Journal of Sound and Vibration 329 (2010) 5321–5345

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Nonlinear dynamics of higher-dimensional system for an axially accelerating viscoelastic beam with in-plane and out-of-plane vibrations

L.H. Chen, W. Zhang *, F.H. Yang

College of Mechanical Engineering, Beijing University of Technology, Beijing 100022, PR China

ARTICLE INFO

Article history:
Received 29 March 2010
Received in revised form
23 June 2010
Accepted 26 June 2010
Handling Editor: L.G. Tham
Available online 27 July 2010

ABSTRACT

In this paper, the bifurcations and chaotic motions of higher-dimensional nonlinear systems are investigated for the nonplanar nonlinear vibrations of an axially accelerating moving viscoelastic beam. The Kelvin viscoelastic model is chosen to describe the viscoelastic property of the beam material. Firstly, the nonlinear governing equations of nonplanar motion for an axially accelerating moving viscoelastic beam are established by using the generalized Hamilton's principle for the first time. Then, based on the Galerkin's discretization, the governing equations of nonplanar motion are simplified to a six-degree-of-freedom nonlinear system and a three-degree-of-freedom nonlinear system with parametric excitation, respectively. At last, numerical simulations, including the Poincare map, phase portrait and Lyapunov exponents are used to analyze the complex nonlinear dynamic behaviors of the axially accelerating moving viscoelastic beam. The bifurcation diagrams for the in-plane and out-of-plane displacements via the mean axial velocity, the amplitude of velocity fluctuation and the frequency of velocity fluctuation are respectively presented when other parameters are fixed. The Lyapunov exponents are calculated to identify the existence of the chaotic motions. From the numerical results, it is indicated that the periodic, quasi-periodic and chaotic motions occur for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam. Observing the in-plane nonlinear vibrations of the axially accelerating moving viscoelastic beam from the numerical results, it is found that the nonlinear responses of the six-degree-of-freedom nonlinear system are much different from that of the three-degree-of-freedom nonlinear system when all parameters are same.

1. Introduction

Axially moving beams can represent many engineering devices, such as band saws and serpentine belts, and have been widely utilized to transmit power and motion. With the development of engineering materials, axially moving beams are usually composed of some metallic or reinforcement materials like glass-cord and polymeric materials which have long-chain molecules and viscoelastic behavior, for example, rubber. The modeling of dissipative mechanisms also is an important research topic in the nonlinear vibrations of axially moving viscoelastic materials. The theory of viscoelasticity is

* Corresponding author.
E-mail address: sandyzhang0@yahoo.com (W. Zhang).

0022-460X/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2010.06.023

an effective approach to model the dissipative mechanism in the nonlinear vibrations of axially moving viscoelastic materials. To accurately describe the material property of axially moving viscoelastic beams, viscoelastic constitutive relations such as the differential or integral constitutive relations must be employed. Axially moving acceleration often appears in axially moving viscoelastic beams. For example, using an axially moving beam models a belt on a pair of rotating pulleys, the rotation vibration of the pulleys will result in a small fluctuation in the axial speed of the belt. The nonlinear dynamical characteristics of axially accelerating moving viscoelastic beams have significant influence on the stability and reliability of mechanical systems. The transverse nonlinear oscillations of axially accelerating moving viscoelastic beams occupy main aspects of the nonlinear dynamical characteristics. The influence of the periodic fluctuation of the rotating speed and the torsional vibration of the pulleys on the transverse nonlinear oscillations of axially accelerating moving viscoelastic beams should be taken into account. Therefore, the investigation on the transverse nonlinear vibrations of axially accelerating moving viscoelastic beams is a challenging subject and is still of interest today.

Considerable attention has been paid to research works on the nonlinear dynamics of axially moving beams or strings by many investigators in the past forty years. With the constant mean axial transport velocity, the dynamic responses of axially moving beams have been studied extensively. For axially accelerating moving beams, Pakdemirli et al. [1] investigated the dynamic stability for transverse vibration of an axially accelerating string based on the Galerkin truncation. Pellicano and Zirilli [2] analyzed the boundary layers and nonlinear vibrations of an axially moving beam with vanishing, flexural stiffness and weak nonlinearities. Pakdemirli and Ulsoy [3] applied the discretization-perturbation method and the direct-perturbation method for analyzing the stability of an axially accelerating string. Oz et al. [4] employed the method of multiple scales to study the dynamic stability and nonlinear responses of an axially accelerating beam with small bending stiffness. Using the one-term Galerkin discretization, Ravindra and Zhu [5] analyzed the chaotic behaviors of axially accelerating beams. Özkaya and Pakdemirli [6] utilized the method of multiple scales and the method of matched asymptotic expansions to construct non-resonant boundary layer solutions of an axially accelerating beam with small bending stiffness. Oz and Pakdemirli [7] and Oz [8] used the method of multiple scales to analytically calculate the stability boundaries of an axially accelerating beam under simply supported conditions and fixed–fixed conditions, respectively. Parker and Lin [9] adopted the one-term Galerkin discretization and the perturbation method to study the dynamic stability of an axially accelerating moving beam subjected to a tension fluctuation. Oz et al. [10] employed the method of multiple scales to determine the steady-state transverse nonlinear responses and their stability of axially accelerating moving beams. Ozkaya and Oz [11] applied artificial neural network algorithm for analyzing the stability of an axially accelerating beam.

Throughout the history of research, all above-mentioned researchers assumed axially accelerating moving strings or beams under their consideration to be elastic and did not take into account any dissipative mechanisms for axially accelerating moving materials. Nevertheless, the modeling of dissipative mechanisms is an important research topic for the nonlinear vibrations of axially accelerating moving materials. The nonlinear parametric vibrations of axially moving viscoelastic strings have been investigated by several researchers. Using the three-term Galerkin discretization, Marynowski [12] and Marynowski and Kapitaniak [13] gave a comparison among the Kelvin model, the Maxwell model and the Bugers model. They used numerical method to simulate the nonlinear responses of an axially moving beam excited by a changing tension and found that all models yield similar results in the case of small damping. Marynowski [14] further numerically studied the nonlinear dynamical behaviors of an axially moving viscoelastic beam with time-dependent tension using the four-term Galerkin discretization. However, the literatures related to axially accelerating moving viscoelastic beams are relatively limited. Based on the second-term Galerkin discretization, Chen et al. [15] analyzed the stability of axially accelerating linear beams. Yang and Chen [16,17] studied the steady-state nonlinear responses, bifurcations and chaos of an axially accelerating moving viscoelastic beam. Although several research works on the relevant topic in the field have been carried out in recent years, effort in analyzing the nonplanar nonlinear transverse vibrations of axially accelerating moving viscoelastic beams is lacking. This situation necessitates the need for the study presented in this paper. Through the observation of our experimental results, it is found that the nonlinear vibrations of axially accelerating moving viscoelastic beams are not located in the in-plane. Mockensturm and Guo [18] utilized the material time derivative in the viscoelastic constitutive relation to analyze the nonlinear dynamic responses of parametrically excited, axially moving viscoelastic belts and gave a comparison to previous work.

Recently, Chen et al. [19] established the governing equations of motion for the nonplanar nonlinear vibrations of an axially moving viscoelastic belt by using the generalized Hamilton's principle for the first time. The problem for using the generalized Hamilton's principle to establish the governing equations of motion for the nonplanar nonlinear vibrations of an axially moving viscoelastic belt was solved. Based on the results mentioned above, Liu et al. [20] investigated the periodic and chaotic oscillations of an axially moving viscoelastic belt with one-to-one internal resonance in three-dimensional space. Zhang and Song [21] studied higher-dimensional periodic and chaotic oscillations for a parametrically excited viscoelastic moving belt with multiple internal resonances. Marynowski and Kapitaniak [22] investigated the modeling and nonlinear responses of an axially moving viscoelastic beam with time-dependent tension and Zener internal damping. Chen et al. [23] gave a survey on the advances in the nonlinear dynamics of transverse motion of axially moving strings belts. Gao et al. [24] modified the higher-dimensional Melnikov method to investigate the single-pulse homoclinic orbits and chaotic dynamics for the six-dimensional nonlinear system for an axially moving viscoelastic belt. Chen and

Ding [25] analyzed the steady-state transverse responses in coupled planar nonlinear vibrations of an axially moving viscoelastic beam.

This paper investigates the complex dynamic behaviors for the nonplanar nonlinear transverse vibrations of an axially accelerating moving viscoelastic beam. In Section 2, the governing equations of motion for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam are established by using the generalized Hamilton's principle for the first time. The governing equations of motion, which are changed to a set of ordinary differential governing equations with three-degree-of-freedom and six-degree-of-freedom, are obtained in Section 3. Section 4 describes investigation of the nonlinear dynamical behaviors of the axially accelerating moving viscoelastic beam in a general way, including the bifurcation diagrams, the largest Lyapunov exponents, Poincare maps and phase portraits. Finally, the results and conclusions are given.

