] developed a boundary control scheme for an axially moving string system in which adaptive boundary control laws are employed on a mass–damper–spring (MDS) mechanism to suppress vibrations and to update online estimation values of unknown parameters. Li and Rahn [10] and Li et al. [13] introduced an adaptive isolation scheme for axially moving systems, which are divided into two spans by a transverse force actuator, to reduce the transverse vibration of the controlled span to zero asymptotically under bounded disturbances in the uncontrolled span. Using an MDS system to provide an actuation force, Chao and Lai [32] presented intelligent control schemes, namely fuzzy sliding-mode control and fuzzy neutral network control, for reduction of transverse vibration. Yang et al. [14] introduced a method of robust adaptive boundary control of an axially moving string under spatiotemporally varying tension and unknown boundary disturbance. They assumed that some varying-tension-related constants, the lower bound, the upper bound and the bounds of the time-derivate, are *a priori* known. Adaptation laws were derived to estimate the values of the mass per unit length of the moving string and unknown disturbance resulting from the uncontrolled span. Uniform stability was proven by application of the semigroup theory. Chen and Zhang [19] designed an adaptive control law for a tensioner, which can be considered as a boundary control actuator, to reduce the transverse vibration of an axially moving string. They assumed that the values of the tension of the axially moving string and the parameters of the tensioner, which are the mass moment of inertia and the rotational stiffness, are unknown. In this case, the unknown system parameters appear only in the ordinary differential equation describing the motion of the tensioner. In this paper, starting from the fact that the performance of adaptive boundary control depends on the estimation of the parameter values of the string and the actuator, we developed a robust adaptive boundary control scheme for an axially moving string that can compensate for the uncertainties of both the string itself and the boundary actuator under spatially varying tension and boundary disturbance.

In this paper, the considered plant (an axially moving string system) for control is coupled partial and ordinary differential equations (PDE and ODE), where the nonlinear PDE represents the string dynamics and the ODE the actuator dynamics including a hydraulic actuator and a damper (Fig. 1). In practice, axially moving systems have varying tensions due to the strain resulting from the displacement of the string, the eccentricity of a support roll, external disturbances, and other factors [12,14,20,22,33]. This again leads to a change in the dynamic response of a moving string. Therefore, the control law has to be designed with consideration for the variations of the tension. The boundary control scheme discussed in this paper can be described as follows. The control law generates a required signal for the hydraulic actuator. The control force supplied by the hydraulic actuator is applied to the string at the right boundary, and the estimation of the unknown parameter values or the adaptation of changing values in the controller as well as in the plant, as well as the unknown bound of the boundary disturbance, are updated online. Collocation of the sensor and actuator at the right boundary is carried out. The axially moving string system under the proposed control law is a closed-loop system in that the control signal uses the transverse displacement and slope information of the string at the right boundary.

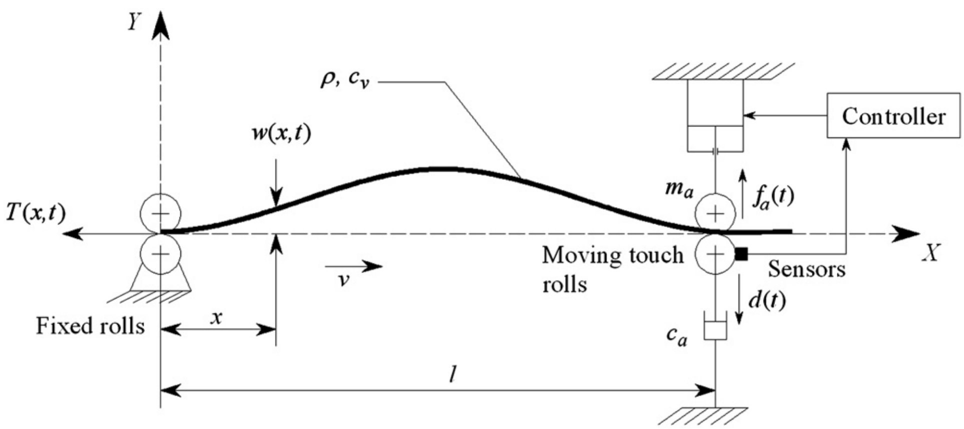

Fig. 1. Schematic of proposed boundary control of an axially moving string.

In this paper, a robust adaptive (and active) control problem for an axially moving string, approached using a boundary control method, was investigated. Nonlinear string and spatially varying tension were considered in the dynamic model. Using the Lyapunov redesign method, a robust adaptive boundary control was derived to regulate the transverse vibration of the axially moving string to zero. Included in the control were adaptation laws for online updating of the unknown parameter values (the mass per unit length of the string, the lumped mass of the hydraulic actuator, and the damping coefficient of the damper) and the unknown bound of the boundary disturbance. Numerical simulations using a finite difference scheme were performed for four cases (i.e., without control; with closed-loop control and exactly known system parameter values; with closed-loop control and estimated system parameter values; and with closed-loop control and estimated system parameter values as well as the estimated bound of the unknown boundary disturbance). Comparing the simulation results for these four cases, we show that the convergence speed of the transverse vibration of the controlled axially moving string is significantly improved. Compared with the work in Ref. [14], the improvements were made as follows. First, the assumption of uncertainties is more general, because along with the boundary disturbance, the three unknown system parameters are revealed (whereas only the mass per unit length of the string and the boundary disturbance were revealed in Ref. [14]). Second, the spatially varying tension is considered explicitly for robust control design. Third, the proposed robust adaptive control law improves the control performance, as shown in the computer simulation results.

2. Problem formulation

Fig. 1 shows a schematic of the axially moving string system with a control mechanism located at the right boundary. The control mechanism includes a hydraulic actuator, a damper, and touch rolls. The left boundary is fixed in the sense that the movement of the string in the vertical direction is restricted. Conversely, at the right boundary, the control mechanism allows for transverse (vertical) movement of the string in accordance with the dynamics of the hydraulic actuator.

Let t be the time, x the spatial coordinate along the longitude of motion, $w(x,t)$ the transverse displacement at the spatial coordinate x and time t, l the distance between two supporting rolls, ρ the mass per unit length of the string, c_v the viscous damping coefficient of the string, and v the traveling speed of the string (assumed constant). Moreover, let m_a be the lumped mass of the hydraulic actuator and c_a be the damping coefficient of the actuator. The variable $T(x,t)$ describes the spatially varying string tension, and $d(t)$ denotes the unknown disturbance force exerted on the actuator due to the transverse vibration of the exterior-span of the string where control is not focused. Finally, the control force $f_a(t)$ supplied from the hydraulic actuator of the control mechanism is applied to the touch rolls to suppress the transverse vibrations.

The governing equation and boundary conditions of the closed-loop system in Fig. 1 are given as [14]

$$\rho w_{tt}(x,t)+2\rho v w_{xt}(x,t)+\rho v^2 w_{xx}(x,t)-(T(x,t)w_x(x,t))_x+c_v(w_t(x,t)+v w_x(x,t))=0, \quad 0\le x\le l, \tag{1}$$

$$w(x,0)=w_0(x), \quad w_t(x,0)=w_{t0}(x), \tag{2}$$

$$w(0,t)=0, \tag{3}$$

and

$$f_a(t)=m_a w_{tt}(l,t)+(c_a-\rho v)w_t(l,t)+(T(l,t)-\rho v^2)w_x(l,t)+d(t), \tag{4}$$

where $(\cdot)_t=\partial(\cdot)/\partial t$, $(\cdot)_{tt}=\partial^2(\cdot)/\partial t^2$, $(\cdot)_{xt}=\partial^2(\cdot)/\partial x\partial t$, $(\cdot)_x=\partial(\cdot)/\partial x$, and $(\cdot)_{xx}=\partial^2(\cdot)/\partial x^2$ denote the partial derivatives with respective to t and x, respectively. Eq. (1) governs the transverse displacement $w(x,t)$ of the axially moving string. The initial conditions are given by Eq. (2), and the boundary conditions are provided by Eqs. (3)–(4). Eq. (4) also describes the dynamics of the hydraulic actuator in compliance with the external force $f_a(t)$, which is specified to dissipate the vibration energy. As shown in Eqs. (1)–(4), the control mechanism attached to the right boundary of the string is coupled to the string system. Therefore, to achieve the stability of the coupled system (1)–(4), the convergence of the motion of the hydraulic actuator to zero should also be satisfied. In this paper, the tension $T(x,t)$ is presented in the formula

$$T(x,t)=T_0+n(x)w_x^2(x,t), \tag{5}$$

where T_0 is the tension of the undisturbed string, and the scalar function $n(x)$ is positive for all $x\in[0,l]$ (see Table 1 and Refs. [12,33]). T_0 is known because the uniformity of the string is assumed, and is sufficiently large. For notational convenience, instead of $w_x(x,t)$ and $w_t(x,t)$, w_x and w_t will be used, with similar abbreviations employed subsequently.

