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Structural optimization for the avoidance of self-excited vibrations based on analytical models

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ABSTRACT

Self-excited vibrations are a severe problem in many technical applications. In many cases they are caused by friction as for example in disk and drum brakes, clutches, saws and paper calenders. The goal to suppress self-excited vibrations can be reached by active and passive techniques, the latter ones being preferable due to the lower costs. Among design engineers it is known that breaking the symmetries of structures is sometimes helpful to avoid self-excited vibrations. This has been verified from an analytical point of view in a recent paper. The goal of the present paper is to use this analytical insight for a systematic structural optimization of rotors in frictional contact. The first system investigated is a simple discrete model of a rotor in frictional contact. As a continuous example a rotating beam in frictional contact is considered and optimized with respect to its bending stiffness. Finally a brake disk is optimized giving some attention to the feasibility of the modifications for the production process.

1. Introduction

Friction induced vibrations occur in many technical applications as for example in disk and drum brakes, clutches, saws and paper calenders. For brake squeal an overview about the topic can be found in the survey papers [1–3], for squealing clutches we mention [4]. Especially for the problem of brake squeal many recent papers consider the excitation mechanism (see e.g. [5–10]) and despite differences in modeling agree that it is caused by sliding friction. While the excitation mechanism for brake squeal seems to be well understood, for the problem of paper calendering this is not the case [11,12]. However, the paper of Brommundt [13] indicates that sliding friction is a possible excitation mechanism for this application as well.

In the literature in addition to sliding friction also the effect of negative damping induced by a falling friction characteristic, stick slip and sprag slip effects [14] and nonlinear saturation effects [15] are proposed as excitation mechanisms [16,17], which, however, do not play a role in the systems addressed in this paper.

For the suppression of self-excited vibrations active, semi-active and passive measures have been suggested. Usually active measures are expensive and less reliable, since states of the system have to be measured and actuators have to be operated. Nevertheless they have been successfully used to suppress brake squeal in the laboratory [17–21]. Semi-active measures have for example been developed in [22,23] using piezo ceramics attached to the brake pads.

An approach for the passive assignment of poles in a nonconservative systems has been suggested by Ouyang [24]. It is noted, however, that the assignment of real parts of eigenvalues is much harder than the assignment of eigenfrequencies.

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The assignment of eigenfrequencies through passive modifications has been successfully done for example in [25–28] the objective being an avoidance of resonances.

Whereas most of the approaches mentioned above rely on the active and semi-active increase of dissipation, in this paper we want to make systematic use of the results of [29], showing that structures with rotational symmetry are more sensitive to self-excited vibrations than systems without rotational symmetry and confirming observations made in [30,31], where it was noted that at least two modes of similar order of the brake disk were necessary to capture squeal in pin-on-disk models.

The approach of braking symmetries in order to prevent squeal was followed by Fieldhouse and coworkers who developed modified disk brakes in which problematic modes were hindered by structural modifications. The results presented in [32,33] include experimental verification of the concept. In the above mentioned papers, however, no explanation for the stabilizing effect of the modifications is given and the excitation mechanism for squeal is not included in the calculations.

The goal of the current paper is to explain the mechanism and to provide a systematic goal for an optimization to avoid self-excited vibrations. We concentrate on structures where terms arising from frictional contacts are small compared to elastic restoring terms. We systematically split up eigenfrequencies of the conservative problem without friction terms and demonstrate that this results in a stabilizing effect. As examples three characteristic systems are studied. At first we study a rigid body model of a rotor in frictional contact which can be interpreted as a drum brake or a centrifugal clutch. To address the problem of structural optimization in continuous systems, a rotating beam in frictional contact is investigated. An analytic procedure to split up the eigenfrequencies shows possibilities and limitations of the structural optimization. Finally, a process for optimization of disk brakes is suggested, which is also concerned about the feasibility of the modifications in the context of manufacturing.

2. Mathematical background

In this section we briefly describe the mathematical background as a motivation for the following structural optimization. By explicitly calculating results for a three parameter setting we slightly generalize the results of [29].

We consider the differential equation of a conservative system perturbed with linear parametric periodic excitation, which, without loss of generality, can be written as

$$\mathbf{M}\ddot{\mathbf{q}}+\Delta\mathbf{D}(t,\varepsilon)\dot{\mathbf{q}}+(\mathbf{K}+\Delta\mathbf{K}(t,\varepsilon))\mathbf{q}=\mathbf{0}, \tag{1a}$$

$$\Delta\mathbf{D}(t,\varepsilon)=\Delta\mathbf{D}(t+2\pi,\varepsilon), \quad \Delta\mathbf{K}(t,\varepsilon)=\Delta\mathbf{K}(t+2\pi,\varepsilon), \tag{1b}$$

$$\mathbf{M}=\mathrm{diag}(1,1,\ldots,1), \quad \mathbf{K}=\mathrm{diag}(\omega_1^2,\ldots,\omega_N^2). \tag{1c}$$

We assume that $\Delta\mathbf{D}(t,\varepsilon)$ and $\Delta\mathbf{K}(t,\varepsilon)$ depend smoothly on the parameter ε and can be expanded as

$$\Delta\mathbf{D}(t,\varepsilon)=\Delta\mathbf{D}_1(t)\varepsilon+\Delta\mathbf{D}_2(t)\varepsilon^2+\cdots, \tag{2a}$$

$$\Delta\mathbf{K}(t,\varepsilon)=\Delta\mathbf{K}_1(t)\varepsilon+\Delta\mathbf{K}_2(t)\varepsilon^2+\cdots. \tag{2b}$$