2. Formula of axially accelerating moving viscoelastic beam

When the bending stiffness of an axially accelerating moving viscoelastic belt cannot be neglected, we use an axially accelerating moving viscoelastic beam to model the belt driving device, as shown in Fig. 1. Therefore, the governing equations of motion for the nonlinear vibrations of the axially accelerating moving viscoelastic belt are based on an axially moving beam model. In the axially accelerating moving viscoelastic beam, it is indicated that A is cross-sectional area of the beam, L is free length of the beam between two supports, ρ is the mass density of the beam and $c(t)$ is the axial velocity of the beam. A Cartesian coordinate system, $Oxyz$, is adopted. Another coordinate system is a moving coordinate fixed on the beam. The u, v and w are the displacements with respect to moving coordinate in the x, y and z directions, respectively. It is assumed that the tension P is characterized as a small periodic perturbation $P_1 \cos\omega t$ on the steady-state tension P_0, namely, $P=P_0+P_1\cos\omega t$. It is also assumed that the moving speed of the beam is characterized as a simple harmonic change about the constant mean speed, namely, $c(t)=c_0+c_1\cos\omega t$. This assumption has its physical meaning. For example, if we use the axially moving beam to model a belt on a pair of rotating pulleys, the rotation vibration of the pulleys will result in a small fluctuation in the axial velocity of the belt.

The displacement components of any point (x,y,z) in the axially accelerating moving viscoelastic beam are obtained as follows:

$$u(x,t)=u_0(x,t)+z\frac{\partial w}{\partial x}+y\frac{\partial v}{\partial x}, \tag{1a}$$

$$w(x,t)=w_0(x,t), \tag{1b}$$

$$v(x,t)=v_0(x,t), \tag{1c}$$

where u_0, v_0 and w_0 are the displacement components in the x, y and z directions on the axial line of the axially accelerating moving beam.

In this paper, the Lagrangian strain for the beam is employed as a finite measure of the strain. The geometric nonlinearity depending on finite stretching is considered through the Lagrangian strain. The axial Lagrangian strain of the axially accelerating moving beam is given by

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \tag{2}$$

It is assumed that the viscoelastic material of the axially accelerating moving beam is homogeneous and satisfies the Kelvin model. Therefore, the constitution relation is given as follows:

$$\sigma=E_0\varepsilon_x+\eta\frac{\partial\varepsilon_x}{\partial t}=\sigma_e+\sigma_v, \tag{3}$$

where η is the dynamic viscous coefficient, σ_e=$E_0\varepsilon$ and σ_v=$\eta(\partial\varepsilon/\partial t)$, respectively, represent the constant stress and the time-varying stress.

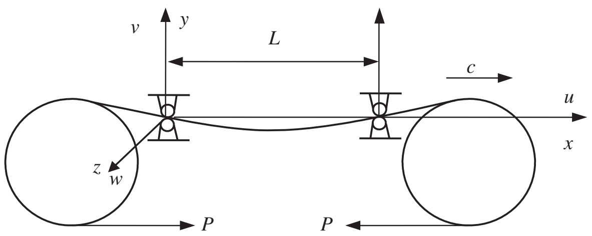

Fig. 1. The model of an axially moving viscoelastic beam is given.

The axially moving velocity of the beam is represented by $c(t)$ which is varying with respect to the time t. Then, the absolute velocity of any point for the axially moving beam is given as follows:

$$\frac{dx}{dt}=c+c\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}, \tag{4}$$

$$\frac{dy}{dt}=c\frac{\partial v}{\partial x}+\frac{\partial v}{\partial t}, \tag{5}$$

$$\frac{dz}{dt}=c\frac{\partial w}{\partial x}+\frac{\partial w}{\partial t}. \tag{6}$$

In 2007, Chen et al. [19] used the generalized Hamilton's principle to establish the governing equations of motion for the nonplanar nonlinear vibrations of the axially moving viscoelastic belt for the first time. In the following analysis, the generalized Hamilton's principle [19,26] will be used to obtain the nonlinear governing equations of motion for the axially accelerating moving viscoelastic beam. The governing equation of motion in the context of the Newtonian mechanics can be derived for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam. However, this derivation requires a free body diagram for a differential element of mass and the use of sign rule for forces and moments. Therefore, it is considered that this derivation is not always easy. Using the generalized Hamilton's principle, we can avoid such problems. The mathematical statement of the generalized Hamilton's principle is given as follows:

$$\int_{t_1}^{t_2}(\delta T-\delta U+\delta W)dt=0, \tag{7}$$

where δT represents the variation of the kinetic energy, δU indicates the variation of the elastic strain energy or elastic potential energy, and δW denotes the virtual work performed by both external and viscous dissipative forces.

From Eq. (3), we know that the variation of elastic potential energy is of the form

$$\delta U=\int_V \sigma_e \delta\varepsilon\, dV=\int_0^L\int_A \sigma_e\delta\varepsilon\, dA\, dx. \tag{8}$$

The virtual work performed by both external forces and viscous dissipative damping for the axially accelerating moving viscoelastic beam is represented as

$$\delta W=\delta W_{ext}+\delta W_{vis}=-\int_0^L\int_A P\delta\varepsilon\, dA\, dx-\int_0^L\int_A \sigma_v A\delta\varepsilon\, dA\, dx, \tag{9}$$

where $\sigma_v A$ is the viscous dissipative force corresponding to the time-varying stress or the past history stress at any position (x,y,z) for the axially accelerating moving beam.

The kinetic energy of the axially accelerating moving viscoelastic beam is given as follows:

$$T=\int_0^L\frac{1}{2}\rho A\left[\dot{x}^2+\dot{y}^2+\dot{z}^2\right]dx=\int_0^L\frac{1}{2}\rho A\left[(c+cu_{,x}+u_{,t})^2+(cv_{,x}+v_{,t})^2+(cw_{,x}+w_{,t})^2\right]dx. \tag{10}$$

For the expression's convenience, we drop the subscript '0', which denotes the displacement on the axial line of the axially accelerating moving viscoelastic beam. The variation of the kinetic energy is obtained as follows:

$$\delta T=\delta\int_0^L\frac{1}{2}\rho A\left[(c+cu_{,x}+u_{,t})^2+(cv_{,x}+v_{,t})^2+(cw_{,x}+w_{,t})^2\right]dx. \tag{11}$$

We first derive the governing equations of three-dimensional nonlinear vibrations for the axially accelerating moving viscoelastic beam. Substituting Eqs. (8), (9) and (11) into Eq. (7) yields

$$\int_{t_1}^{t_2}\int_0^L\frac{\rho A}{2}\delta\left[(c+cu_{,x}+u_{,t})^2+(cv_{,x}+v_{,t})^2+(cw_{,x}+w_{,t})^2\right]dx\,dt-\int_{t_1}^{t_2}\int_0^L\int_A(P\delta\varepsilon+\sigma_e\delta\varepsilon+\sigma_v\delta\varepsilon)\,dx\,dt=0. \tag{12}$$

Using Eq. (3), then, Eq. (12) can be rewritten as

$$\int_{t_1}^{t_2}\int_0^L\frac{\rho A}{2}\delta\left[(c+cu_{,x}+u_{,t})^2+(cv_{,x}+v_{,t})^2+(cw_{,x}+w_{,t})^2\right]dx\,dt-\int_{t_1}^{t_2}\int_0^L\int_A(P\delta\varepsilon+\sigma\delta\varepsilon)\,dx\,dt=0. \tag{13}$$

It is assumed that the variation and differentiation are interchangeable. The boundary conditions with respect to time are obtained as

$$\delta u(x,t_1)=\delta u(x,t_2)=0,\quad \delta v(x,t_1)=\delta v(x,t_2)=0,\quad \delta w(x,t_1)=\delta w(x,t_2)=0. \tag{14}$$

The boundary conditions with respect to displacement are denoted as

$$u(0,t)=u(L,t)=0,\quad v(0,t)=v(L,t)=0,\quad w(0,t)=w(L,t)=0, \tag{15}$$

$$v_{,xx}(0,t)=v_{,xx}(L,t)=0,\quad w_{,xx}(0,t)=w_{,xx}(L,t)=0. \tag{16}$$

The governing equations of three-dimensional nonlinear vibrations for the axially accelerating moving viscoelastic beam are obtained in the following three equations:

$$\rho A(u_{,tt}+2cu_{,xt}+c^2u_{,xx}+c_{,t}u_{,x}+c_{,t})-\int_A \sigma_{,x}\,dA=0,\quad 0<x<L, \tag{17a}$$

$$\rho A(v_{,tt}+2cv_{,xt}+c^2v_{,xx}+c_{,t}v_{,x})-PAv_{,xx}-\int_A [(\sigma v_{,x})_{,x}+y\sigma_{,xx}]\,dA=0,\quad 0<x<L, \tag{17b}$$

$$\rho A(w_{,tt}+2cw_{,xt}+c^2w_{,xx}+c_{,t}w_{,x})-PAw_{,xx}-\int_A [(\sigma w_{,x})_{,x}+z\sigma_{,xx}]\,dA=0,\quad 0<x<L. \tag{17c}$$

Eq. (17) is a general form of the governing equations of three-dimensional nonlinear vibrations for the axially accelerating moving viscoelastic beam, which can be satisfied by both the differential and integral constitutive relations of viscoelastic materials.