Assumption 1. The function $n(x)$ and its partial derivative $n_x(x)$ are uncertain, but their bounds (constant lower and upper bounds) are known, and for all $x\in[0,l]$,

$$n_{\min}\le n(x)\le n_{\max}, \tag{6}$$

$$n_{x,\min}\le n_x(x)\le n_{x,\max}. \tag{7}$$

In Section 4, $n(x)=EA/2+300\sin(\pi x/l)$ and $T(x,t)=1000+n(x)w_x^2(x,t)$ will be used.

Table 1
System parameter values used in numerical simulation.

Parameter values in plant	Values
Mass per unit length, ρ	2.7 kg/m
Cross-section area, A	1.4×0.002 m^2
Elastic modulus, E	1.3×10^7 N/m^2
Traveling speed, v	2 m/s
Distance between two pulleys, l	20 m
Viscous damping coefficient, c_v	0.001 N m^{2}s
Lump mass of hydraulic actuator, m_a	10 kg
Damping coefficient of actuator, c_a	0.25 N s/m
Spatially varying string tension, $T(x,t)$	$1000+(EA/2+300\sin(\pi x/l))w_x^2(x,t)$N
Boundary disturbance, $d(t)$	$50\sin(20\pi t)$N

Assumption 2. The disturbance force at the right boundary, $d(t)$, is unknown but is bounded by an unknown positive constant μ_d (this unknown bound will be estimated); that is,

$$|d(t)| \le \mu_d. \tag{8}$$

To achieve the control objective, the vibration energy of the axially moving string should be regulated such that it decays to zero. This decay requires the existence of a solution such that $w(x,t)\to 0$ as $t\to\infty$. In the case of varying tension, the transverse displacement of the string, even with viscous damping, might not go to zero without control (see Eq. (15) and the discussions in Ref. [14]). The varying tension due to the eccentricity of a roll or from the span of the string where control is not focused can add energy to the span of the string where control is focused. Therefore, the proposed control method should be able to handle such an effect. The solution to this problem is provided in Eqs. (22)–(28), where the boundary control force $f_a(t)$ includes terms that eliminate the effects of the varying tension.

In implementing the boundary control algorithm, the actuator displacement $w(l,t)$ and the slope of the string $w_x(l,t)$ can be measured by adding laser sensors and an encoder at the actuator [9,10,13]. The actuator velocity $w_t(l,t)$, the actuator acceleration $w_{tt}(l,t)$, and the time-rate of the slope of the string $w_{xt}(l,t)$ can then be obtained through the backward differencing of such signals. In this study, the length of the string l and the traveling speed of the string v were assumed to be known, whereas there is no exact knowledge of the values of the other system parameters including the mass per unit length of the string ρ, the lumped mass of the hydraulic actuator m_a, and the damping coefficient of the damper c_a. Therefore, estimation laws were employed to account for the uncertainties associated with the unknown parameters and the unknown boundary disturbance.

3. Control formulation

The control objective is to stabilize the axially moving string in the presence of the unknown system parameters (i.e., the mass per unit length, the lumped mass of the hydraulic actuator, and the damping coefficient of the damper) and the unknown disturbance force at the right boundary. To achieve robust stability of the closed-loop system, a robust adaptive boundary control with adaptation laws is derived using the Lyapunov redesign method. Let

$$\boldsymbol{\theta} = \begin{bmatrix} \rho & m_a & c_a \end{bmatrix}^{\mathrm{T}} \tag{9}$$

be the unknown system parameter vector, and

$$\hat{\boldsymbol{\theta}}(t) = \begin{bmatrix} \hat{\rho}(t) & \hat{m}_a(t) & \hat{c}_a(t) \end{bmatrix}^{\mathrm{T}} \tag{10}$$

be the estimate vector of the vector $\boldsymbol{\theta}$. The parameter error vector $\boldsymbol{\varphi}(t)$ is defined as

$$\boldsymbol{\varphi}(t) = \hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}. \tag{11}$$

Similarly, for estimation of the bound value of the boundary disturbance, the error value is given as

$$\tilde{\mu}_d(t) = \hat{\mu}_d(t) - \mu_d, \tag{12}$$

where $\hat{\mu}_d(t)$ is the estimated value of μ_d. Based on the total mechanical energy of the axially moving string, a function

$$V(t) = \alpha V_0(t) + 2\int_0^l \rho x \beta(x) w_x (w_t + v w_x)\,dx + \frac{1}{2\lambda_d}\tilde{\mu}_d^2(t) + \frac{1}{2}\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t) \tag{13}$$

is introduced, where α is a positive real constant, $\beta(x)$ is a positive scalar function strictly increasing for all $x\in[0,l]$, λ_d is a positive adaptation gain for disturbance estimation, and $\boldsymbol{\Lambda}$ is a positive definite matrix such as

$$\boldsymbol{\Lambda} = \mathbf{diag}\{\lambda_1 \quad \lambda_2 \quad \lambda_3\}, \tag{14}$$

where λ_i (i=1, 2, 3) are positive adaptation gains. Finally, $V_0(t)$ is defined as

$$V_0(t)=\frac{1}{2}\int_0^l \rho(w_t+vw_x)^2\,dx+\frac{1}{2}\int_0^l T_0 w_x^2\,dx+\frac{1}{4}\int_0^l n(x)w_x^4\,dx+\frac{1}{2}m_a[w_t(l,t)+(v+2\beta(l)l/\alpha)w_x(l,t)]^2. \tag{15}$$

Eq. (13) can be rewritten as

$$V(t)=\alpha\left\{\frac{1}{2}\int_0^l \rho(w_t+vw_x)^2\,dx+\frac{1}{2}\int_0^l T_0 w_x^2\,dx+\frac{1}{4}\int_0^l n(x)w_x^4\,dx+\frac{1}{2}m_a[w_t(l,t)+(v+2\beta(l)l/\alpha)w_x(l,t)]^2\right\}$$
$$+\int_0^l \rho x\beta(x)(w_x+(w_t+vw_x))^2\,dx-\int_0^l \rho x\beta(x)w_x^2\,dx-\int_0^l \rho x\beta(x)(w_t+vw_x)^2\,dx+\frac{1}{2\lambda_d}\tilde{\mu}_d^2(t)+\frac{1}{2}\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t). \tag{16}$$

From Eq. (16), $V(t)$ will be a positive definite function if the conditions

$$2\beta(l)l<\alpha, \tag{17}$$

$$\rho<T_0 \tag{18}$$

are satisfied. Under conditions (17) and (18), $V(t)$ given by Eq. (13) becomes a Lyapunov function candidate. The time derivative of $V(t)$ is derived as

$$\begin{aligned}\dot{V}(t)&=\alpha\dot{V}_0(t)+\frac{d}{dt}\left[2\int_0^l \rho x\beta(x)w_x(w_t+vw_x)\,dx\right]+\tilde{\mu}_d(t)\dot{\tilde{\mu}}_d(t)/\lambda_d+\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\varphi}}(t)\\&=\alpha\dot{V}_0(t)+\rho[x\beta(x)(w_t+vw_x)^2]_0^l-\rho\int_0^l(\beta(x)+x\beta_x(x))(w_t+vw_x)^2\,dx\\&\quad+2\int_0^l x\beta(x)w_x((Tw_x)_x-c_v(w_t+vw_x))\,dx+\tilde{\mu}_d(t)\dot{\tilde{\mu}}_d(t)/\lambda_d+\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\varphi}}(t),\end{aligned} \tag{19}$$

where $\dot{V}_0(t)$ is given as

$$\begin{aligned}\dot{V}_0=&-c_v\int_0^l(w_t+vw_x)^2\,dx+\int_0^l vn(x)w_x^3w_{xx}\,dx+\frac{1}{2}\int_0^l vn_x(x)w_x^4\,dx\\&-vT(0,t)w_x^2(0,t)+vT(l,t)w_x^2(l,t)+T(l,t)w_t(l,t)w_x(l,t)\\&+[f_a(t)-(c_a-\rho v)w_t(l,t)+\rho v^2w_x(l,t)-T(l,t)w_x(l,t)+m_a(v+2\beta(l)l/\alpha)w_{xt}(l,t)]\\&\times[w_t(l,t)+(v+2\beta(l)l/\alpha)w_x(l,t)]-d(t)[w_t(l,t)+(\alpha v+2\beta(l)l/\alpha)w_x(l,t)].\end{aligned} \tag{20}$$