The parameter ε can be seen as a norm of the perturbation. It is well known from Floquet theory that the stability of the trivial solution depends on the eigenstructure of the monodromy matrix, which we expand in terms of the perturbation parameter ε,

$$\mathbf{X}(2\pi,\varepsilon)=\mathbf{X}(2\pi,0)+\left.\frac{\partial\mathbf{X}(2\pi,0)}{\partial\varepsilon}\right|_{\varepsilon=0}\varepsilon+\cdots. \tag{3}$$

From the perturbation theory developed in [34,35] it is known that the derivative of the monodromy matrix can be calculated as

$$\left.\frac{\partial\mathbf{X}(2\pi,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}=\mathbf{X}(2\pi,0)\mathbf{H}, \tag{4a}$$

$$\mathbf{H}=\int_0^{2\pi}\mathbf{Y}(t,0)^{\mathrm{T}}\left.\frac{\partial\mathbf{A}(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}\mathbf{X}(t,0)\,\mathrm{d}t, \tag{4b}$$

an expression which depends only on the derivative of the system matrix of the corresponding first-order system $\mathbf{A}$ with respect to ε, the monodromy matrix of the unperturbed problem $\mathbf{X}$ and its adjoint $\mathbf{Y}$.

The spectrum of the unperturbed problem can be either simple or semi-simple. Semi-simple eigenvalues can occur due to the symmetry of the structure, i.e. $\omega_i=\omega_j$, or due to internal or combination resonances. According to [36] simple and semi-simple eigenvalues can be expanded in terms of ε as

$$\rho=\rho_0+\left.\frac{\partial\rho}{\partial\varepsilon}\right|_{\varepsilon=0}\varepsilon+\cdots, \tag{5}$$

where the derivative of the Floquet multiplier can be analytically calculated [37]. The derivative of the modulus can then be obtained [34,35] from

$$\frac{\partial|\rho|}{\partial\varepsilon}=\frac{1}{|\rho_0|}\mathrm{Re}\left(\overline{\rho}_0\frac{\partial\rho}{\partial\varepsilon}\right). \tag{6}$$

The eigenvalues ρ_{0j}, eigenvectors $\mathbf{u}_j$ of the monodromy matrix $X(2\pi,0)$ of the unperturbed problem and the eigenvectors $\mathbf{v}_j$ of its adjoint $Y(2\pi,0)$ have a particularly simple form given by

$$\rho_{0j}=\cos 2\pi\omega_j+\mathrm{i}\sin 2\pi\omega_j \tag{7}$$

and

$$\mathbf{u}_j=\frac{1}{\sqrt{2}}\left(0,\ldots,0,\frac{\mathrm{i}}{\omega_j},0,\ldots,0,1,0,\ldots,0\right)^{\mathrm{T}}, \tag{8a}$$

$$\mathbf{v}_j=\frac{1}{\sqrt{2}}(0,\ldots,0,-\mathrm{i}\omega_j,0,\ldots,0,1,0,\ldots,0)^{\mathrm{T}}, \tag{8b}$$

where the nonzero entries appear in the j-th and $N+j$–th position. The eigenvectors are normalized such that $\mathbf{v}_j^{\mathrm{T}}\mathbf{u}_j=1$. The derivative of a semi-simple Floquet multiplier can be calculated from [34,35,29]

$$\det\begin{bmatrix}\rho_{0j}\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_j-\dfrac{\partial\rho_j}{\partial\varepsilon} & \rho_{0j}\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_k\\ \rho_{0k}\mathbf{v}_j^{\mathrm{T}}\mathbf{H}\mathbf{u}_j & \rho_{0k}\mathbf{v}_k^{\mathrm{T}}\mathbf{H}\mathbf{u}_k-\dfrac{\partial\rho_j}{\partial\varepsilon}\end{bmatrix}=0. \tag{9}$$

For a system with $\omega_i=\omega_j$ therefore only the matrix entries k, l with $k,l\in\{i,j\}$ enter the first derivative of the Floquet multiplier. The derivative of a semi-simple Floquet multiplier can therefore be calculated from an equivalent two by two system of the type (1a).

We now consider the case of such a two by two system with three parameters κ, δ, γ where

$$\Delta\mathbf{D}(t,\varepsilon)=\delta(\varepsilon)\mathbf{D}(t),\quad \Delta\mathbf{K}(t,\varepsilon)=\kappa(\varepsilon)\mathbf{K}(t)+\gamma(\varepsilon)\mathbf{N}(t), \tag{10}$$

where $\mathbf{D}(t)$, $\mathbf{K}(t)$ are periodic, and positive semi-definite and $\mathbf{N}(t)$ is skew-symmetric and periodic. According to the assumptions made in the context (1a) the parameters can be expanded in terms of ε as

$$\kappa(\varepsilon)=\kappa_1\varepsilon+\kappa_2\varepsilon^2+\cdots, \tag{11a}$$

$$\delta(\varepsilon)=\delta_1\varepsilon+\delta_2\varepsilon^2+\cdots, \tag{11b}$$

$$\gamma(\varepsilon)=\gamma_1\varepsilon+\gamma_2\varepsilon^2+\cdots. \tag{11c}$$

The derivative of the Floquet multiplier can be calculated from

$$\det\left(\int_0^{2\pi}\frac{-\rho_0}{2}\left(\mathbf{D}(t)+\frac{\mathrm{i}}{\omega}(\mathbf{K}(t)+\mathbf{N}(t))\right)\mathrm{d}t-\frac{\partial\rho}{\partial\varepsilon}\mathbf{I}\right)=0. \tag{12}$$