The inertia moments of a cross-sectional area for the axially accelerating moving viscoelastic beam are given as follows:

$$J_{yz}=\int_A yz\,dA,\quad J_y=\int_A z^2\,dA,\quad J_z=\int_A y^2\,dA. \tag{18}$$

Using the Kelvin type viscoelastic constitutive law and substituting Eq. (3) into Eq. (17), we obtain the governing equations of three-dimensional nonlinear vibrations for the axially accelerating moving viscoelastic beam as

$$\rho A(u_{,tt}+2cu_{,xt}+c^2u_{,xx}+c_{,t}u_{,x}+c_{,t})=E_0A(u_{,xx}+v_{,x}v_{,xx}+w_{,x}w_{,xx})+\eta A(u_{,xxt}+v_{,xt}v_{,xx}+v_{,x}v_{,xxt}+w_{,xt}w_{,xx}+w_{,x}w_{,xxt}), \tag{19a}$$

$$\begin{aligned}&\rho A(v_{,tt}+2cv_{,xt}+c^2v_{,xx}+c_{,t}v_{,x})+J_z(E_0v_{,xxxx}+\eta v_{,xxxxt})+J_{yz}(E_0w_{,xxxx}+\eta w_{,xxxxt})\\&\quad=PAv_{,xx}+E_0A\left(v_{,x}u_{,xx}+\frac{3}{2}v_{,x}^2v_{,xx}+v_{,x}w_{,x}w_{,xx}+v_{,xx}u_{,x}+\frac{1}{2}w_{,x}^2v_{,xx}\right)\\&\qquad+\eta A(v_{,x}u_{,xxt}+2v_{,x}v_{,xt}v_{,xx}+v_{,x}^2v_{,xxt}+v_{,x}w_{,xt}w_{,xx}+v_{,x}w_{,x}w_{,xxt}+v_{,xx}u_{,xt}+v_{,xx}w_{,xt}w_{,x}),\end{aligned} \tag{19b}$$

$$\begin{aligned}&\rho A(w_{,tt}+2cw_{,xt}+c^2w_{,xx}+c_{,t}w_{,x})+J_y(E_0w_{,xxxx}+\eta w_{,xxxxt})+J_{yz}(E_0v_{,xxxx}+\eta v_{,xxxxt})\\&\quad=PAw_{,xx}+E_0A\left(w_{,x}u_{,xx}+w_{,x}v_{,x}v_{,xx}+\frac{3}{2}w_{,x}^2w_{,xx}+w_{,xx}u_{,x}+\frac{1}{2}v_{,x}^2w_{,xx}\right)\\&\qquad+\eta A(w_{,x}u_{,xxt}+w_{,x}v_{,xt}v_{,xx}+w_{,x}v_{,x}v_{,xxt}+2w_{,x}w_{,xt}w_{,xx}+w_{,x}^2w_{,xxt}+w_{,xx}u_{,xt}+v_{,x}v_{,xt}w_{,xx}).\end{aligned} \tag{19c}$$

Eqs. (19a)–(19c) respectively describe the longitudinal vibration, the in-plane and the out-of-plane transverse nonlinear vibrations of the axially accelerating moving viscoelastic beam with the Kelvin type viscoelastic constitutive law.

Generally, the transverse nonlinear vibrations are primary motions of the axially accelerating moving viscoelastic beam. Therefore, the longitudinal vibration can be ignored in some studies. The governing equations of the nonplanar nonlinear transverse vibrations, including the in-plane and out-of-plane nonlinear vibrations, are given by

$$\begin{aligned}&\rho A(v_{,tt}+2cv_{,xt}+c^2v_{,xx}+c_{,t}v_{,x})+J_z(E_0v_{,xxxx}+\eta v_{,xxxxt})+J_{yz}(E_0w_{,xxxx}+\eta w_{,xxxxt})\\&\quad=PAv_{,xx}+E_0A\left(\frac{3}{2}v_{,x}^2v_{,xx}+v_{,x}w_{,x}w_{,xx}+\frac{1}{2}w_{,x}^2v_{,xx}\right)\\&\qquad+\eta A(2v_{,x}v_{,xt}v_{,xx}+v_{,x}^2v_{,xxt}+v_{,x}w_{,xt}w_{,xx}+v_{,x}w_{,x}w_{,xxt}+v_{,xx}w_{,xt}w_{,x}),\end{aligned} \tag{20a}$$

$$\begin{aligned}&\rho A(w_{,tt}+2cw_{,xt}+c^2w_{,xx}+c_{,t}w_{,x})+J_y(E_0w_{,xxxx}+\eta w_{,xxxxt})+J_{yz}(E_0v_{,xxxx}+\eta v_{,xxxxt})\\&\quad=PAw_{,xx}+E_0A\left(w_{,x}v_{,x}v_{,xx}+\frac{3}{2}w_{,x}^2w_{,xx}+\frac{1}{2}v_{,x}^2w_{,xx}\right)\\&\qquad+\eta A(w_{,x}v_{,xt}v_{,xx}+w_{,x}v_{,x}v_{,xxt}+2w_{,x}w_{,xt}w_{,xx}+w_{,x}^2w_{,xxt}+v_{,x}v_{,xt}w_{,xx}).\end{aligned} \tag{20b}$$

In order to compare the nonplanar nonlinear transverse vibrations with the in-plane nonlinear transverse vibrations, the governing equation of the in-plane nonlinear transverse vibrations for the axially accelerating moving viscoelastic beam is given by

$$\rho A(v_{,tt}+2cv_{,xt}+c^2v_{,xx}+c_{,t}v_{,x})+J_y(E_0v_{,xxxx}+\eta v_{,xxxxt})=PAv_{,xx}+\frac{3}{2}E_0Av_{,x}^2v_{,xx}+\eta A(2v_{,x}v_{,xt}v_{,xx}+v_{,x}^2v_{,xxt}). \tag{21}$$

3. Galerkin discretization

Using the three-term Galerkin discretization, the nonplanar transverse displacements v and w can be expressed as follows [15]:

$$v=\sin\frac{\pi x}{l}q_1(t)+\sin\frac{2\pi x}{l}q_2(t)+\sin\frac{3\pi x}{l}q_3(t), \tag{22a}$$

$$w=\sin\frac{\pi x}{l}q_4(t)+\sin\frac{2\pi x}{l}q_5(t)+\sin\frac{3\pi x}{l}q_6(t). \tag{22b}$$

Then, substituting Eq. (22) into Eq. (20) and applying the Galerkin approach to system (20), we obtain the following nonlinear ordinary differential governing equations of motion for the axially accelerating moving viscoelastic beam with six-degrees-of-freedom

$$\begin{aligned}
&\ddot{q}_1+\frac{\pi^4 J_y\eta}{\rho A l^4}\dot{q}_1+\frac{\pi^2}{\rho A l^2}\left(\frac{\pi^2 J_y E_0}{l^2}-\rho A c^2+P_0+P_1\cos\omega t\right)q_1+\frac{8}{3l}c_1\omega\sin\omega t q_2\\
&-\frac{16}{3l}(c_0+c_1\cos\omega t)\dot{q}_2+\frac{\pi^4 J_{yz}}{\rho A l^4}(E_0 q_4+\eta\dot{q}_4)+\frac{\pi^4 E_0}{\rho l^4}\left(\frac{3}{8}q_1^3+\frac{9}{8}q_1^2q_3+3q_2^2q_1+\frac{27}{4}q_3^2q_1\right.\\
&\left.+\frac{3}{8}q_4^2q_1+\frac{3}{4}q_1q_4q_6+q_5^2q_1+\frac{9}{4}q_6^2q_1+\frac{9}{2}q_2^2q_3+2q_2q_4q_5+3q_2q_6q_5+\frac{3}{8}q_4^2q_3+\frac{9}{2}q_3q_4q_6+\frac{3}{2}q_5^2q_3\right)\\
&+\frac{2\pi^4\eta}{\rho l^4}\left(\frac{3}{8}q_1^2\dot{q}_3+\frac{3}{8}q_1^2\dot{q}_1+2q_1q_2\dot{q}_2+\frac{9}{2}\dot{q}_3q_3q_1+\frac{3}{4}\dot{q}_1q_3q_1+\frac{3}{8}q_1q_4\dot{q}_4+\frac{3}{8}q_1q_4\dot{q}_6\right.\\
&+q_1q_5\dot{q}_5+\frac{3}{8}q_1q_6\dot{q}_4+\frac{9}{4}q_1q_6\dot{q}_6+\frac{3}{2}q_2^2\dot{q}_3+q_2^2\dot{q}_1+3\dot{q}_2q_2q_3+q_2q_4\dot{q}_5+\frac{3}{2}q_2q_5\dot{q}_6\\
&\left.+q_2q_5\dot{q}_4+\frac{3}{2}q_2q_6\dot{q}_5+\frac{3}{2}q_3^2\dot{q}_1+\frac{9}{4}\dot{q}_6q_4q_3+\frac{3}{8}q_3q_4\dot{q}_4+\frac{9}{4}q_3q_6\dot{q}_4+\frac{3}{2}q_3q_5\dot{q}_5\right)=0,
\end{aligned} \tag{23a}$$