Using the inequality $2ab\le\sigma\ a^2+b^2/\sigma$ for $\forall\sigma>0$, the following inequality is obtained:

$$2\int_0^l xw_x(w_t+vw_x)\,dx\le l\sigma\int_0^l w_x^2\,dx+\frac{l}{\sigma}\int_0^l(w_t+vw_x)^2\,dx. \tag{21}$$

Using Eq. (21), the time derivative of $V(t)$ is then evaluated as follows:

$$\begin{aligned}\dot{V}(t)\le&-\phi(t)+[\alpha v+2\beta(l)l]n(l)w_x^4(l,t)/4+\alpha T(l,t)w_t(l,t)w_x(l,t)\\&+[\alpha v+\beta(l)l]T(l,t)w_x^2(l,t)+\beta(l)\rho l[w_t(l,t)+vw_x(l,t)]^2\\&-[\alpha w_t(l,t)+(\alpha v+2\beta(l)l)w_x(l,t)]T(l,t)w_x(l,t)+[f_a(t)-(c_a-\rho v)w_t(l,t)\\&+\rho v^2w_x(l,t)+m_a(v+2\beta(l)l/\alpha)w_{xt}(l,t)]\times[\alpha w_t(l,t)+(\alpha v+2\beta(l)l)w_x(l,t)]\\&-d(t)[\alpha w_t(l,t)+(\alpha v+2\beta(l)l)w_x(l,t)]+\tilde{\mu}_d(t)\dot{\tilde{\mu}}_d(t)/\lambda_d+\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\varphi}}(t),\end{aligned} \tag{22}$$

where

$$\begin{aligned}\phi(t)=&\int_0^l[\alpha c_v+\rho(\beta(x)+x\beta_x(x))-\beta(x)c_vl/\sigma](w_t+vw_x)^2\,dx\\&+\int_0^l[(\beta(x)+x\beta_x(x))T_0-c_v\sigma l\beta(x)]w_x^2\,dx\\&+\int_0^l[3(\beta(x)+x\beta_x(x))n(x)/2-(x\beta(x)+\alpha v/2)n_x(x)/2]w_x^4\,dx.\end{aligned} \tag{23}$$

To guarantee the uniform stability of the closed-loop system, $\dot{V}(t)\le 0$ should be assured. That is, the control force $f_a(t)$ must be able to eliminate the positive terms in Eq. (22). Therefore, the control law

$$f_a(t)=-\mathbf{B}(t)\hat{\boldsymbol{\theta}}(t)-\mathrm{sgn}(\overline{w}(l,t))\hat{\mu}_d(t)+k_1w_x(l,t)-k_2w_t(l,t)-k_3\overline{w}^3(l,t) \tag{24}$$

is proposed, where $\mathbf{B}(t)$ is a vector including three components such that

$$\mathbf{B}(t)=\left[\begin{matrix}v(w_t(l,t)+vw_x(l,t)) & (v+2\beta(l)l/\alpha)w_{xt}(l,t) & -w_t(l,t)\end{matrix}\right], \tag{25}$$

where k_i (i=1, 2, 3) are the positive control gains, and $\overline{w}(l,t)=\alpha w_t(l,t)+(\alpha v+2\beta(l)l)w_x(l,t)$. The term $-\text{sgn}(\overline{w}(l,t))\hat{\mu}_d(t)$ is introduced to cope with the unknown boundary disturbance. Note that $-d(t)\overline{w}(l,t)\le\mu_d|\overline{w}(l,t)|$. Substituting Eq. (24) into Eq. (22) yields

$$\begin{aligned}\dot{V}(t)\le &-\phi(t)+n(l)[\alpha v-2l\beta(l)]w_x^4(l,t)/4-k_3[\alpha w_t(l,t)+(\alpha v+2\beta(l)l)w_x(l,t)]^4\\ &-[k_2\alpha-\rho l\beta(l)]w_t^2(l,t)-[T_0 l\beta(l)-k_1(\alpha v+2\beta(l)l)-\rho l v^2\beta(l)]w_x^2(l,t)\\ &+[2\beta(l)\rho l v+k_1\alpha-k_2(\alpha v+2\beta(l)l)]w_t(l,t)w_x(l,t)\\ &+\tilde{\mu}_d(t)[\dot{\hat{\mu}}_d(t)-|\overline{w}(l,t)|\lambda_d]/\lambda_d+\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}[\dot{\boldsymbol{\varphi}}(t)-\overline{w}(l,t)\boldsymbol{\Lambda}\mathbf{B}^{\mathrm{T}}(t)].\end{aligned}\tag{26}$$

To make $\dot{V}(t)$ negative semi-definite, adaptation laws for updating $\hat{\boldsymbol{\theta}}(t)$ and $\hat{\mu}_d(t)$ in Eq. (23) online are introduced as follows, respectively:

$$\dot{\hat{\boldsymbol{\theta}}}(t)=-\lambda_r\hat{\boldsymbol{\theta}}(t)+\overline{w}(l,t)\boldsymbol{\Lambda}\mathbf{B}^{\mathrm{T}}(t),\tag{27}$$

$$\dot{\hat{\mu}}_d(t)=-\lambda_r\hat{\mu}_d(t)+\lambda_d|\overline{w}(l,t)|,\tag{28}$$

where the adaptation gain λ_r is positive. In Eqs. (27) and (28), based on the robust control strategy, the terms $-\lambda_r\hat{\boldsymbol{\theta}}(t)$ and $-\lambda_r\hat{\mu}_d(t)$ are inserted to ensure that the estimated values do not become unbounded [35]. The following six variables are introduced as

$$P_1=\frac{2}{\rho}\left[\alpha c_v+\rho(\beta(x)+x\beta_x(x))-\frac{\beta(x)c_v l}{\sigma}\right],\tag{29}$$

$$P_2=\frac{2}{T_0}\left[(\beta(x)+x\beta_x(x))T_0-c_v\sigma l\beta(x)\right],\tag{30}$$

$$P_3=\frac{4}{n(x)}\left[\frac{3}{2}(\beta(x)+x\beta_x(x))n(x)-\frac{1}{2}(x\beta(x)+\alpha v/2)n_x(x)\right],\tag{31}$$

$$P_4=\frac{2\alpha[T_0 l\beta(l)-k_1(\alpha v+2\beta(l)l)-\rho l v^2\beta(l)]}{m_a(\alpha v+2\beta(l)l)^2},\tag{32}$$

$$\eta=\min\{\min_{x\in[0,l]}P_1,\ \min_{x\in[0,l]}P_2,\min_{x\in[0,l]}P_3,\ P_4\},\tag{33}$$

$$\eta_v=\frac{\eta}{\alpha+2\beta(l)l}.\tag{34}$$

Preliminarily to an analysis of the stability of the closed-loop system, four lemmas are established as follows.

Lemma 1. *Consider*

$$V_m(t)=V(t)-\frac{1}{2\lambda_d}\tilde{\mu}_d^2(t)-\frac{1}{2}\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t).\tag{35}$$

The following relationship is then obtained:

$$(\alpha-2\beta(l)l)V_0(t)\le V_m(t)\le(\alpha+2\beta(l)l)V_0(t).\tag{36}$$

Proof. Using the elementary inequality $(a^2+b^2)/2\ge ab$, the proof of this lemma can easily be verified. □

Remark 1. Lemma 1 states that $V_m(t)$ and $V_0(t)$ are equivalent. It should be note that the function $V(t)$ contains two factors: the transverse vibration of the axially moving string and the uncertainties.

Lemma 2. *Given* $u(x,t):[0,l]\times\Re^+\to\Re$, *if* $u(0,t)$=0,

$$u^2(x,t)\le l\int_0^l u_x^2(x,t)\,dx,\tag{37}$$

$$\int_0^l u^2(x,t)\,dx\le l^2\int_0^l u_x^2(x,t)\,dx.\tag{38}$$

Proof. See Appendix A. □

Lemma 3. *Consider the scalar equation*

$$\dot{p}(t)=-\kappa p(t)+q(t),\tag{39}$$

where κ is a positive constant. Let $|q(t)|\le K$. Then, $\forall p_d\in\Re^+$, there exists κ, $q(t)$, and $p(0)$ such that $p(t)\ge p_d$, for all $t\in[0,\infty)$.