Assuming that either $\int_0^{2\pi}\mathbf{D}(t)\,\mathrm{d}t$ or $\int_0^{2\pi}\mathbf{K}(t)\,\mathrm{d}t$ is positive definite the number of parameters in (12) can be reduced by performing an orthogonal transformation

$$\det\left(\mathbf{Q}^{\mathrm{T}}\left(\int_0^{2\pi}\frac{-\rho_0}{2}\left(\mathbf{D}(t)+\frac{\mathrm{i}}{\omega}(\mathbf{K}(t)+\mathbf{N}(t))\right)\mathrm{d}t\right)\mathbf{Q}-\frac{\partial\rho}{\partial\varepsilon}\mathbf{I}\right) \tag{13a}$$

$$=\det\left(\frac{-\rho_0}{2}\left(\begin{bmatrix}d & 0\\ 0 & d+\Delta_d\end{bmatrix}+\frac{\mathrm{i}}{\omega}\begin{bmatrix}k & 0\\ 0 & k+\Delta_k\end{bmatrix},+\begin{bmatrix}0 & n\\ -n & 0\end{bmatrix}\right)-\frac{\partial\rho}{\partial\varepsilon}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right), \tag{13b}$$

where $\mathbf{Q}$ consists of the eigenvectors, which simultaneously diagonalize $\int_0^{2\pi}\mathbf{D}(t)\,\mathrm{d}t$ and $\int_0^{2\pi}\mathbf{K}(t)\,\mathrm{d}t$. The derivative of the Floquet multiplier can thus be calculated as

$$\left.\frac{\partial\rho}{\partial\varepsilon}\right|_{\varepsilon=0}=\rho_0\left(-\delta(d+\Delta_d)-\frac{\mathrm{i}}{\omega}\kappa(k+\Delta_k)\pm\frac{1}{2\omega}\sqrt{4\gamma^2n^2-(\kappa\Delta_k-\mathrm{i}\delta\Delta_d)^2}\right). \tag{14}$$

The first approximation of the stability boundary $\partial|\rho|/\partial\varepsilon=0$ with Eq. (6) reads

$$0=-\delta(d+\Delta_d)+\mathrm{Re}\left(\frac{1}{2\omega}\sqrt{4\gamma^2n^2-(\kappa\Delta_k-\mathrm{i}\delta\Delta_d\omega)^2}\right) \tag{15a}$$

$$=-\delta(d+\Delta_d)+|z|\cos\left(\frac{1}{2}\arg(z)\right), \tag{15b}$$

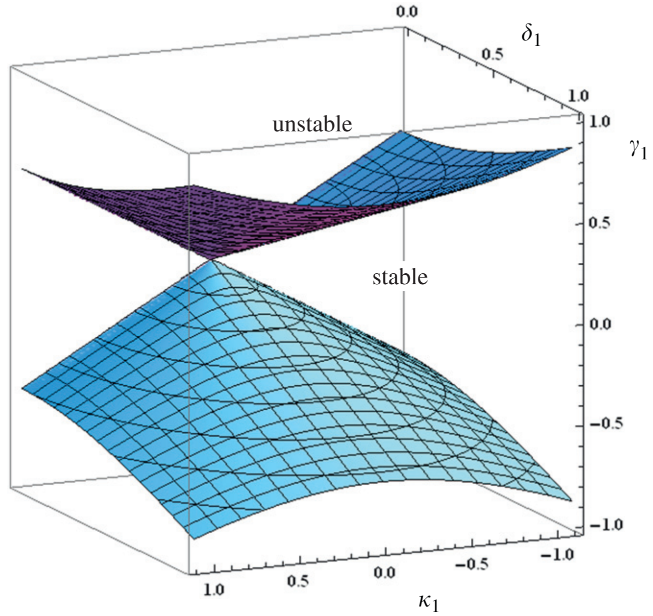

Fig. 1. Approximation to the stability domain for $\Delta_k=\pi$, $\Delta_d=0.3\pi$.

$$|z|=\frac{1}{2\omega}\sqrt{(4\gamma^2n^2-\kappa^2\Delta_k^2+\delta^2\Delta_d^2\omega^2)^2+(2\delta\kappa\Delta_d\Delta_k)^2}, \tag{15c}$$

$$\arg(z)=\arctan\left(\frac{2\kappa\delta\Delta_d\Delta_k}{4\gamma^2n^2-\kappa^2\Delta_k^2+\delta^2\Delta_d^2\omega^2}\right). \tag{15d}$$

The corresponding graph is shown in Fig. 1 for $d=\pi$, $n=\pi$, $k=\pi$, $\Delta_k=\pi$, $\Delta_d=0.3\pi$. For $\Delta_d=0$ the expression for the stability boundary simplifies a great deal and can be written as

$$\gamma n=\sqrt{\kappa^2\Delta_k^2+\delta^2d^2\omega^2}. \tag{16}$$

The corresponding stability boundary is given in Fig. 2 and coincides with the numerical example

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\ddot{\mathbf{q}}+\delta(\varepsilon)\cos^2t\begin{bmatrix}1&0\\0&1\end{bmatrix}\dot{\mathbf{q}}+\left(\begin{bmatrix}\omega^2&0\\0&\omega^2+\kappa(\varepsilon)\end{bmatrix}+\gamma(\varepsilon)\cos^2t\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right)\mathbf{q}=\mathbf{0} \tag{17}$$

studied in [29] (i.e. k=0, $\Delta_k=2\pi$, $d=\pi$, $\Delta_d=0, n=\pi$) in a three parameter setting.

By comparison of Figs. 1 and 2 it is seen that they are qualitatively equivalent. In Fig. 3 cuts through the stability domain at $\kappa_1=0$ and 0.13 ($\omega_2/\omega_1\approx1.15$) are shown in comparison with a numerical evaluation of the Floquet multipliers. In Fig. 4 cuts through the stability region at $\delta_1=0$ and 0.3 are shown.

It is seen that for small perturbations the approximation of the stability boundary is in very good agreement with the numerical results especially in the technically relevant range with small damping and a moderate splitting of the frequencies to up to 10 percent.