$$\begin{aligned}
&\ddot{q}_2+\frac{16\pi^4 J_y\eta}{\rho A l^4}\dot{q}_2+\frac{2\pi^2}{\rho A l^2}\left(\frac{8\pi^2 J_y E_0}{l^2}-2\rho A c^2+2(P_0+P_1\cos\omega t)\right)q_2-\frac{8}{3l}c_1\omega\sin\omega t q_1\\
&+\frac{16}{3l}(c_0+c_1\cos\omega t)\dot{q}_1+\frac{24}{5l}\omega c_1\sin\omega t q_3-\frac{48}{5l}(c_0+c_1\cos\omega t)\dot{q}_3+\frac{16\pi^4 J_{yz}}{\rho A l^4}(E_0q_5+\eta\dot{q}_5)\\
&+\frac{2\pi^4 E_0}{\rho l^4}\left(\frac{3}{2}q_1^2q_2+\frac{9}{2}q_1q_2q_3+q_1q_4q_5+\frac{3}{2}q_1q_5q_6+3q_2^3+\frac{27}{2}q_3^2q_2+\frac{1}{2}q_4^2q_2+\frac{3}{2}q_2q_4q_6\right.\\
&\left.+3q_2q_5^2+\frac{9}{2}q_2q_6^2+\frac{3}{2}q_3q_4q_5+9q_3q_5q_6\right)+\frac{2\pi^4\eta}{\rho l^4}(q_1^2\dot{q}_2+2q_1q_2\dot{q}_1+3q_1q_2\dot{q}_3+3\dot{q}_2q_3q_1\\
&+\dot{q}_5q_4q_1+q_1q_5\dot{q}_4+\frac{3}{2}q_1q_5\dot{q}_6+3q_2q_3\dot{q}_1+18q_2q_3\dot{q}_3+q_2q_4\dot{q}_4+\frac{3}{2}q_2q_4\dot{q}_6+6q_2q_5\dot{q}_5\\
&+\frac{3}{2}\dot{q}_4q_2q_6+9q_2q_5\dot{q}_5+\frac{3}{2}q_1q_6\dot{q}_5+9q_3^2\dot{q}_2+\frac{3}{2}\dot{q}_5q_4q_3+\frac{3}{2}q_3q_5\dot{q}_4\\
&+9q_3q_6\dot{q}_5+9q_3q_5\dot{q}_6)=0,
\end{aligned} \tag{23b}$$

$$\begin{aligned}
&\ddot{q}_3+\frac{81\pi^4 J_y\eta}{\rho A l^4}\dot{q}_3+\frac{\pi^2}{\rho A l^2}\left(\frac{81\pi^2 J_y E_0}{l^2}-9\rho A c^2+9(P_0+P_1\cos\omega t)\right)q_3-\frac{24}{5l}\omega c_1\sin\omega t q_2\\
&+\frac{48}{5l}(c_0+c_1\cos\omega t)\dot{q}_2+\frac{81\pi^4 J_{yz}}{\rho A l^4}(E_0q_5+\eta\dot{q}_6)+\frac{2\pi^4 E_0}{\rho l^4}\left(\frac{3}{16}q_1^3+\frac{27}{8}q_1^2q_3+\frac{9}{4}q_2^2q_1\right.\\
&+\frac{3}{16}q_4^2q_1+\frac{3}{4}q_1q_5^2+\frac{9}{4}q_4q_6q_1+\frac{27}{2}q_2^2q_3+\frac{3}{2}q_2q_4q_5+9q_2q_6q_5+\frac{243}{16}q_3^3+\frac{9}{8}q_4^2q_3\\
&\left.+\frac{243}{16}q_6^2q_3\right)+\frac{2\pi^4\eta}{\rho l^4}\left(\frac{9}{4}q_1^2\dot{q}_3+\frac{3}{8}q_1^2\dot{q}_1+3q_1q_2\dot{q}_2+\frac{9}{2}\dot{q}_1q_3q_1+\frac{9}{4}\dot{q}_6q_4q_1+\frac{3}{8}q_1q_4\dot{q}_4\right.\\
&+\frac{9}{4}q_1q_6\dot{q}_4+\frac{3}{2}q_1q_5\dot{q}_5+\frac{3}{2}q_2^2\dot{q}_1+9q_2^2\dot{q}_3+18\dot{q}_2q_2q_3+\frac{3}{2}q_2q_4\dot{q}_5+\frac{3}{2}q_2q_5\dot{q}_4+9q_2q_5\dot{q}_6\\
&+9q_2q_6\dot{q}_5+\frac{243}{8}q_3^2\dot{q}_3+9\dot{q}_5q_5q_3+\frac{9}{4}q_3q_4\dot{q}_4+\frac{243}{8}q_3q_6\dot{q}_6)=0,
\end{aligned} \tag{23c}$$

$$\begin{aligned}
&\ddot{q}_4+\frac{\pi^4 J_z\eta}{\rho A l^4}\dot{q}_4+\frac{\pi^2}{\rho A l^2}\left(\frac{\pi^2 J_z E_0}{l^2}-\rho A c^2+P_0+P_1\cos\omega t\right)q_4-\frac{16}{3l}(c_0+c_1\cos\omega t)\dot{q}_5\\
&+\frac{8}{3l}\omega c_1\sin\omega t q_5+\frac{\pi^4 J_{zy}}{\rho A l^4}(E_0q_1+\eta\dot{q}_1)+\frac{2\pi^4 E_0}{\rho l^4}\left(\frac{3}{16}q_1^2q_6+\frac{3}{16}q_1^2q_4+q_1q_2q_5+\frac{9}{4}q_1q_3q_6\right.\\
&+\frac{3}{8}q_1q_3q_4+\frac{1}{2}q_2^2q_4+\frac{3}{4}q_2^2q_6+\frac{3}{2}q_2q_3q_5+\frac{9}{8}q_3^2q_4+\frac{3}{16}q_4^3+\frac{9}{16}q_4^2q_6+\frac{3}{2}q_5^2q_4+\frac{27}{8}q_6^2q_4\\
&\left.+\frac{9}{4}q_5^2q_6\right)+\frac{2\pi^4\eta}{\rho l^4}\left(\frac{3}{8}q_1q_4\dot{q}_1+\frac{3}{8}\dot{q}_3q_4q_1+\frac{9}{4}q_1q_6\dot{q}_3+\frac{3}{8}q_1q_6\dot{q}_1+\dot{q}_2q_2q_4+\frac{3}{2}q_2q_5\dot{q}_3+\frac{3}{8}q_4^2\dot{q}_4\right.\\
&+q_2q_5\dot{q}_1+\frac{3}{2}q_6q_2\dot{q}_2+\frac{9}{4}q_4q_3\dot{q}_3+\frac{3}{8}\dot{q}_1q_4q_3+\frac{3}{2}q_3q_5\dot{q}_2+\frac{9}{4}q_3q_6\dot{q}_1+\frac{3}{8}q_4^2\dot{q}_6+\dot{q}_2q_5q_1\\
&+2q_4q_5\dot{q}_5+\frac{3}{4}q_4q_6\dot{q}_4+\frac{9}{2}q_4q_6\dot{q}_6+q_5^2\dot{q}_4+\frac{3}{2}q_5^2\dot{q}_6+3\dot{q}_5q_5q_6+\frac{9}{4}q_5^2\dot{q}_4)=0,
\end{aligned} \tag{23d}$$

$$\ddot{q}_5+\frac{16\pi^4 J_z\eta}{\rho A l^4}\dot{q}_5+\frac{2\pi^2}{\rho A l^2}\left(\frac{8\pi^2 J_z E_0}{l^2}-2\rho A c^2+2(P_0+P_1\cos\omega t)\right)q_5-\frac{8}{3l}\omega c_1\sin\omega t q_4$$

$$+\frac{16}{3l}(c_0+c_1\cos\omega t)\dot q_4+\frac{24}{5l}\omega c_1\sin\omega t q_6-\frac{48}{5l}(c_0+c_1\cos\omega t)\dot q_6+\frac{16\pi^4 J_{zy}}{\rho A l^4}(E_0 q_2+\eta\dot q_2)$$
$$+\frac{2\pi^4 E_0}{\rho l^4}\left(\frac{1}{2}q_1^2q_5+q_1q_2q_4+\frac{3}{2}q_1q_2q_6+\frac{3}{2}q_1q_3q_5+3q_2^3q_5+9q_2q_3q_6+\frac{3}{2}q_2q_3q_4\right.$$
$$\left.+\frac{9}{2}q_5q_3^2+\frac{3}{2}q_5q_4^2+\frac{9}{2}q_4q_5q_6+3q_5^3+\frac{27}{2}q_5q_6^2\right)+\frac{2\pi^4\eta}{\rho l^4}\left(q_1q_4\dot q_2+q_1q_5\dot q_1+9q_6^2\dot q_5+\frac{3}{2}q_1q_5\dot q_3\right.$$
$$+\frac{3}{2}\dot q_2q_6q_1+\dot q_1q_4q_2+\frac{3}{2}\dot q_3q_2q_4+6q_2q_5\dot q_2+\frac{3}{2}q_2q_6\dot q_1+9q_2q_6\dot q_3$$
$$+\frac{3}{2}\dot q_2q_3q_4+3\dot q_4q_5q_6+9q_3q_5\dot q_3+9q_3q_6\dot q_2+q_4^2\dot q_5+3q_4q_5\dot q_6+2q_4q_5\dot q_4$$
$$\left.+3q_6q_4\dot q_5+6q_5^2\dot q_5+18q_6q_5\dot q_6+\frac{3}{2}\dot q_1q_3q_5\right)=0, \tag{23e}$$