Proof. See Appendix B. □

Lemma 4. (*Fung et al.* [31]). *If* $u(x,t): [0,l] \times \Re^+ \to \Re$ *is uniformly bounded,* $\{u(x)\}_{x\in[0,l]}$ *is equi-uniformly continuous in t, and* $\lim_{t\to\infty}\int_0^t \|u(\tau)\|^2 d\tau$ *exists and is finite, then* $\lim_{t\to\infty}\|u(t)\| = 0$.

Property 1. (*Queiroz et al.* [34]). *If the kinetic energy of the axially moving string given as* $E_k(t) = \frac{1}{2}\int_0^l \rho(w_t + v w_x)^2 dx$ *is bounded, then* w_{xt} *is bounded.*

Property 2. (*Queiroz et al.* [34]). *If* $E_p(t) = \frac{1}{2}\int_0^l T_0 w_x^2 dx$ *is bounded, then* w_x *and* w_{xx} *are bounded.*

Theorem 1. *Consider the system* (1) *with the boundary conditions* (3) and (4), *where the system parameters* ρ, m_a, *and* c_a *are unknown and the boundary disturbance d(t) is* bounded. *The control gains* k_i *(i=1, 2, 3) in Eq.* (24) *and the adaptation gain* λ_r *in Eqs.* (27) and (28) *are selected to satisfy the following conditions*:

$$k_1 < \frac{T_0\beta(l)l - \beta(l)l\rho v^2}{\alpha v + 2\beta(l)l}, \tag{40}$$

$$k_2 = \frac{\eta m_a}{\alpha} + \frac{k_1\alpha + 2\beta(l)lv}{v\alpha + 2\beta(l)l}, \tag{41}$$

$$k_3 > \frac{n(l)(v\alpha - 2\beta(l)l)}{4(v\alpha + 2\beta(l)l)}, \tag{42}$$

$$\lambda_r > \eta_v. \tag{43}$$

Then, the boundary control (24) *with the adaptation laws* (27) and (28) *guarantees the robust stability of the closed-loop system in the sense that all of the signals (i.e., the transverse vibration and the estimation errors) are uniformly and ultimately bounded.*

Proof. The substitution of Eqs. (27) and (28) into Eq. (26) yields

$$\begin{aligned}\dot V(t) \le & -\phi(t) + n(l)[\alpha v - 2l\beta(l)]w_x^4(l,t)/4 - k_3[\alpha w_t(l,t) + (\alpha v + 2\beta(l)l)w_x(l,t)]^4 \\ & - [k_2\alpha - \rho l\beta(l)]w_t^2(l,t) - [T_0 l\beta(l) - k_1(\alpha v + 2\beta(l)l) - \rho l v^2\beta(l)]w_x^2(l,t) \\ & + [2\beta(l)lv + k_1\alpha - k_2(\alpha v + 2\beta(l)l)]w_t(l,t)w_x(l,t) \\ & - \frac{\lambda_r}{2}\left[\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t) + \frac{\tilde\mu_d^2(t)}{\lambda_d}\right] + \frac{\lambda_r}{2}\left[\boldsymbol{\theta}\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta} + \frac{\mu_d^2}{\lambda_d}\right].\end{aligned} \tag{44}$$

Since the value of T_0 is sufficiently large, there exists sufficiently small σ, sufficiently large α, and $\beta(x)$ such that the following inequalities hold for all $x\in[0,l]$:

$$\alpha c_v + \rho(\beta(x) + x\beta_x(x)) - \beta(x)c_v l/\sigma > 0, \tag{45}$$

$$(\beta(x) + x\beta_x(x))T_0 - c_v\sigma l\beta(x) > 0, \tag{46}$$

$$3(\beta(x) + x\beta_x(x))n(x)/2 - (x\beta(x) + \alpha v/2)n_x(x)/2 > 0. \tag{47}$$

The following inequality is then obtained:

$$\dot V(t) \le -\eta V_0(t) - \frac{\lambda_r}{2}\left[\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t) + \frac{\tilde\mu_d^2(t)}{\lambda_d}\right] + \frac{\lambda_r}{2}\left[\boldsymbol{\theta}\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta} + \frac{\mu_d^2}{\lambda_d}\right]. \tag{48}$$

Using Lemma 1, Eq. (48) is then rewritten as

$$\dot V(t) \le -\eta_v V(t) - \frac{(\lambda_r - \eta_v)}{2}\left[\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t) + \frac{\tilde\mu_d^2(t)}{\lambda_d}\right] + \frac{\lambda_r}{2}\left[\boldsymbol{\theta}\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta} + \frac{\mu_d^2}{\lambda_d}\right]. \tag{49}$$

Define a new variable as

$$\omega(t) = \dot V(t) + \eta_v V(t) - \varepsilon, \tag{50}$$

where

$$\varepsilon = \frac{\lambda_r}{2}\left[\boldsymbol{\theta}\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta} + \frac{\mu_d^2}{\lambda_d}\right]. \tag{51}$$

From Eq. (50), it follows that $\omega(t) \le 0$. Solving Eq. (50) yields

$$\begin{aligned}V(t) &= V(0)e^{-\eta_v t} + \int_0^t e^{-\eta_v(t-\tau)}(\omega(\tau) + \varepsilon)d\tau \\ &\le V(0)e^{-\eta_v t} + \frac{\varepsilon}{\eta_v}(1 - e^{-\eta_v t}) \le V(0) + \frac{\varepsilon}{\eta_v}.\end{aligned} \tag{52}$$

Eq. (52) implies that

$$V(t) \le V(0)e^{-\eta_v t} + \frac{\varepsilon}{\eta_v}(1-e^{-\eta_v t}) \to \frac{\varepsilon}{\eta_v} \tag{53}$$

as the time approaches infinity. Utilizing Eqs. (15) and (35)–(37), we obtain

$$\frac{T_0}{2l}w^2 \le \frac{1}{2}\int_0^l T_0 w_x^2\,dx \le V_0(t) \le \frac{V_m(t)}{(\alpha-2\beta(l)l)} \le \frac{V(t)}{(\alpha-2\beta(l)l)}. \tag{54}$$

Therefore, we have that $w(x,t)$ is bounded. From Eqs. (52) to (54), it is concluded that $V(t)$ and $V_0(t)$ are uniformly and ultimately bounded for all $t\in[0,\infty)$. □

Remark 2. Theorem 1 shows that $V(t)$ converges to the ball of radius ε/η_v; that is, $V(t)$ can be pushed in an arbitrarily small boundedness region by setting a sufficiently small λ_r, a sufficiently large λ_i (i=1, 2, 3), and a sufficiently large λ_d. Since $V_0(t)$ is bounded, the kinetic energy of the axially moving string $E_k(t)$ is bounded. From Eq. (15), $w_t(x,t)$ and $w_x(x,t)$ are bounded and, using Property 1, $w_{xt}(x,t)$ is also bounded. Therefore, Eqs. (27) and (28) imply that $\hat{\boldsymbol{\theta}}(t)$ and $\hat{\mu}_d(t)$ are bounded. Finally, it is shown that all of the signals in the boundary control force (24) are bounded.

Remark 3. From Theorem 1, if the system parameters are known, and if the unknown boundary disturbance is ignored, that is, $\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t)=0$ and $\tilde{\mu}_d^2(t)/\lambda_d=0$, then $V(t)\le V(0)e^{-\eta_v t}$ is obtained. This implies that the exponential stability of the closed-loop system is achieved with the boundary control law (24) and the known parameters.