3. Minimal model for self-excited vibrations of a rotor in frictional contact

In this section we consider the example of a minimal model for a discrete rotor model in frictional contact depicted in Fig. 5. The model can be interpreted as a minimal model for a drum brake or a centrifugal clutch. We consider a rotor of mass M rotating at constant angular velocity Ω. The bearings are modeled as elastic springs with stiffness c_x, c_y, respectively. The rotor is in contact with a circular guidance through massless pins which are pressed onto it by linear elastic springs with spring constant k and prestressed with the force N_0. Between the ring and the pins friction occurs which we describe using Coulomb's law with constant μ. We assume that the pins are always in contact with the outer ring and that there is always a nonvanishing relative velocity between the ring and the pins. There are two straightforward ways to define the two coordinates of the system. The first one is to express the displacement in the frame B attached to the rotor i.e. to define the displacement vector as

$$\mathbf{r}=\vec{\mathrm{OP}}=\overline{q}_1\mathbf{b}_1+\overline{q}_2\mathbf{b}_2. \tag{18}$$

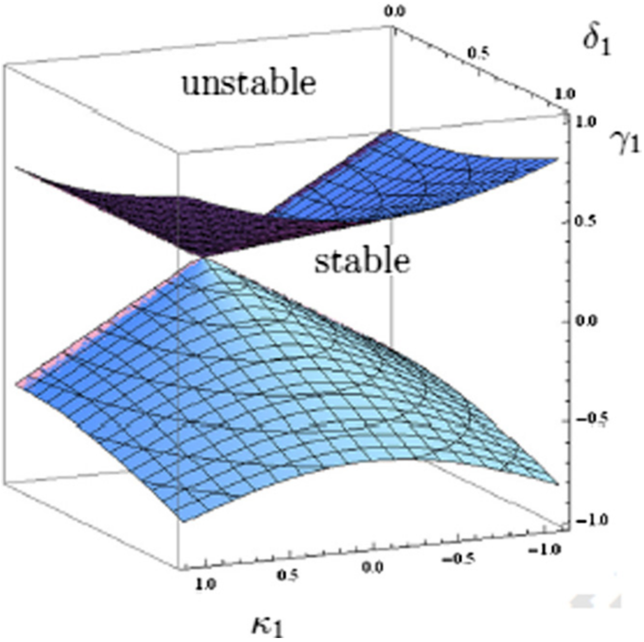

Fig. 2. Approximation to the stability boundary for $\Delta_d = 0$.

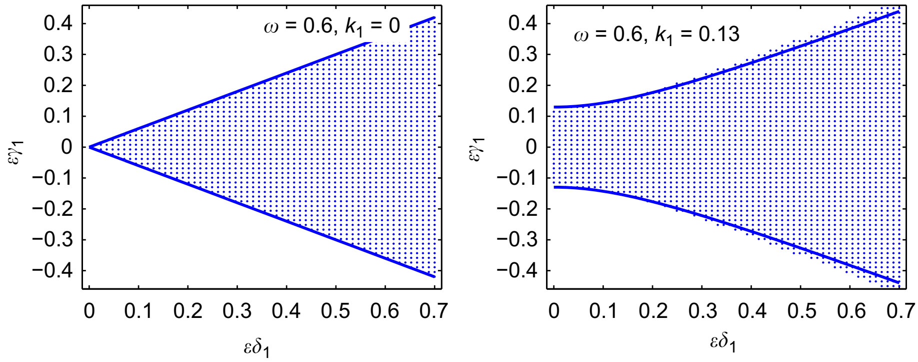

Fig. 3. Stable regions for a symmetric and an asymmetric system (dot: stable).

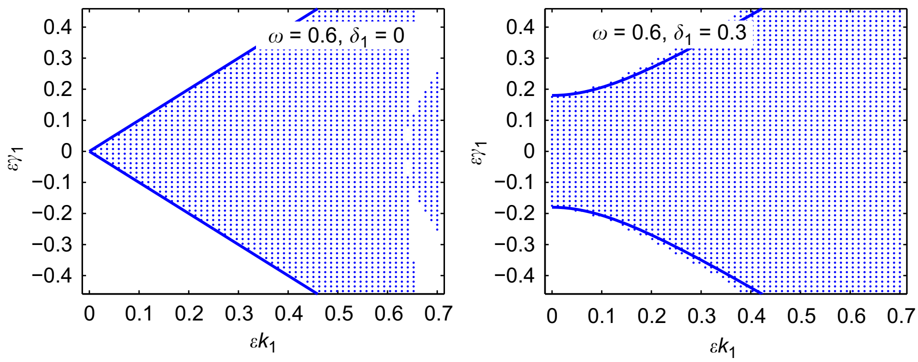

Fig. 4. Stable regions for a symmetric and an asymmetric system (dot: stable).

The linearized equations of motion are then given by

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \ddot{\bar{q}}_1 \\ \ddot{\bar{q}}_2 \end{bmatrix} + \begin{bmatrix} 0 & -2M\Omega \\ 2M\Omega & 0 \end{bmatrix}\begin{bmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 \end{bmatrix} + \left(\begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix} + \begin{bmatrix} \Delta k_{11} & \Delta k_{12} \\ \Delta k_{21} & \Delta k_{22} \end{bmatrix} \right)\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} = \mathbf{0}, \tag{19}$$

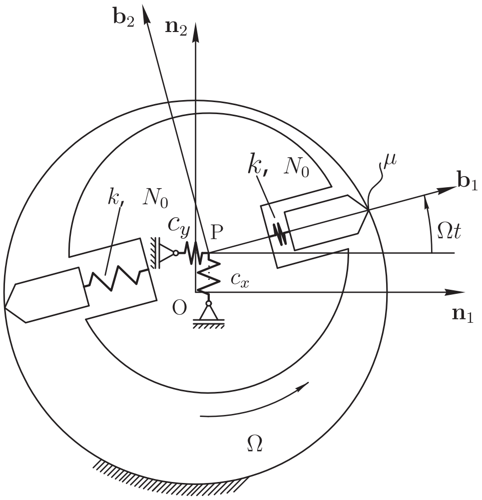

Fig. 5. Minimal model for a rotor in frictional contact.