$$\ddot q_6+\frac{81\pi^4 J_z\eta}{\rho A l^4}\dot q_6+\frac{\pi^2}{\rho A l^2}\left(\frac{81\pi^2 J_z E_0}{l^2}-9\rho A c^2+9(P_0+P_1\cos\omega t)\right)q_6+\frac{24}{5l}\omega c_1\sin\omega t q_5$$
$$+\frac{48}{5l}(c_0+c_1\cos\omega t)\dot q_5+\frac{81\pi^4 J_{zy}}{\rho A l^4}(E_0q_3+\eta\dot q_3)+\frac{2\pi^4E_0}{\rho l^4}\left(\frac{3}{16}q_1^2q_4+\frac{9}{8}q_1^2q_6+\frac{3}{2}q_2q_5q_1\right.$$
$$+\frac{9}{4}q_4q_3q_1+\frac{3}{4}q_4q_2^2+\frac{9}{2}q_2^2q_6+9q_2q_3q_5+\frac{243}{16}q_3^2q_6+\frac{3}{16}q_4^3+\frac{27}{8}q_4^2q_6+\frac{9}{4}q_5^2q_4$$
$$\left.+\frac{27}{2}q_5^2q_6+\frac{243}{16}q_6^3\right)+\frac{2\pi^4\eta}{\rho l^4}\left(\frac{3}{8}\dot q_1q_4q_1+\frac{9}{4}\dot q_3q_4q_1+\frac{3}{2}\dot q_2q_5q_1+\frac{9}{4}\dot q_1q_6q_1+\frac{3}{2}q_2q_4\dot q_2\right.$$
$$+\frac{3}{2}q_2q_5\dot q_1+9q_2q_5\dot q_3+9q_2q_6\dot q_2+\frac{9}{4}\dot q_1q_4q_3+9q_3q_5\dot q_2+\frac{243}{8}q_3q_6\dot q_3+\frac{9}{4}q_4^2\dot q_6+\frac{3}{8}q_4^2\dot q_4$$
$$\left.+\frac{9}{2}\dot q_4q_4q_6+3q_5q_4\dot q_5+18q_5q_6\dot q_5+9q_5^2\dot q_6+\frac{3}{2}q_5^2\dot q_4+\frac{243}{8}q_5^2\dot q_6\right)=0. \tag{23f}$$

System (23) can be used to describe the in-plane and out-of-plane nonlinear vibrations of the axially accelerating moving viscoelastic beam simultaneously.

For the in-plane nonlinear vibrations, substituting Eq. (22a) into Eq. (21), we obtain the following nonlinear governing equations of motion for the axially accelerating moving viscoelastic beam with three-degrees-of-freedom:

$$\ddot q_1+\frac{\pi^4J_y\eta}{\rho A l^4}\dot q_1+\frac{\pi^2}{\rho A l^2}\left(\frac{\pi^2J_yE_0}{l^2}-\rho A c^2+P_0+P_1\cos\omega t\right)q_1+\frac{8}{3l}\omega c_1\sin\omega t q_2$$
$$-\frac{16}{3l}(c_0+c_1\cos\omega t)\dot q_2+\frac{\pi^4E_0}{\rho l^4}\left(\frac{3}{8}q_1^3+\frac{9}{8}q_1^2q_3+3q_2^2q_1+\frac{27}{4}q_3^2q_1+\frac{9}{2}q_2^2q_3\right)+\frac{2\pi^4\eta}{\rho l^4}\left(\frac{3}{8}q_1^2\dot q_3\right.$$
$$\left.+\frac{3}{8}q_1^2\dot q_1+2q_1q_2\dot q_2+\frac{9}{2}\dot q_3q_3q_1+\frac{3}{4}\dot q_1q_3q_1+\frac{3}{2}q_2^2\dot q_3+q_2^2\dot q_1+3\dot q_2q_2q_3+\frac{3}{2}q_3^2\dot q_1\right)=0, \tag{24a}$$

$$\ddot q_2+\frac{16\pi^4J_y\eta}{\rho A l^4}\dot q_2+\frac{2\pi^2}{\rho A l^2}\left(\frac{8\pi^2J_yE_0}{l^2}-2\rho A c^2+2(P_0+P_1\cos\omega t)\right)q_2-\frac{8}{3l}\omega c_1\sin\omega t q_1$$
$$+\frac{16}{3l}(c_0+c_1\cos\omega t)\dot q_1+\frac{24}{5l}\omega c_1\sin\omega t q_3-\frac{48}{5l}(c_0+c_1\cos\omega t)\dot q_3+\frac{2\pi^4E_0}{\rho l^4}\left(\frac{3}{2}q_1^2q_2\right.$$
$$\left.+\frac{9}{2}q_1q_2q_3+3q_2^3+\frac{27}{2}q_3^2q_2\right)+\frac{2\pi^4\eta}{\rho l^4}(q_1^2\dot q_2+2q_1q_2\dot q_1+3q_1q_2\dot q_3$$
$$+3\dot q_2q_3q_1+6q_2^2\dot q_2+3q_2q_3\dot q_1+18q_2q_3\dot q_3+9q_3^2\dot q_2)=0, \tag{24b}$$

$$\ddot q_3+\frac{81\pi^4J_y\eta}{\rho A l^4}\dot q_3+\frac{\pi^2}{\rho A l^2}\left(\frac{81\pi^2J_yE_0}{l^2}-9\rho A c^2+9(P_0+P_1\cos\omega t)\right)q_3+\frac{24}{5l}\omega c_1\sin\omega t q_2$$
$$+\frac{48}{5l}(c_0+c_1\cos\omega t)\dot q_2+\frac{2\pi^4E_0}{\rho l^4}\left(\frac{3}{16}q_1^3+\frac{27}{8}q_1^2q_3+\frac{9}{4}q_2^2q_1+\frac{27}{2}q_2^2q_3+\frac{243}{16}q_3^3\right)+\frac{2\pi^4\eta}{\rho l^4}$$
$$\left(\frac{9}{4}q_1^2\dot q_3+\frac{3}{8}q_1^2\dot q_1+3q_1q_2\dot q_2+\frac{9}{2}\dot q_1q_3q_1+\frac{3}{2}q_2^2\dot q_1+9q_2^2\dot q_3+18\dot q_2q_2q_3+\frac{243}{8}q_3^2\dot q_3\right)=0. \tag{24c}$$

System (24) can be utilized to describe the in-plane nonlinear vibrations of the axially accelerating moving viscoelastic beam.

4. Numerical simulations

In the present numerical investigation, a straight polymeric beam, namely $J_{yz}=J_{zy}=0$, is considered with length $L=1.0$ m, width $b=0.016$ m, density $\rho=1000\ \mathrm{kg/m^3}$, Young's modulus along the axial direction $E=1.5\times10^8\ \mathrm{N/m^2}$, viscoelastic coefficient $\eta=4.0\times10^5\ \mathrm{N\,s/m^2}$, and initial tension $P=100$ N. The displacement and the velocity on the midpoint of the axially accelerating moving viscoelastic beam, namely $x=0.5$ m, can be numerically obtained from Eqs. (23) or (24).

The complex dynamic responses of six-degrees-of-freedom nonlinear system, namely Eq. (23), are firstly investigated.

4.1. Case of beam with square or circular cross-sectional area

It is first considered that the beam is of the square or circular cross-sectional area. In this case, two flexural stiffness coefficients of the in-plane and out-of-plane are same, such as J_y=J_z. For the different parameters and initial conditions, the complicated nonlinear phenomena are observed from the numerical results. Fig. 2 illustrates that the in-plane motion of the axially accelerating moving viscoelastic beam, with square cross-sectional area 0.01 m × 0.01 m, is same as that of the out-of-plane when the motions of the system are the nonlinear periodic vibrations, where v represents the in-plane motion and w denotes the out-of-plane motion. This means that in this case, the phase angle of the in-plane motion is also same as that of the out-of-plane motion. It is seen from Fig. 2(e) that the ratio of the amplitude for the in-plane motion to the amplitude for the out-of-plane motion is a constant, which means the trajectory of motion of any point on the axially accelerating moving viscoelastic beam is a line and the beam oscillates in a plane.