When the unknown boundary disturbance is neglected, the function

$$V_s(t) = \alpha V_{0s}(t) + 2\int_0^l \rho x\beta(x)w_x(w_t+vw_x)\,dx + \frac{1}{2}\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t) \tag{55}$$

is considered, where $V_{0s}(t)$ is defined as

$$V_{0s}(t) = \frac{1}{2}\int_0^l \rho(w_t+vw_x)^2\,dx + \frac{1}{2}\int_0^l T_0 w_x^2\,dx + \frac{1}{4}\int_0^l n(x)w_x^4\,dx + \frac{1}{2}m_a(w_t(l,t)+vw_x(l,t))^2. \tag{56}$$

If conditions (17) and (18) hold, then $V_s(t)$ becomes a Lyapunov function candidate. It follows from the proof of Lemma 1 that

$$(\alpha-2\beta(l)l)V_{0s}(t) \le V_{ms}(t) \le (\alpha+2\beta(l)l)V_{0s}(t), \tag{57}$$

where

$$V_{ms}(t) = V_s(t) - \frac{1}{2}\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t). \tag{58}$$

The boundary control force is rewritten as

$$f_a(t) = \mathbf{C}(t)\hat{\boldsymbol{\theta}}(t) + g_1(w_t(l,t)+vw_x(l,t)) + g_2(w_t(l,t)+vw_x(l,t))^3, \tag{59}$$

where the vector $\mathbf{C}(t)$ is defined as

$$\mathbf{C}(t) = \Big[\gamma v w_t(l,t)/\beta(l)l \quad ((\beta(l)l+v\alpha)w_{tt}(l,t)+v^2\alpha w_{xt}(l,t))/\beta(l)l \quad w_t(l,t)\Big], \tag{60}$$

where γ is a positive adaptation gain, and g_i (i=1, 2) are positive control gains. The estimate vector $\hat{\boldsymbol{\theta}}(t)$ used in the control law (59) is obtained by means of the adaptation law

$$\dot{\hat{\boldsymbol{\theta}}}(t) = -\lambda_r\hat{\boldsymbol{\theta}}(t) + \{(w_t(l,t)+vw_x(l,t))\beta(l)l/v\}\boldsymbol{\Lambda}\mathbf{C}^{\mathrm{T}}(t). \tag{61}$$

Remark 4. It follows from Lemma 3 that the initial values ($\hat{\rho}(0)$, $\hat{m}_a(0)$, and $\hat{c}_a(0)$) and the adaptation gains (λ_i (i=1, 2, 3) and λ_r) in Eq. (61) can be selected such that the estimation errors ($\tilde{\rho}(t)$, $\tilde{m}_a(t)$, and $\tilde{c}_a(t)$) are larger or equal to zero, for all $t\in[0,\infty)$, which is explained as follows. Define a vector $\boldsymbol{\Omega}(t)=\{(w_t(l,t)+vw_x(l,t))\beta(l)l/v\}\boldsymbol{\Lambda}\mathbf{C}^{\mathrm{T}}(t)$ including three components. From the estimation scheme (61), we obtain three estimation equations taking the same form as Eq. (39), where each component of $\boldsymbol{\Omega}(t)$ and the adaptation gain λ_r play roles as the input $q(t)$ and the gain κ in Eq. (39), respectively. Therefore, we can apply Lemma 3 to each estimation equation obtained from Eq. (61). It follows from the proof of Lemma 3 that, to make the estimation errors positive, the inequalities (B.2)–(B.4) must be satisfied. In combining estimation and control, when λ_i (i=1, 2, 3) and λ_r vary, $w_t(l,t)$, $w_x(l,t)$, $w_{tt}(l,t)$, and $w_{xt}(l,t)$ also change. Moreover, the bounds of the components of $\boldsymbol{\Omega}(t)$, which are considered as the bound K of the input $q(t)$ in Lemma 3, can be adjusted by changing the adaptation gains λ_i (i=1, 2, 3). Therefore, it is possible to search the adaptation gains (λ_i (i=1, 2, 3) and λ_r) such that inequalities (B.2) and (B.3) are satisfied. In practice, the ranges (lower and upper bounds) of the unknown parameter values can be estimated. Therefore, we can select the initial values ($\hat{\rho}(0)$, $\hat{m}_a(0)$, and $\hat{c}_a(0)$) such that they are larger than the unknown parameter values, which implies that inequality (B.4) can be satisfied.

Theorem 2. *Consider the system* (1) *with the boundary conditions* (3)–(4), *where the boundary disturbance is neglected* ($d(t)=0$) *and the system parameters* ρ, m_a, *and* c_a *are unknown. The control gains* g_i ($i=1, 2$) *in Eq.* (59) *and the adaptation gains* λ_r *and* γ *in Eq.* (61) *are chosen according to the conditions*

$$g_1 > [(v+1)v\alpha/\beta(l)l+2v+1]T_0+\rho\gamma v^2(v+1)/\beta(l)l, \tag{62}$$

$$g_2 > [(5v+4)v\alpha/\beta(l)l+10v+4]n(l)/4, \tag{63}$$

$$\lambda_r > \xi/(\alpha+2\beta(l)l), \tag{64}$$

$$\gamma > \xi m_a/2\rho, \tag{65}$$

where

$$\xi = \min\{\min_{x\in[0,l]}P_1,\ \min_{x\in[0,l]}P_2,\ \min_{x\in[0,l]}P_3\}. \tag{66}$$

Then,

(i) *The boundary control* (59) *using the adaptation law* (61) *guarantees the uniform asymptotic convergence of the transverse vibration of the axially moving string*;
(ii) *If adaptation gains* (λ_i ($i=1, 2, 3$) *and* λ_r) *and the initial values* ($\hat{\rho}(0)$, $\hat{m}_a(0)$ *and* $\hat{c}_a(0)$) *are chosen such that the estimation errors* $\tilde{\rho}(t)$, $\tilde{m}_a(t)$, *and* $\tilde{c}_a(t)$ *are larger than or equal to zero for all* $t\in[0,\infty)$ *(see Remark 4), the estimate values of the unknown parameters will converge to the true values.*

Proof. Consider the Lyapunov function candidate (55). Taking the derivative of $V_s(t)$ and using Eq. (4), we arrive at

$$\begin{aligned}\dot{V}_s(t) \le & -\phi(t)-\rho\gamma(w_t(l,t)+vw_x(l,t))^2+n(l)[\alpha v+2l\beta(l)]w_x^4(l,t)/4\\ & +(\alpha+\beta(l)l/v)T(l,t)w_t(l,t)w_x(l,t)+[\alpha v+2\beta(l)l]T(l,t)w_x^2(l,t)\\ & +(w_t(l,t)+vw_x(l,t))\times\{\gamma w_t(l,t)\rho+[(\beta(l)l/v+\alpha)w_{tt}(l,t)+v\alpha w_{xt}(l,t)]m_a\\ & +\beta(l)lw_t(l,t)c_a/v-\beta(l)l f_a/v\}+\rho\gamma vw_x(l,t)(w_t(l,t)+vw_x(l,t))\\ & +\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\varphi}}(t).\end{aligned} \tag{67}$$

We have the following inequalities:

$$\begin{aligned}(\alpha+\beta(l)l/v)T(l,t)w_t(l,t)w_x(l,t) \le & (\alpha/2+\beta(l)l/2v)[T_0w_t^2(l,t)+T_0w_x^2(l,t)\\ & +n(l)w_t^4(l,t)/2+3n(l)w_t^4(l,t)/2],\end{aligned} \tag{68}$$

$$\rho\gamma vw_x(l,t)(w_t(l,t)+vw_x(l,t)) \le \rho\gamma(v^2+v/2)w_x^2(l,t)+\rho\gamma vw_t^2(l,t)/2. \tag{69}$$

Using Eqs. (59)–(61) and (68)–(69), we obtain

$$\begin{aligned}\dot{V}_s(t) \le & -\phi(t)-\rho\gamma(w_t(l,t)+vw_x(l,t))^2+[(5v+3)\alpha+(3/v+10)\beta(l)l]n(l)w_x^4(l,t)/4\\ & +(\alpha+\beta(l)l/v)n(l)w_t^4(l,t)/4+[(\alpha+\beta(l)l/v)T_0/2+\gamma\rho v/2]w_t^2(l,t)\\ & +[((v+1/2)\alpha+(v/2+2)\beta(l)l)T_0+(v+1/2)v\gamma\rho]w_x^2(l,t)\\ & -g_1\beta(l)l(w_t(l,t)+vw_x(l,t))^2/v-g_2\beta(l)l(w_t(l,t)+vw_x(l,t))^4/v\\ & -\lambda_r(\boldsymbol{\varphi}^T(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t))/2+\lambda_r(\boldsymbol{\theta}\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta})/2).\end{aligned} \tag{70}$$

If σ, α, and $\beta(x)$ are chosen according to the inequalities (45)–(47), we obtain the inequality

$$\dot{V}_s(t) \le -\xi V_{0s}(t)-\lambda_r(\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t))/2+\lambda_r(\boldsymbol{\theta}\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta})/2, \tag{71}$$

which is negative definite except for the positive constant $\lambda_r(\boldsymbol{\theta}\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta})/2$. Therefore, the boundedness of $V_s(t)$, $V_{0s}(t)$, $w(x,t)$, $w_t(x,t)$, $w_x(x,t)$, and $w_{xt}(x,t)$ can be proved in a manner similar to the case for Theorem 1 (see Eqs. (50)–(54)). Using Property 2, we obtain the boundedness of $w_{xx}(x,t)$. Applying Eq. (1) and the above statements (the boundedness of $w_t(x,t)$, $w_x(x,t)$, $w_{xt}(x,t)$, and $w_{xx}(x,t)$), we conclude that $w_{tt}(x,t)$ is also bounded. At this point, we have shown that all of the signals of the boundary control law (59) and the adaptation law (61) are bounded. Then, assertions (i) and (ii) of Theorem 2 are proved as follows:

(i) Eq. (71) is rewritten as

$$\dot{V}_s(t) \le -\xi V_{0s}(t)+\lambda_r\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\theta}}(t). \tag{72}$$

Define a norm $\|w(x,t)\| = \left(\int_0^l w^2(x,t)\,dx\right)^{1/2}$. Using Lemma 2, the following inequality is obtained:

$$\|w\|^2+\|\dot{w}\|^2 \le 2(1/\rho+l^2/T_0)V_{0s}(t)<\infty. \tag{73}$$

Eqs. (72) and (73) yield

$$\dot{V}_s(t) \le -\xi\delta\left\{\|w\|^2+\|\dot{w}\|^2\right\}+\lambda_r\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\theta}}(t), \tag{74}$$

where $\delta=(1/\rho+l^2/T_0)^{-1}/2>0$. Since the inequality $\left|\int_0^\infty \lambda_r\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\theta}}(t)\,dt\right|<\infty$ is confirmed (the proof is given in Appendix C), Eq. (74) implies that

$$\int_0^\infty \|w\|^2\,dt \le \left(V_s(0)-V_s(\infty)+\int_0^\infty \lambda_r\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\theta}}(t)\,dt\right)/\xi\delta<\infty. \tag{75}$$

And since

$$\frac{d}{dt}\left(\|w\|^2\right)=\frac{d}{dt}\left(\int_0^l w^2\,dx\right)=\int_0^l 2w\dot{w}\,dx \le \int_0^l (w^2+\dot{w}^2)\,dx \le \|w\|^2+\|\dot{w}\|^2<\infty \quad (\text{from Eq. (73)}), \tag{76}$$

$\{w(x,t)\}_{x\in[0,l]}$ is uniformly bounded and equi-uniformly continuous in t. Using Lemma 4, Eq. (76) implies that $\lim_{t\to\infty}\|w(x,t)\|=0$.

(ii) Eq. (71) is rewritten as

$$\dot{V}_s(t) \le -\xi V_{0s}(t)-\lambda_r\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t)-\lambda_r\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta}. \tag{77}$$

Since $\lambda_r\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\theta}\ge 0$, the following inequality is obtained:

$$\int_0^\infty \boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\boldsymbol{\varphi}(t)\,dt \le (V_s(0)-V_s(\infty))/\lambda_r<\infty. \tag{78}$$

It should be noted that $\boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\varphi}}(t)$ is bounded. It follows from Barbalat's lemma [36, p. 192] that the convergence of the estimated values to the true values, that is, $\hat{\boldsymbol{\theta}}(t)\to\boldsymbol{\theta}$ as $t\to\infty$, is assured. □

4. Numerical simulations

The finite difference method is employed to find an approximate solution for the PDE with the initial and boundary conditions given by Eqs. (1)–(4). The convergence scheme is based on the central (for the string span) and forward/backward (for the left/right boundary) difference methods [37, Chapter 9]. The dynamic responses of the axially moving string were simulated in four cases. In the first case, no control is considered, and only the viscous damping of the string reduces the vibration energy of the axially moving string. In the second case, the boundary control law (24) is applied to the closed-loop system under the assumption that all parameter values are exactly known and the disturbance is negligible. Therefore, the control law uses these known values. In the third case, the proposed robust adaptive boundary control law (59) using the adaptation scheme (61) is simulated. In the fourth case, the unknown boundary disturbance is considered together with the unknown system parameters. The response of the closed-loop system is accomplished using the robust boundary control (24) with the adaptation schemes (27) and (28) in order to compensate for the unknown system parameters and the unknown boundary disturbance.

The system parameters used in the simulations are listed in Table 1. Let the initial conditions of the string be $w(x,0)=0.5\sin(\pi x/l)$ and $w_t(x,0)=0$. The positive values (α and σ) and the function $\beta(x)$ are chosen according to the inequalities (45)–(47) as follows: $\alpha=6.2$, $\sigma=1$, and $\beta(x)=0.1+0.01x$. To select the control gains, first, the case of no boundary disturbance is considered. Using the boundary control force (24) (with $\hat{\mu}_d(t)=0$), the boundary condition (4) can be rewritten as

$$\upsilon(t)+k_1w_x(l,t)-k_2w_t(l,t)=T_0w_x(l,t), \tag{79}$$

where

$$\upsilon(t)=-(\hat{\rho}-\rho)v(w_t(l,t)+vw_x(l,t))-\hat{m}_a(v+2\beta(l)l/\alpha)w_{xt}(l,t)-m_aw_{tt}(l,t)+(\hat{c}_a-c_a)w_t(l,t)-k_3\overline{w}^3(l,t)-n(l)w_x^3(l,t). \tag{80}$$

It can be assumed that, compared with $w_x(l,t)$ and $w_t(l,t)$, the function $\upsilon(t)$ is very small when $\hat{\boldsymbol{\theta}}(t)\to\boldsymbol{\theta}$. The boundary condition (4) can then be approximated as

$$w_x(l,t)=-k_fw_t(l,t), \tag{81}$$

where $k_f=k_2/(T_0-k_1)$. It follows from [8, Section 4.3] that the optimal value of k_f should be

$$k_f=\sqrt{T_0\overline{\rho}} \tag{82}$$

to maximize energy dissipation at the right boundary, where $\overline{\rho}$ can be taken from a certain range of ρ. Therefore, the control gain k_1 can be selected to be close to the value given by Eq. (82). The control gains k_i (i=1, 2, 3) in Eq. (24) must satisfy the inequalities (40)–(42): k_1=993, k_2=329, and k_3=540. The control gains g_i (i=1, 2) are selected according to Eqs. (62) and (63): g_1=1200 and g_2=4200. The adaptation gains $\lambda_i(i=1,2,3)$ should be chosen to be large so that the estimated values converge quickly. It follows from Remark 2 that the leakage gain λ_r should be small so that $V(t)$ is ultimately bounded by a small region. Therefore, the adaptation gains are selected as follows: λ_1=33, λ_2=40, λ_3=40, and λ_r=1. Following Lemma 3, the initial values for the adaptation scheme (61) should be chosen to be greater than the true values. The adaptation gain γ in Eq. (61) are set to satisfy the inequalities given by Eq. (65): γ=10. In the case where the boundary disturbance is considered, the adaptation gain selected for the estimation of μ_d is λ_d=30. Other adaptation gains are maintained as in the case of no boundary disturbance.

Figs. 2–5 show the transverse displacements of the axially moving string at $x=l/2$ for the four cases mentioned above, respectively. As seen in Fig. 2 (no control), the axially moving string can be stabilized if the viscous damping is sufficiently large, but this type of stabilization requires a great amount of time: in our case, it took almost 20 s. As seen in Fig. 3 (with the exactly known parameter values), the transverse vibrations can be suppressed within one second if the proposed boundary control with known values is applied. This simulation result is consistent with the theoretical one inferred in Remark 3, where the exponential stability of the closed-loop system can be achieved with exactly known system parameter values and the proposed boundary control law (24). However, if the parameter values are unknown, their

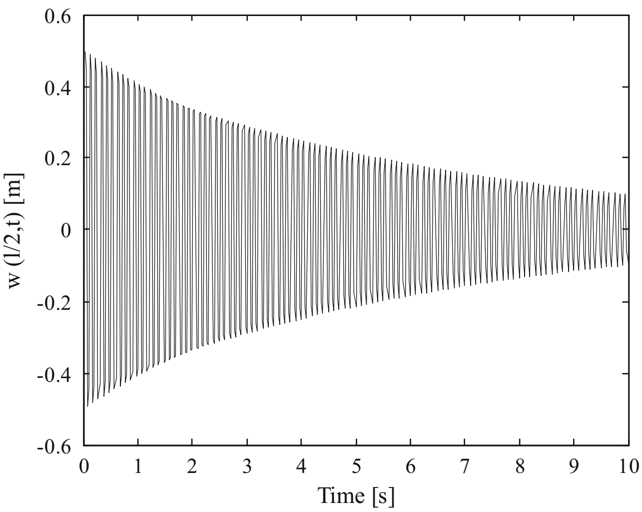

Fig. 2. Transverse displacement of the string at $x=l/2$ (open loop, without control).