with

$$k_{11}(t)=c_x-M\Omega^2+(c_y-c_x)\sin^2\Omega t,\quad k_{12}(t)=(c_y-c_x)\cos\Omega t\sin\Omega t,$$

$$k_{21}(t)=(c_y-c_x)\cos\Omega t\sin\Omega t,\quad k_{22}(t)=c_y-M\Omega^2+(c_x-c_y)\sin^2\Omega t$$

and

$$\Delta k_{11}=2k,\quad \Delta k_{12}=0,$$

$$\Delta k_{21}=2k\mu,\quad \Delta k_{22}=2\frac{N_0}{r}(1+\mu^2).$$

We observe that the equations of motion have a particularly simple form if the bearing stiffnesses c_x and c_y are equal. Then we obtain

$$k_{11}=c+2k-M\Omega^2,\quad k_{12}=0,\quad k_{21}=2k\mu,\quad k_{22}=c-M\Omega^2+2\frac{N_0}{r}(1+\mu^2)$$

such that the equations of motion have constant coefficients and can be written in the form

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{G}\dot{\mathbf{q}}+(\mathbf{K}+\mathbf{N})\mathbf{q}=\mathbf{0},$$

$$\mathbf{M}=\mathbf{M}^{\mathrm{T}}>0,\quad \mathbf{G}=-\mathbf{G}^{\mathrm{T}},\quad \mathbf{K}=\mathbf{K}^{\mathrm{T}},\quad \mathbf{N}=-\mathbf{N}^{\mathrm{T}}. \tag{20}$$

Since we are dealing with a two by two system it can be shown analytically that the trivial solution of (20) is unstable [5,38,39]. The self-excited vibrations arising can be interpreted as the reason for squeal.

In case of asymmetric bearing stiffness it is no longer obvious weather the trivial solution of (20) is stable since the equations of motion have periodic coefficients. In this case it is beneficial to define the coordinates of the system with respect to the inertial frame N i.e. to define the displacement vector as

$$\mathbf{r}=\overrightarrow{\mathrm{OP}}=q_1\mathbf{n}_1+q_2\mathbf{n}_2. \tag{21}$$

The equations of motion then read

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\left(\begin{bmatrix} c_x & 0 \\ 0 & c_y \end{bmatrix}+\begin{bmatrix} \Delta k_{11}(t) & \Delta k_{12}(t) \\ \Delta k_{21}(t) & \Delta k_{22}(t) \end{bmatrix}\right)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\mathbf{0}, \tag{22}$$

with

$$\Delta k_{11}(t)=2k\cos^2\Omega t+2\frac{N_0}{r}(1+\mu^2)\sin^2\Omega t-k\mu\sin 2\Omega t,$$

$$\Delta k_{12}(t)=2k\cos\Omega t\sin\Omega t-2\frac{N_0}{r}(1+\mu^2)\cos\Omega t\sin\Omega t-2k\mu\sin^2\Omega t,$$

$$\Delta k_{21}(t)=2k\mu\cos^2\Omega t+2k\cos\Omega t\sin\Omega t-2\frac{N_0}{r}(1+\mu^2)\cos\Omega t\sin\Omega t,$$

$$\Delta k_{22}(t)=2\frac{N_0}{r}(1+\mu^2)\cos^2\Omega t-2k\cos^2\Omega t+k\mu\sin 2\Omega t$$

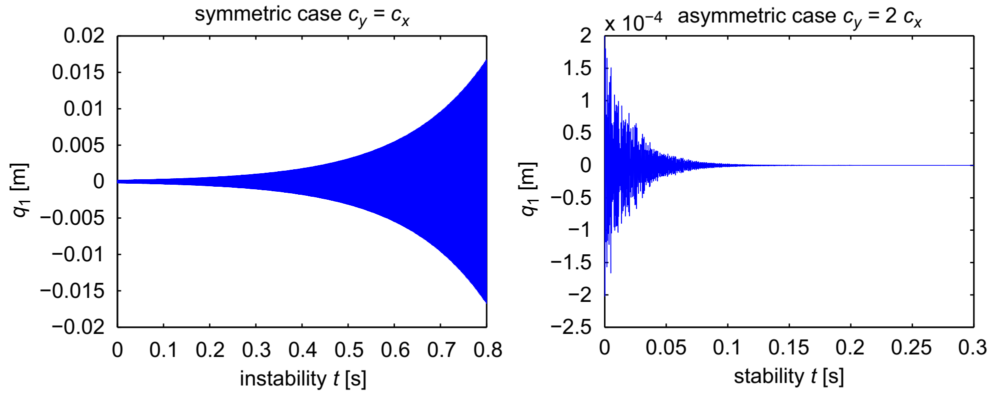

Fig. 6. Numerical integration of the equations of motion.

and are a special case of Eqs. (1a) if the terms from the pins are interpreted as perturbations. The analysis of Section 2 shows that asymmetric bearing stiffnesses have a stabilizing effect. This remains true also if we drop the assumption that the mass m of the pins is zero, which is more realistic for applications and is easily verified by numerical integration as can be seen in Fig. 6. The results were calculated using the parameters

$$c_x = 10^8\,\mathrm{N/m},\quad c_y = 2\times 10^8\,\mathrm{N/m},\quad N_0 = 2\,\mathrm{N},\quad k = 10^8\,\mathrm{N/m},\quad M = 1\,\mathrm{kg},$$

$$\mu = 0.8,\quad \Omega = 500\,1/\mathrm{s},\quad r = 0.1\,\mathrm{m},\quad m = 0.2\,\mathrm{kg}$$

for the asymmetric case. For the symmetric case $c_x = c_y = 10^8\,\mathrm{N/m}$ was used.