If the motions of the axially accelerating moving viscoelastic beam are the quasi-periodic or chaotic vibrations, the amplitude of the in-plane motion is same as that of the out-of-plane motion, but the phase angle of the in-plane motion is different from that of the out-of-plane motion, as shown in Fig. 3. In this case, it is observed from Fig. 3(e) that the

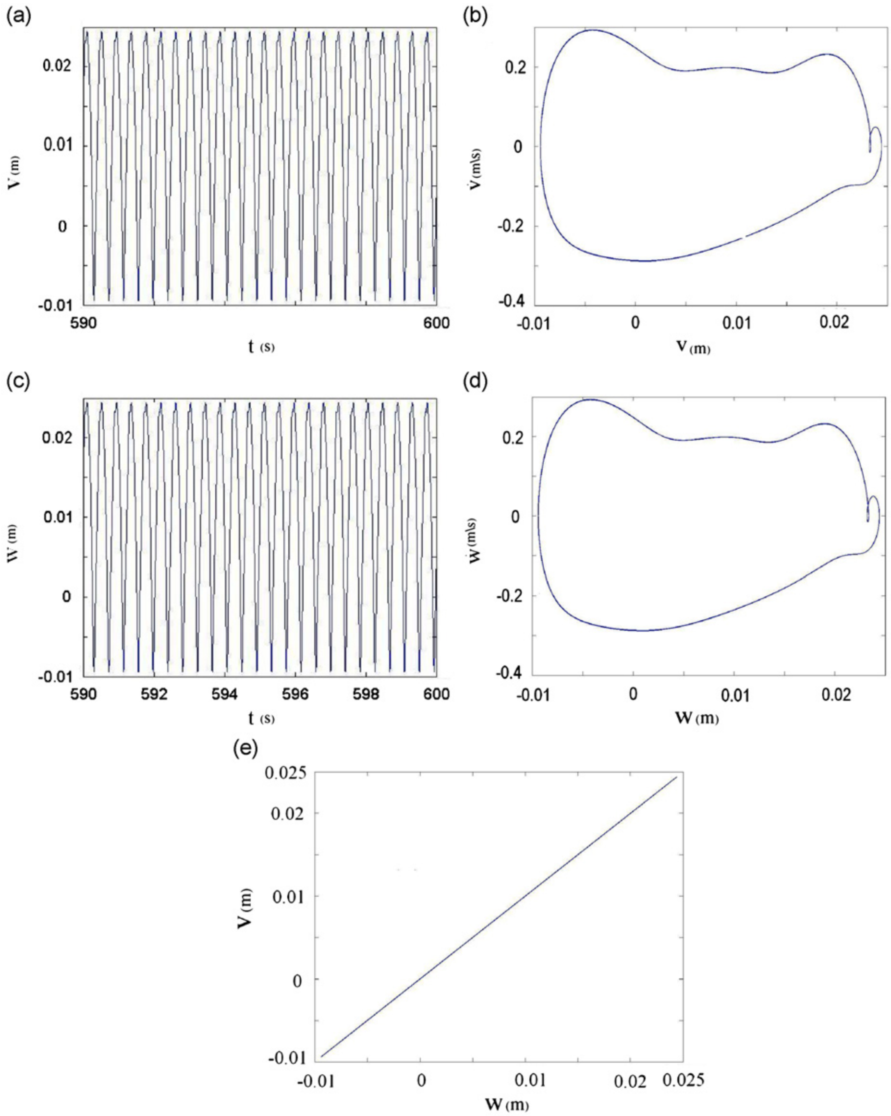

Fig. 2. The periodic motion of system (23) is depicted: (a) waveform of v, (b) phase portrait of v and $\dot{v}$, (c) waveform of w, (d) phase portrait of w and $\dot{w}$, and (e) trajectory of transverse motion for any point on the beam.

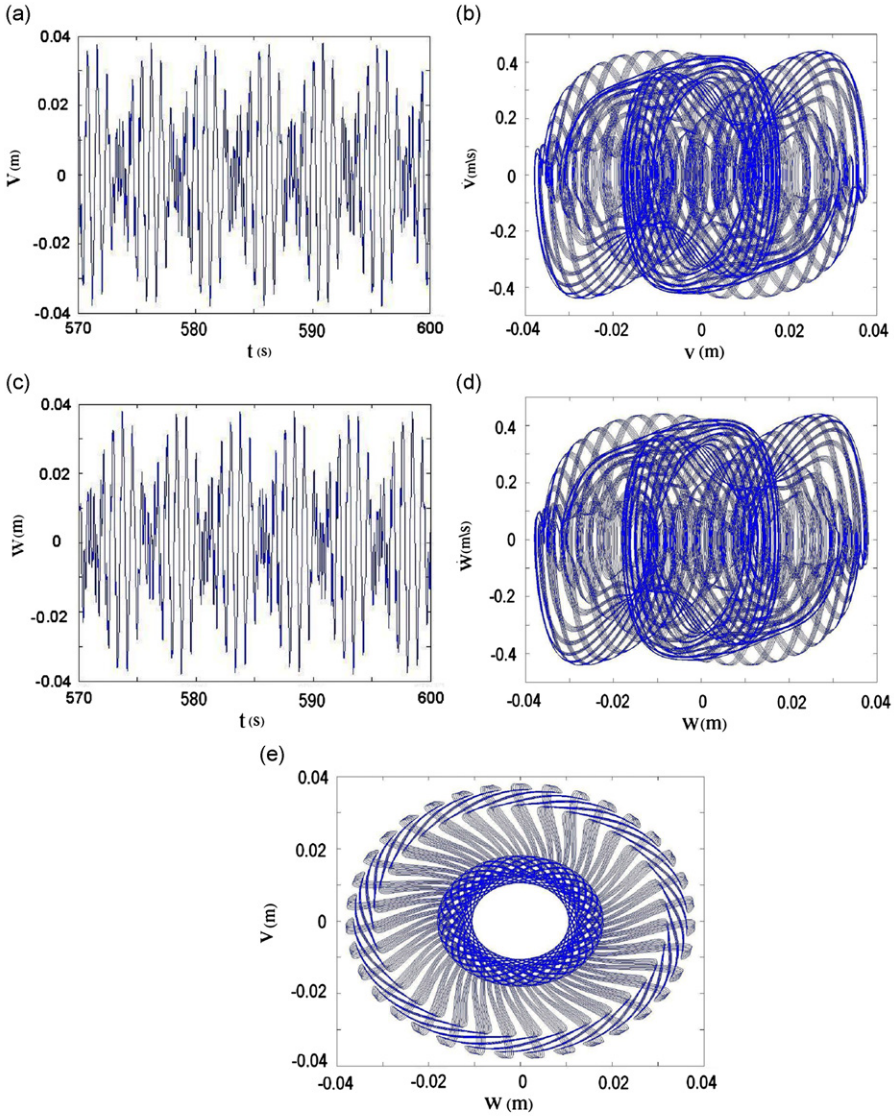

Fig. 3. The quasi-periodic response of system (23) is depicted: (a) waveform of v, (b) phase portrait of v and $\dot{v}$, (c) waveform of w, (d) phase portrait of w and $\dot{w}$, and (e) trajectory of transverse motion for any point on the beam.

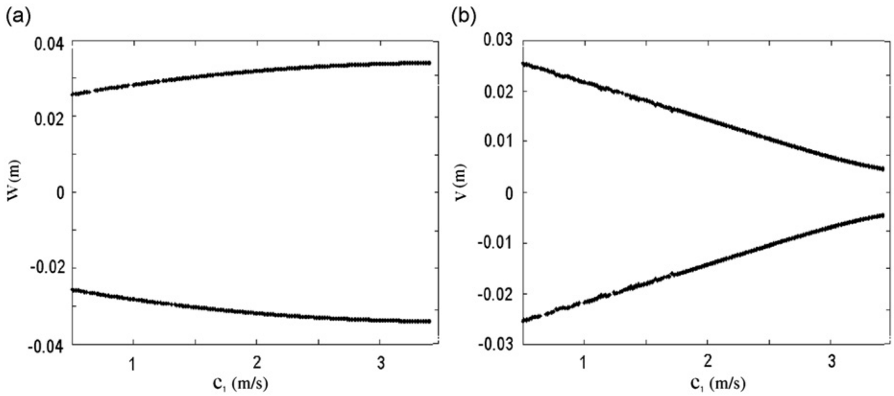

Fig. 4. The bifurcation diagrams for the two displacements v and w via the amplitude of the axial velocity fluctuation c_1 are presented when $c_0 = 30$ m/s and ω=15: (a) bifurcation diagrams for the displacement w via c_1 and (b) bifurcation diagrams for the displacement v via c_1.