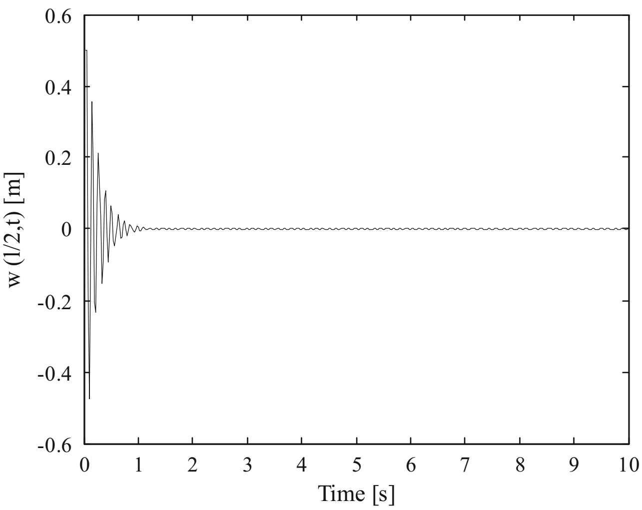

Fig. 3. Transverse displacement at $x=l/2$: closed-loop control using (24) with exactly known system parameter values.

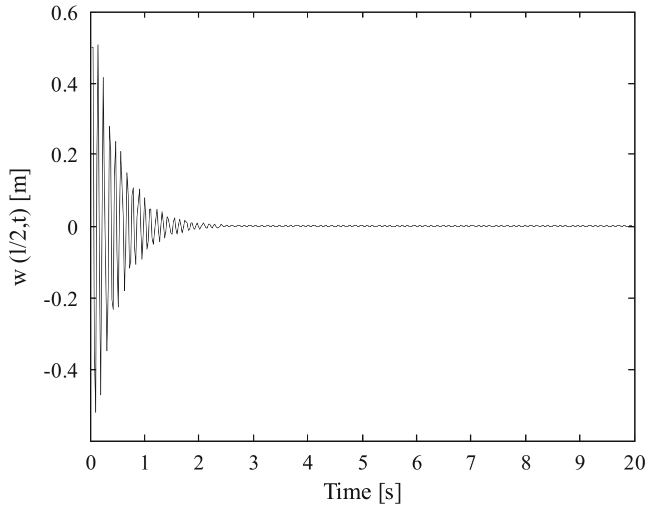

Fig. 4. Transverse displacement at $x=l/2$: closed-loop control with estimated system parameter values, where boundary disturbance is not considered (the control performance is slightly inferior to Fig. 3 but generally acceptable when compared with Fig. 2, considering that ρ, m_a, and c_a are unknown).

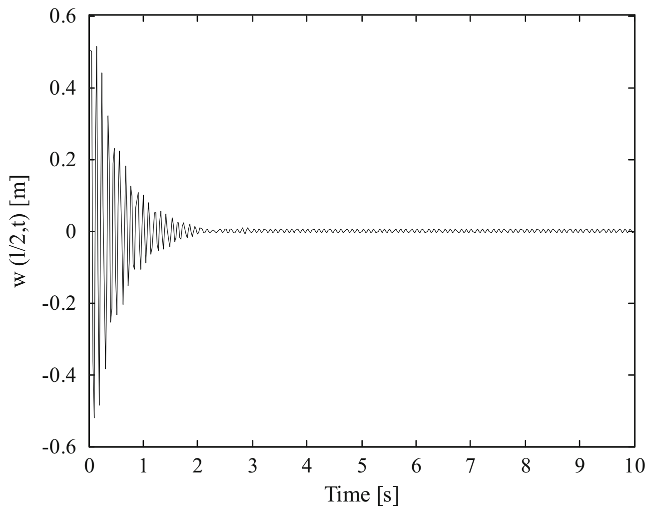

Fig. 5. Transverse displacement at $x=l/2$: closed-loop control with estimated system parameter values, where boundary disturbance is considered.

estimated values, which are obtained by using the adaptation scheme (61), must be used. Therefore, the control performance will deteriorate, as shown in Fig. 4, where the stabilization is achieved in 3 s. However, the overall control performance was acceptable compared with that shown in Fig. 2. When the boundary disturbance appears, as shown in Fig. 5, the robust boundary control (24) provides similar control performance to that shown in the case of no boundary disturbance, in which the transverse vibration is also suppressed in three seconds. However, asymptotic stabilization is not achieved because the boundary control (24) and the adaptation law (28) cope only with the bound of the boundary disturbance. As shown in Fig. 5, the transverse vibration is ultimately bounded by a small value. As shown in Fig. 6(A)–(C), the convergence of the estimated values to the true values is demonstrated in the case of no boundary disturbance, as the theoretical proof in Theorem 2.

5. Conclusions

In this paper, a robust adaptive boundary control scheme for suppressing the transverse vibration of a nonlinear axially moving string with unknown system parameter values and unknown boundary disturbance under spatially varying

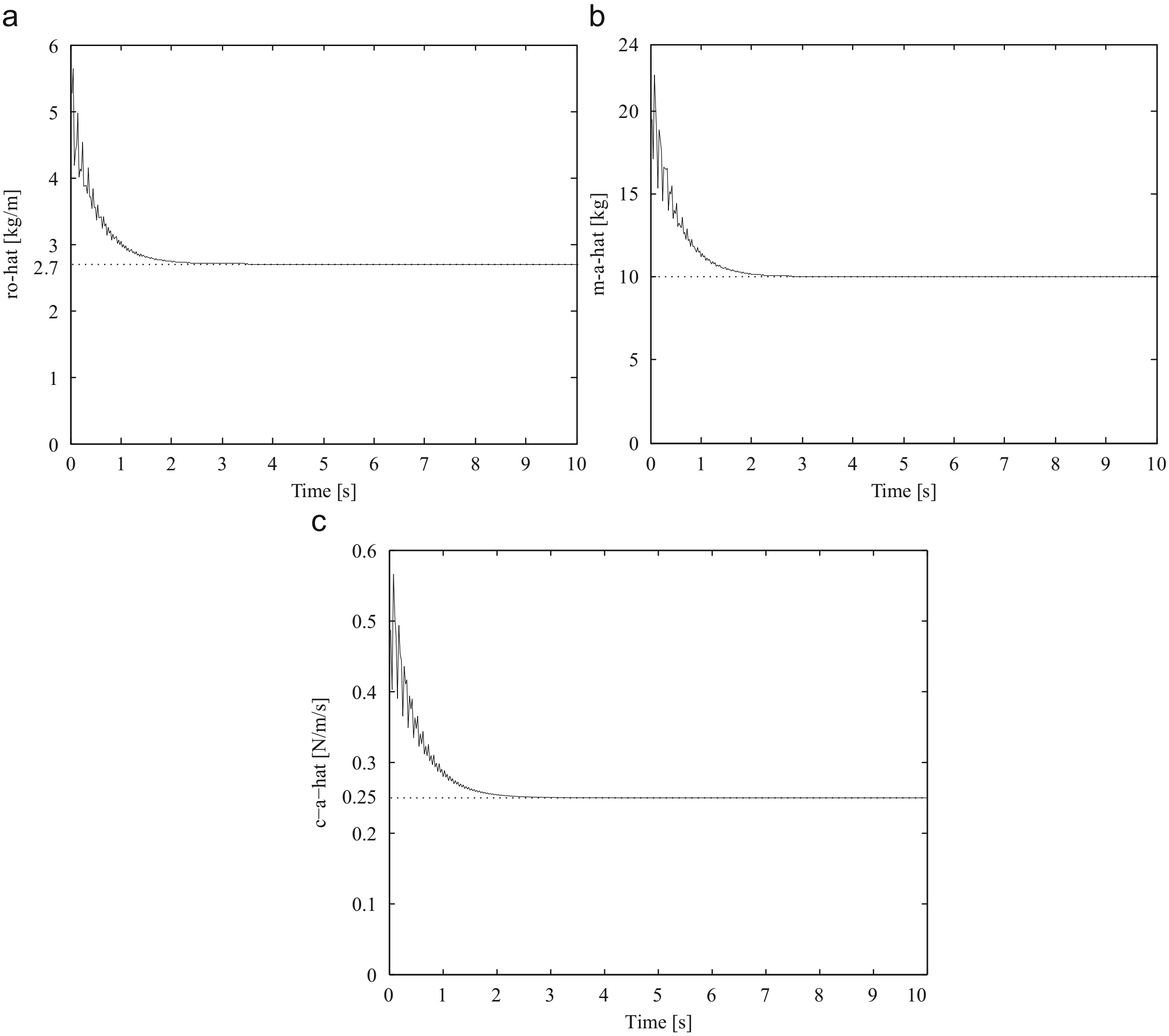

Fig. 6. Convergence of estimated parameter values: (a) convergence of $\hat{\rho}$; (b) convergence of $\hat{m}_a$; and (c) convergence of $\hat{c}_a$.

tension was developed. The robust boundary control and adaptation laws were derived using the Lyapunov redesign method. The three unknown system parameters were the mass per unit length of the axially moving string, the lumped mass of the hydraulic actuator, and the damping coefficient of the damper. In the case that the system parameter values were exactly known, the exponential stability of the closed-loop system was achieved. When the system parameter values were unknown, estimation schemes were used to estimate the unknown values and the adaptive boundary control achieved uniform asymptotic stability. Moreover, the parameter estimates converged to their true values if the initial values of the estimates and the adaptation gains were properly selected. When an unknown boundary disturbance and unknown parameter values were considered together, the robust stability of the closed-loop system was assured in the sense that all of the signals were uniformly and ultimately bounded. It can be concluded that the proposed robust boundary control scheme can provide a viable solution to problem of the vibration control of axially moving systems with unknown system parameters, varying tension, and bounded disturbances.