Summarizing it can be noted, that for the simple example studied, an easy design modification can be proposed such that the system is stabilized. If a slight damping is considered in the bearings the results also carry over to the nonlinear case since asymptotic stability can be observed from the linear equations. In continuous systems the choice of design modifications is not so obvious, therefore this case is addressed in the following sections.

4. Avoidance of self-excited vibrations in a rotating beam with periodic boundary conditions

One of the easiest continuous models for brake squeal is a rotating beam on an elastic foundation with periodic boundary conditions in contact with idealized friction pads [40]. A picture of the model is shown in Fig. 7 in which the elastic foundation is omitted. In a stationary coordinate system the equations of motion obtained from a Ritz discretization are given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + (\mathbf{K} + \Delta\mathbf{K})\mathbf{q} = \mathbf{0},\quad \mathbf{M} = \mathbf{M}^{\mathrm{T}},\quad \mathbf{G} = \mathbf{G}^{\mathrm{T}},\quad \mathbf{K} = \mathbf{K}^{\mathrm{T}},\quad \Delta\mathbf{K} \neq \Delta\mathbf{K}^{\mathrm{T}}, \tag{23a}$$

with

$$m_{ji} = \rho A \int_0^{2\pi} W_j W_i \,\mathrm{d}\phi, \tag{23b}$$

$$g_{ji} = \frac{\rho A \Omega}{a} \int_0^{2\pi} (W_j W_i' - W_j' W_i)\,\mathrm{d}\phi, \tag{23c}$$

$$k_{ji} = -\frac{\rho A \Omega^2}{a^2} \int_0^{2\pi} W_j' W_i' \,\mathrm{d}\phi + \frac{EI}{a^4} \int_0^{2\pi} W_j'' W_i'' \,\mathrm{d}\phi + \int_0^{2\pi} c W_j W_i \,\mathrm{d}\phi,$$

$$\Delta k_{ji} = -\frac{hk\mu}{a} W_j'(0) W_i(0) + \frac{hN_0}{a^2}(1+\mu^2) W_j'(0) W_i'(0) - \frac{h^2 N_0 \mu}{2a^3} W_j''(0) W_i'(0) \tag{23d}$$

in agreement with the results derived in [5,41]. It can be analytically seen that the stability domain of the trivial solution of (23a) is a set of measure zero in the space of parameters [5,38,39] and therefore self-excited vibrations are to be expected. The equations of motion only have constant coefficients since the beam was assumed to be rotationally symmetric i.e. the bending stiffness EI and the stiffness of the elastic bedding c were constant.

For the asymmetric problem it is beneficial to work in a coordinate system rotating with the beam in which the discretized equations of motion read

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{K} + \Delta\mathbf{K}(t))\mathbf{q} = \mathbf{0},\quad \mathbf{M} = \mathbf{M}^{\mathrm{T}},\quad \mathbf{K} = \mathbf{K}^{\mathrm{T}},\quad \Delta\mathbf{K}(t) \neq \Delta\mathbf{K}^{\mathrm{T}}(t), \tag{24}$$

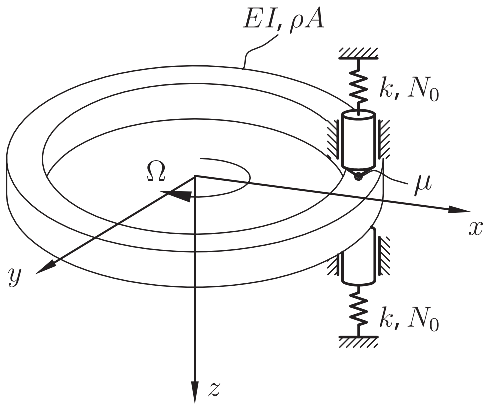

Fig. 7. Rotating beam in frictional contact.

where the matrix entries are given by

$$m_{ji} = \rho A \int_0^{2\pi} W_j W_i \, \mathrm{d}\phi, \tag{25a}$$

$$k_{ji} = \int_0^{2\pi} \frac{EI}{a^4} W_j'' W_i'' \, \mathrm{d}\phi + c \int_0^{2\pi} W_j W_i \, \mathrm{d}\phi, \tag{25b}$$

$$\Delta k_{ji} = 2kW_j(\Omega t)W_i(\Omega t) - \frac{hk\mu}{a} W_j'(\Omega t)W_i(\Omega t) + \frac{hN_0(1+\mu^2)}{a^2} W_j'(\Omega t)W_i'(\Omega t) - \frac{h^2N_0\mu}{2a^3} W_j''(\Omega t)W_i'(\Omega t). \tag{25c}$$

It is interesting to note that for this example gyroscopic terms appear in the stationary frame and not in the rotating frame which is opposite to what was observed in the discrete example studied in Section 3. The reason is that in this example the gyroscopic terms arise due to the transportation of material whereas in the discrete example they were due to Coriolis forces.