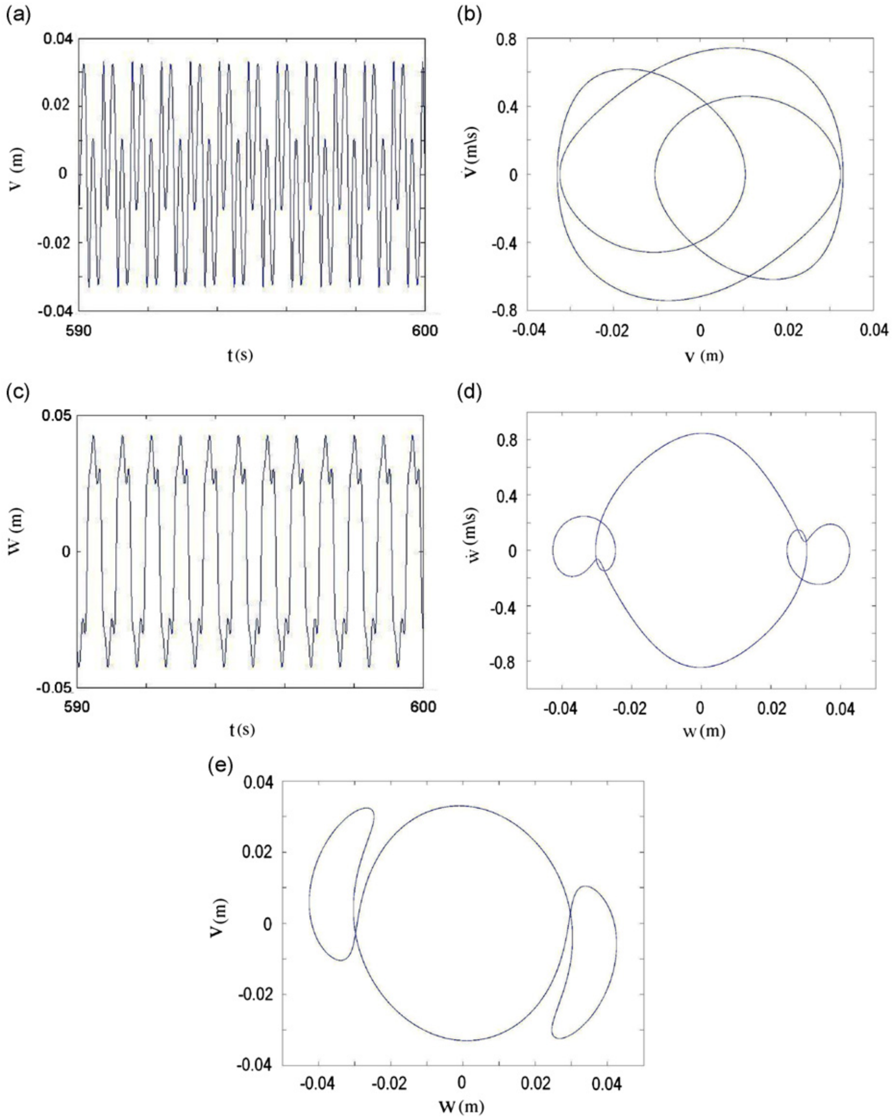

Fig. 5. The periodic-2 motion of system (23) is depicted: (a) waveform of v, (b) phase portrait of v and $\dot{v}$, (c) waveform of w, (d) phase portrait of w and $\dot{w}$ and (e) trajectory of transverse motion for any point on the beam.

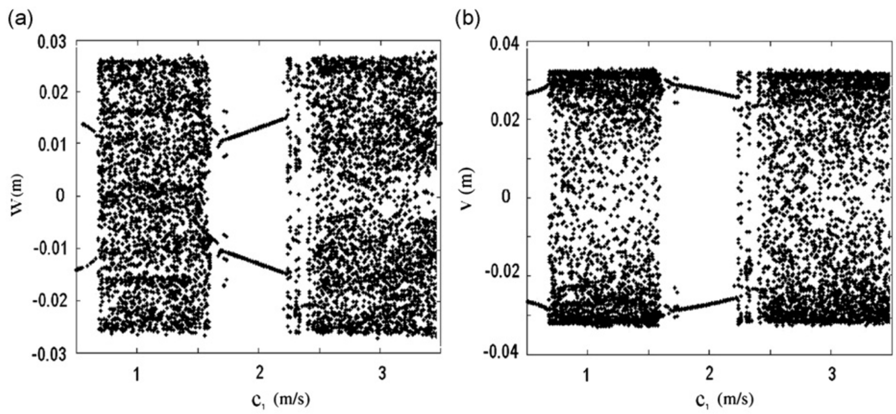

Fig. 6. The bifurcation diagrams for the two displacements v and w via the amplitude of the axial velocity fluctuation c_1 are presented when $c_0 = 20$m and ω=15: (a) bifurcation diagrams for the displacement w via c_1 and (b) bifurcation diagrams for the displacement v via c_1.

trajectory of the motion for any point on the axially accelerating moving viscoelastic beam is a curve, which means that the beam's motion is not located in a plane.

4.2. Case of beam with rectangular cross-sectional area

In the following analysis, we study the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam with a rectangular cross-sectional area which is given by 0.01 m $\times$ 0.02 m. In this case, the in-plane and out-of-plane

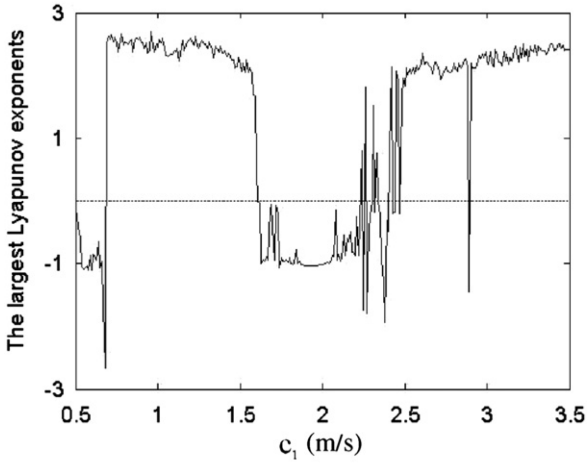

Fig. 7. The largest Lyapunov exponents via the c_1 are calculated when $c_0 = 30\,\mathrm{m/s}$.

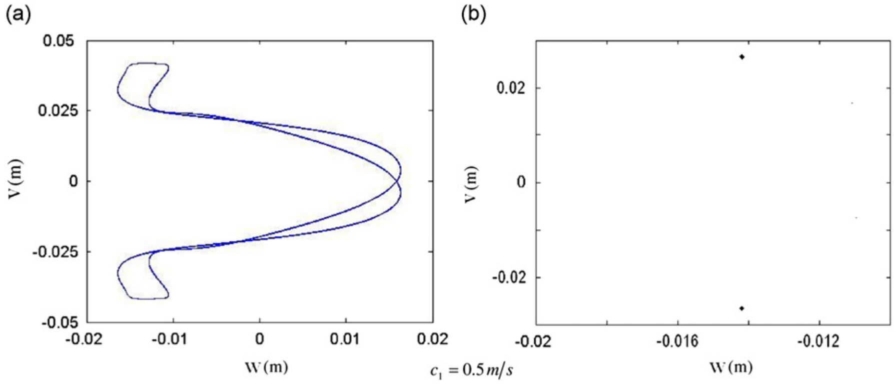

Fig. 8. The periodic-2 response of system (23) is depicted when $c_1 = 0.5\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

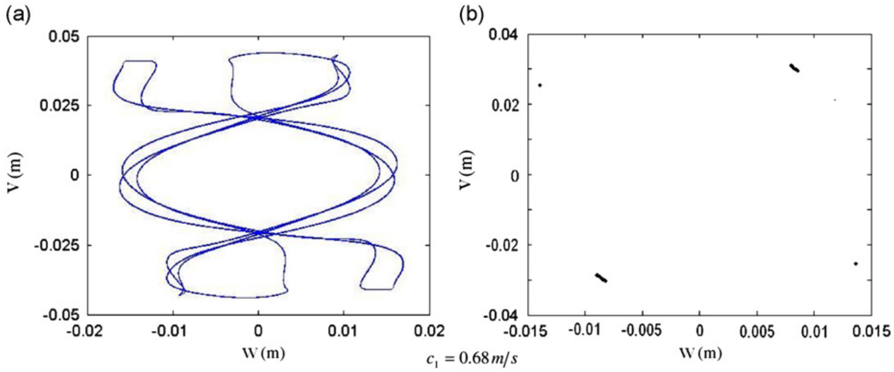

Fig. 9. The quasi-periodic motion of system (23) appears when $c_1 = 0.68\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

motions have great differences because two flexural stiffness coefficients are not equal. We analyze the effect of the mean axial speed, the amplitude of the axial speed fluctuation and the frequency of the axial speed fluctuation on the nonlinear dynamic behaviors of the axially accelerating moving viscoelastic beam, respectively. To give the bifurcation diagrams, we will project the Poincare maps onto a plane which consists of the two displacements v and w. With variation of the mean axial velocity c_0, the amplitude c_1 of the axial velocity fluctuation and the frequency ω of the axial velocity fluctuation c_1, the three types of bifurcation diagrams are respectively presented.