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Appendix A. Proof of Lemma 2

According to the Fundamental Theorem of Calculus, the following equation holds:

$$|u(x,t)| = \int_0^x |u_s(s,t)|\, ds. \tag{A.1}$$

By applying the Cauchy-Schwarz inequality (i.e., $\int hg \le (\int h^2)^{1/2}(\int g^2)^{1/2}$ with $h=1$, $g=|u_s(s,t)|$), the following is obtained:

$$|u(x,t)| \le (x)^{1/2}\left(\int_0^x |u_s(s,t)|^2\, ds\right)^{1/2} \le l^{1/2}\left(\int_0^l |u_s(s,t)|^2\, ds\right)^{1/2}. \tag{A.2}$$

Squaring both sides yields

$$|u(x,t)|^2 \le l\int_0^l u_x^2(x,t)\,dx. \tag{A.3}$$

Finally, Eq. (37) is obtained by integrating the last inequality over $[0,l]$.

Appendix B. Proof of Lemma 3

Consider the following equation:

$$\begin{aligned} p(t)-p_d &= e^{-\kappa t}p(0)-e^{-\kappa t}p_d+\int_0^t e^{-\kappa(t-\tau)}q(\tau)d\tau-p_d(1-e^{-\kappa t}) \\ &= e^{-\kappa t}(p(0)-p_d)+\int_0^t e^{-\kappa(t-\tau)}(q(\tau)-\kappa p_d)d\tau. \end{aligned} \tag{B.1}$$

The gain κ is selected such that

$$0<\kappa \le K/p_d. \tag{B.2}$$

The input $q(t)$ is made to satisfy the following inequality:

$$K \ge q(t) \ge \kappa\, p_d-(p(0)-p_d)e^{-(\kappa+1)t}. \tag{B.3}$$

Finally, if

$$p(0) > p_d, \tag{B.4}$$

we obtain the following inequality:

$$p(t)-p_d \ge e^{-\kappa t}(p(0)-p_d)+\int_0^t e^{-\kappa(t-\tau)}[\kappa p_d-(p(0)-p_d)e^{-(\kappa+1)\tau}-\kappa p_d]d\tau \ge (p(0)-p_d)e^{-(\kappa+1)t} \ge 0 \tag{B.5}$$

for all $t\in[0,\infty)$. From the conditions (B.2)–(B.4), it is possible to choose κ, $q(t)$, and $p(0)$, where $\kappa>0$ and $q(t)$ is bounded as $|q(t)|<K$, such that the inequality $p(t)-p_d \ge 0$ is assured. Then, the proof is complete.

Appendix C. The boundedness of $\int_0^\infty \lambda_r \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\theta}}(t)\,dt$

Using the adaptation law (61), we have

$$\begin{aligned} \left|\int_0^\infty \lambda_r \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\theta}}(t)\,dt\right| &= \left|-\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\int_0^\infty \dot{\hat{\boldsymbol{\theta}}}(t)dt+\boldsymbol{\theta}^{\mathrm{T}}\int_0^\infty \{(w_t(l,t)+vw_x(l,t))\beta(l)l/v\}\mathbf{C}^{\mathrm{T}}(t)\,dt\right| \\ &= \Bigg|\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}(\hat{\boldsymbol{\theta}}(0)-\hat{\boldsymbol{\theta}}(\infty))+\rho\gamma\int_0^\infty (w_t(l,t)+vw_x(l,t))w_t(l,t)\,dt \\ &\quad +m_a\int_0^\infty (w_t(l,t)+vw_x(l,t))\{(\beta(l)l/v+\alpha)w_{tt}(l,t)+v\alpha w_{xt}(l,t)\}\,dt \\ &\quad +c_a\int_0^\infty \{(w_t(l,t)+vw_x(l,t))\beta(l)l/v\}w_t(l,t)\,dt\Bigg| \\ &= \Bigg|\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}(\hat{\boldsymbol{\theta}}(0)-\hat{\boldsymbol{\theta}}(\infty))+(\rho\gamma+c_a\beta(l)l/v)\int_0^\infty w_t^2(l,t)\,dt \\ &\quad +(\gamma v\rho+\beta(l)lc_a)\int_0^\infty w_t(l,t)w_x(l,t)\,dt \\ &\quad +(\beta(l)l/v+\alpha)m_a\int_0^\infty w_t(l,t)w_{tt}(l,t)\,dt \\ &\quad +v\alpha m_a\int_0^\infty w_t(l,t)w_{xt}(l,t)\,dt+v^2\alpha m_a\int_0^\infty w_x(l,t)w_{xt}(l,t)\,dt \\ &\quad +(v\alpha+\beta(l)l)m_a\int_0^\infty w_x(l,t)w_{tt}(l,t)\,dt\Bigg|. \end{aligned} \tag{C.1}$$

It should be noted that

$$\left|\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}(\hat{\boldsymbol{\theta}}(0)-\hat{\boldsymbol{\theta}}(\infty))\right| < \infty \tag{C.2}$$

when $\hat{\boldsymbol{\theta}}(t)$ is bounded. When all internal signals of the string ($w(l,t)$, $w_t(l,t)$, $w_x(l,t)$, $w_{tt}(l,t)$, and $w_{xt}(l,t)$) are bounded, we conclude that

$$\left|\int_0^\infty w_t^2(l,t)\,dt\right| \le \int_0^\infty \left|w_t(l,t)\right|\left|w_t(l,t)\right|dt \le M_1\int_0^\infty \left|w_t(l,t)\right|dt < \infty, \tag{C.3}$$

$$\left|\int_0^\infty w_t(l,t)w_x(l,t)\,dt\right| \le \int_0^\infty \left|w_x(l,t)\right|\left|w_t(l,t)\right|dt < M_2\int_0^\infty \left|w_t(l,t)\right|dt < \infty, \tag{C.4}$$

$$\left|\int_0^\infty w_t(l,t)w_{xt}(l,t)\,dt\right| \le \int_0^\infty \left|w_t(l,t)\right|\left|w_{xt}(l,t)\right|dt \le M_3\int_0^\infty \left|w_t(l,t)\right|dt < \infty, \tag{C.5}$$

$$\left|\int_0^\infty w_x(l,t)w_{tt}(l,t)\,dt\right| \le \int_0^\infty \left|w_x(l,t)\right|\left|w_{tt}(l,t)\right|dt \le M_2\int_0^\infty \left|w_{tt}(l,t)\right|dt < \infty, \tag{C.6}$$

$$\left|\int_0^\infty w_t(l,t)w_{tt}(l,t)\,dt\right| = \left|w_t^2(l,t)/2\big|_0^\infty\right| < \infty, \tag{C.7}$$

$$\left|\int_0^\infty w_x(l,t)w_{xt}(l,t)\,dt\right| = \left|w_x^2(l,t)/2\big|_0^\infty\right| < \infty, \tag{C.8}$$

where $\left|w_t(l,t)\right| \le M_1$, $\left|w_x(l,t)\right| \le M_2$, and $\left|w_{xt}(l,t)\right| \le M_3$. Applying the triangle inequality and utilizing inequalities (C.2)–(C.8) to Eq. (C.1), it is concluded that

$$\left|\int_0^\infty \lambda_r \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\theta}}(t)\,dt\right| < \infty. \tag{C.9}$$

Then, the proof is complete.

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