As shape functions for the Ritz discretization mentioned above we take eigenfunctions of the beam without pads which is described by the boundary value problem

$$\rho A \ddot{w} + \frac{EI}{a^4} w^{IV} + cw = 0, \tag{26a}$$

$$w(0,t) = w(2\pi,t), \quad w'(0,t) = w'(2\pi,t), \tag{26b}$$

$$w''(0,t) = w''(2\pi,t), \quad w'''(0,t) = w'''(2\pi,t). \tag{26c}$$

It is easily shown analytically that the eigenvalues ω_k appear in semi-simple pairs with corresponding eigenfunctions

$$w_k^c = \cos(k\phi), \quad w_k^s = \sin(k\phi).$$

From the structure of Eqs. (24) and the analysis in Section 2 it is clear that one should separate the pairs of semi-simple eigenvalues to avoid self-excited vibrations. One way to achieve this goal is to modify the bending stiffness *EI* of the beam. For mathematical convenience we agree to leave the area of the cross-section constant and to just vary the shape. Furthermore we would like to keep our modifications point symmetric with respect to the axis of rotation in order to keep the structure balanced. The discretized equations of the beam expanded in $w_k^c = \cos(k\phi)$ and $w_k^s = \sin(k\phi)$ read

$$\begin{bmatrix} \mathbf{M}^c & 0 \\ 0 & \mathbf{M}^s \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} \mathbf{K}^c & 0 \\ 0 & \mathbf{K}^s \end{bmatrix} \mathbf{q} = \mathbf{0}, \tag{27a}$$

with

$$m_{ij}^c = \rho A \pi, \quad m_{ij}^s = \rho A \pi, \tag{27b}$$

$$k_{ij}^c = \int_0^{2\pi} \frac{EI(\phi)}{a^4} w_i''^c w_j''^c + c w_i^c w_j^c \, \mathrm{d}\phi, \tag{27c}$$

$$k_{ij}^s = \int_0^{2\pi} \frac{EI(\phi)}{a^4} w_i''^s w_j''^s + c w_i^s w_j^s \, \mathrm{d}\phi. \tag{27d}$$

A good solution for the choice of bending stiffness is given by

$$EI(\phi) = EI_0 + \sum_{k=1}^{N} EI_k \cos^2 k\phi = EI_0 + \sum_{k=1}^{N} EI_k \frac{1}{2}(1+\cos 2k\phi) \tag{28}$$

which is a point symmetric modification.and $EI_k << EI_o$ is assumed.

Making use of the orthogonality relations for trigonometric functions

$$\int_0^{2\pi} \cos(2k\phi)\cos(l\phi)\cos(m\phi)\,\mathrm{d}\phi = \begin{cases} \frac{\pi}{2} & l+m=2k, \\ \frac{\pi}{2} & m-l=2k, \\ \frac{\pi}{2} & l-m=2k, \\ 0 & \text{otherwise}, \end{cases}$$

$$\int_0^{2\pi} \cos(2k\phi)\sin(l\phi)\sin(m\phi)\,\mathrm{d}\phi = \begin{cases} -\frac{\pi}{2} & l+m=2k, \\ \frac{\pi}{2} & m-l=2k, \\ \frac{\pi}{2} & l-m=2k, \\ 0 & \text{otherwise}, \end{cases}$$

$$\int_0^{2\pi} \cos(2k\phi)\sin(l\phi)\cos(m\phi)\,\mathrm{d}\phi = 0,$$

we see that the terms $EI_k\frac{1}{2}(1+\cos 2k\phi)$ influence the matrices corresponding to the sine and the cosine terms in Eq. (27a) differently on the main diagonal of the stiffness matrix.

Since $\mathbf{K}$ is diagonal dominant, the modification $EI_k\cos(2k\phi)$ yields

$$\omega_k^{*c2} \approx \omega_k^2 + \frac{EI_k}{2\rho h_k}\left(\frac{k}{a}\right)^4, \quad \omega_k^{*s2} \approx \omega_k^2 - \frac{EI_k}{2\rho h_k}\left(\frac{k}{a}\right)^4,$$

which can also be derived arguing with perturbation theory and interpreting EI_k as a perturbation parameter. We see that the eigenfrequency of the cosine mode is increased while the eigenfrequency of the sine mode is decreased. Since the functions $\cos k\phi$ form a base of $L^2_{\mathrm{sym}}(0,2\pi)$, the functions $\cos 2k\pi$ form a base of all point symmetric functions on $L^2(0,2\pi)$. So at least up to a constant any point symmetric shape can be constructed by (28).

Although being efficient from a mathematical perspective, the modifications of the circular beam may not be easy to realize from a manufacturing point of view. Therefore the results obtained in this section are more of a theoretical value showing that modifications exist which almost exclusively touch one pair of semi-simple eigenvalues without influencing the other pairs. In the next section the problem is addressed from a more application oriented point of view and a procedure of designing a brake disk is proposed.

5. Semi-analytic structural optimization of a brake disk

The model of the rotating circular beam studied in Section 4 is capable to describe the excitation mechanism of self-excited vibrations due to friction but is not a proper model for brake squeal since an important damping effect due to the three dimensional contact kinematics is missed [5]. Therefore in this section we investigate a disk brake model of a rotating Kirchhoff plate in contact with idealized friction pads developed and analyzed in [5,9] for the case of a rotationally symmetric brake rotor and in [29] for an asymmetric brake rotor. In this paper we concentrate on the equations, preliminary experiments are described in [42]. As discussed in [29], the discretized equations of motion have the form of Eq. (1a). Therefore, to avoid self-excited vibrations, we again try to separate eigenfrequencies of the rotor without pads. The goal of the structural optimization is to search for technically feasible modifications of the disk maintaining it balanced and strong enough to withstand the braking loads. A first approach towards a systematic structural optimization is based on the analytical models presented in [5,9] modeling the brake rotor as a rotating annular Kirchhoff plate clamped at the inner radius. The disk is discretized with a Ritz approach using the eigenfunctions of the nonrotating plate which correspond to eigenfrequencies in the relevant frequency range. Instead of evaluating the occurring integrals over the whole domain in one step, the plate is divided into sectors as demonstrated in Fig. 8. For each sector the height of the cooling rib enters the equations as a factor multiplying the occurring integrals. The discretized eigenvalue problem corresponding to the linearized equations of motion can therefore be written in a relatively simple form as

$$\det\left(\sum_n \mathbf{M}_n(1-a_n)\lambda^2 + \sum_n \mathbf{K}_n(1-a_n^3)\right) = 0, \tag{29}$$

$$(m_{ij})_n = \frac{\partial^2}{\partial \dot{q}_i \partial \dot{q}_j} \int_{-h/2}^{h/2} \int_{\varphi_{n-1}}^{\varphi_n} \int_{r_i}^{r_o} \rho \dot{w}(r,\varphi,t)^2 r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z, \tag{30}$$

$$(k_{ij})_n = \frac{\partial^2}{\partial q_i \partial q_j} \int_{-h/2}^{h/2} \int_{\varphi_{n-1}}^{\varphi_n} \int_{r_i}^{r_o} u_{\mathrm{el}}(r,\varphi,t) r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z \tag{31}$$

in which the heights of the cross-sections appear as analytic variables. We have gained an approximation for the eigenvalue problem depending on the heights of the ribs. At this point we remark that from a physical point of view the formulation only makes sense if the heights of the cross-sections are chosen in such a way that the Kirchhoff hypothesis for the cross-sections stays approximately valid.

Based on the discretized eigenvalue problem (29), we can formulate an optimization problem

$$\max_{a_n} \tau$$

$$\text{s.t.}$$

$$\frac{\omega_{k+1} - \omega_k}{\omega_k} \geq \tau, \quad \omega_{\hat{N}} \geq \omega_{\hat{N}-1} \geq \cdots \geq \omega_1,$$

$$0 \leq a_n \leq \hat{a} \quad \text{for all } n,$$

$$a_n = 0 \ \text{ for selected } n, \tag{32}$$

$$\sum_n a_n \cos\phi_n = 0, \quad \sum_n a_n \sin\phi_n = 0, \quad \phi_n = \left(n - \frac{2\pi}{N}\right).$$

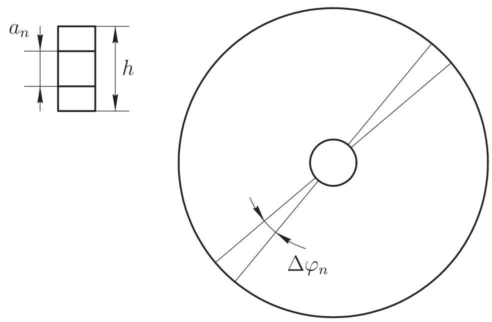

Fig. 8. Division of the brake disk in sectors.

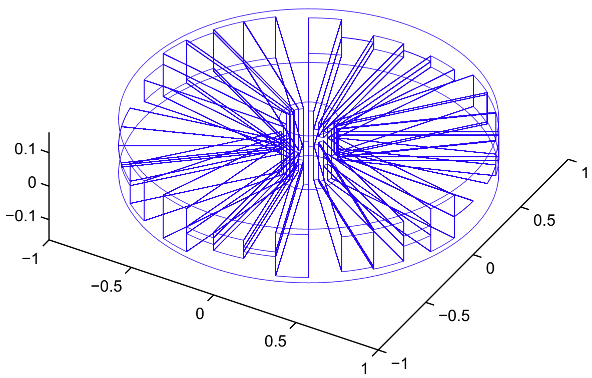

Fig. 9. Optimized disk (for N=36, $\hat{a} = 0.8h$, $\min \frac{\omega_{k+1} - \omega_k}{\omega_k} \approx 3.1\%$, 6 channels placed symmetric over the circumference are kept solid by constraint.

The objective function is to maximize the minimum relative distance between the eigenfrequencies of the rotor. The constraints ensure that the disk stays balanced and that at certain places no ribs occur.

The optimization problem is solved using the Matlab optimization toolbox using the parameters (nondimensionalized with the dimensionless time $\overline{t}=(t/r_o)\sqrt{E/\rho}$ and the radius $\overline{r}=r/r_o$)

$$\overline{r}_i=\frac{r_i}{r_o}=0.154,\quad \overline{r}_o=\frac{r_o}{r_o}=1,\quad \overline{h}=\frac{h}{r_o}=0.1605,\quad \nu=0.3 \tag{33}$$

which are based on parameters for a disk brake used in [5]. They correspond to a brake rotor represented by an equivalent Kirchhoff plate with r_o=0.162 m, E=4.16 $\times 10^{10}$ N/m, $\rho=4846\,\text{kg/m}^3$. These parameters were experimentally identified in [43].

Fig. 9 shows an optimized disk using 39 shape functions corresponding to eigenfrequencies in the range up to 17 kHz of the solid plate, and 36 sectors. Since the eigenfrequencies increase with optimization, this covers the audible frequency range. In this configuration it was possible to obtain a minimum relative difference for the optimized eigenfrequencies of 3.1 percent. We remark that the formulation of the optimization problem in (32) is superior to the one used in [42]. By limiting the optimization to sectors of a plate, the design freedom is only exploited in a very narrow manner and not practical from a manufacturing point of view. Future work therefore aims in the direction of solving an inverse problem for the rotating plate under constraints. Preliminary experiments carried out in [42] and the work of Fieldhouse [32,33] indicate that the effect can also be observed in experiments.

6. Conclusion

In this paper an approach for passive suppression of self-excited vibrations was discussed. For a general conservative linear mechanical system it was shown that the structure is extremely sensitive to skew-symmetric periodic parametrical forcing if multiple semi-simple eigenvalues are present. Approximations to the stability boundaries were calculated and passive constructive measures for the avoidance of squeal were presented using typical mechanical examples. The examples show the effectiveness of the approach suggesting a more sophisticated and practically orientated structural optimization for the avoidance of multiple eigenvalues for future work.

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