4.2.1. Effect of amplitude of axial velocity fluctuation

Fig. 4 presents the bifurcation diagrams for the two displacements v and w via the amplitude of the axial velocity fluctuation c_1 when $c_0 = 30$ m/s and ω=15 Hz. From the bifurcation diagrams, we can see that the nonplanar nonlinear

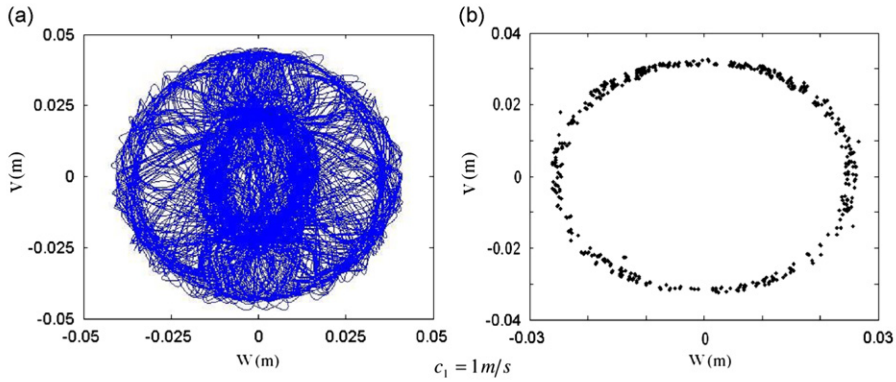

Fig. 10. The chaotic motion of system (23) is depicted when $c_1 = 1$ m/s: (a) phase portrait and (b) Poincare map.

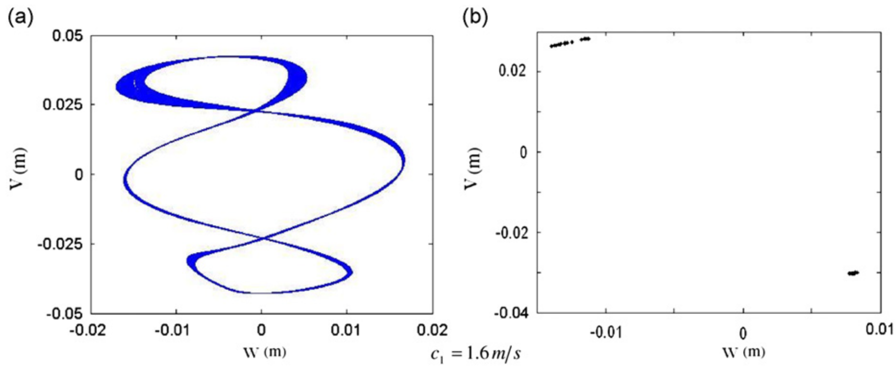

Fig. 11. The quasi-periodic motion of system (23) is depicted when $c_1 = 1.6$ m/s: (a) phase portrait and (b) Poincare map.

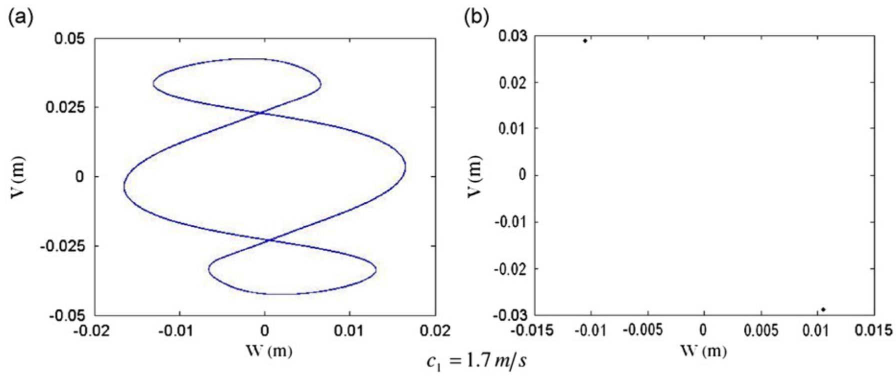

Fig. 12. The periodic-2 motion of system (23) is depicted when $c_1 = 1.7$ m/s: (a) phase portrait and (b) Poincare map.

vibrations of the axially accelerating moving viscoelastic beam only are the periodic-2 motion, which means that there is no bifurcation. Fig. 5 gives the phase portraits and waveforms of the periodic-2 motion. Comparing with the aforementioned case in Section 4.1, it is observed that the trajectory of the transverse motion for any point on the axially accelerating moving viscoelastic beam is not in a plane, as shown in Fig. 5(e). It is also seen that the motions of the in-plane and out-plane are great different.

We change the mean axial velocity to $c_0 = 20$ m/s while other parameters are fixed. The bifurcation diagrams for the two displacements v and w via the amplitude of the axial velocity fluctuation c_1 are presented, as shown in Fig. 6. To further confirm the nonlinear dynamic behaviors which we observe from the bifurcation diagrams given in Fig. 6, the largest Lyapunov exponents are calculated for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam, as shown in Fig. 7. In order to illustrate in detail the bifurcation behaviors in a plane which consists

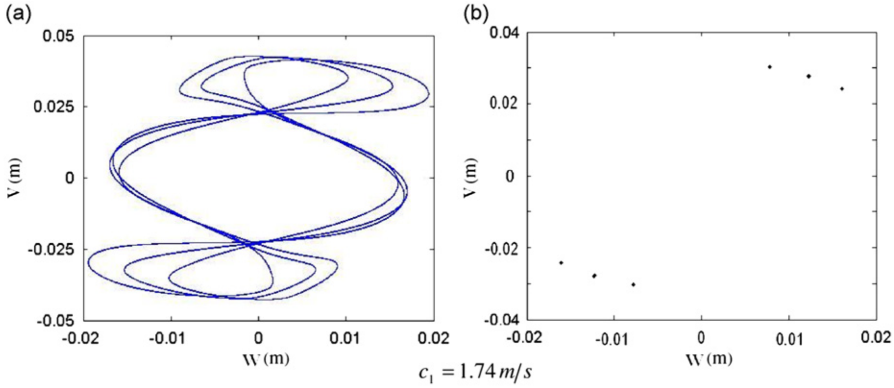

Fig. 13. The periodic-6 response of system (23) is obtained when $c_1 = 1.74$ m/s: (a) phase portrait and (b) Poincare map.

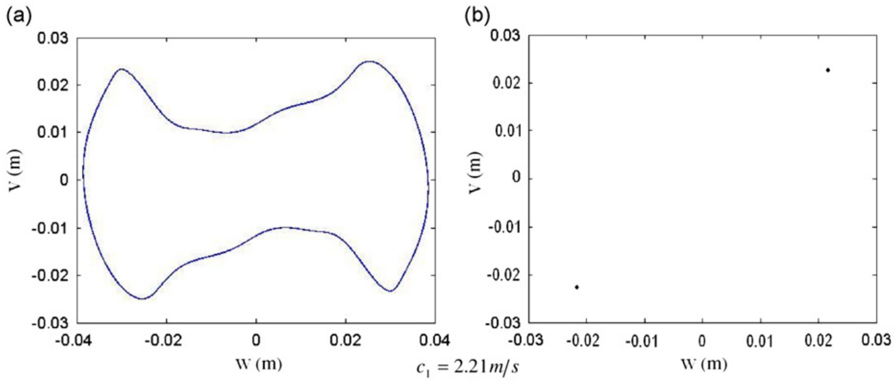

Fig. 14. The periodic-2 motion of system (23) exists when $c_1 = 2.21$ m/s: (a) phase portrait and (b) Poincare map.

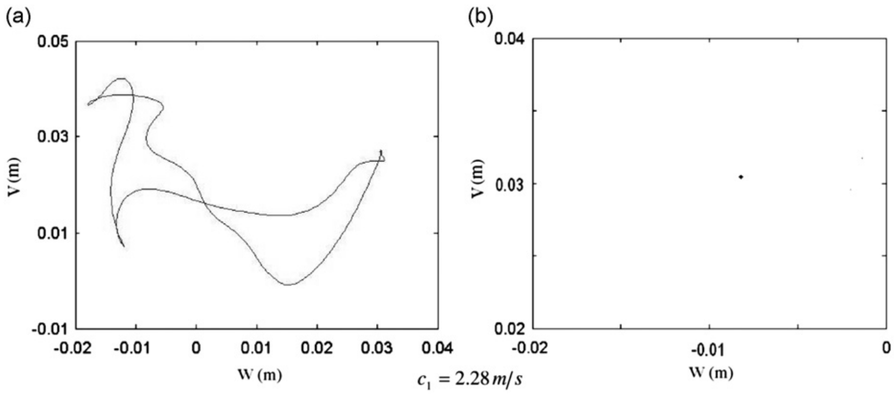

Fig. 15. The periodic response of system (23) is depicted when $c_1 = 2.28$ m/s: (a) phase portrait and (b) Poincare map.

of the two displacements v and w, the Poincare maps and the phase portraits are respectively depicted, as shown in Figs. 8–18. Qualitative explanation of the various bifurcation phenomena is given based on the observation obtained from Figs. 6–18.

There is a window of the periodic-2 motion when the axial velocity fluctuation $c_1 \in [0.5, 0.67]$m/s. The corresponding phase portrait and Poincare map are presented in Fig. 8. Fig. 9 gives a quasi-periodic motion of the axially accelerating moving viscoelastic beam when c_1=0.68 m/s. When we continuously increase the axial velocity fluctuation c_1, it is observed that a large amplitude chaotic motion occurs. Fig. 10 indicates that chaotic motion exists for the axially accelerating moving viscoelastic beam when c_1=1.0 m/s. Based on Figs. 6, 7, 9 and 10, it is found that a quasi-periodic bifurcation given in Fig. 9 results in the chaotic motion. The largest Lyapunov exponents of the chaotic motion are

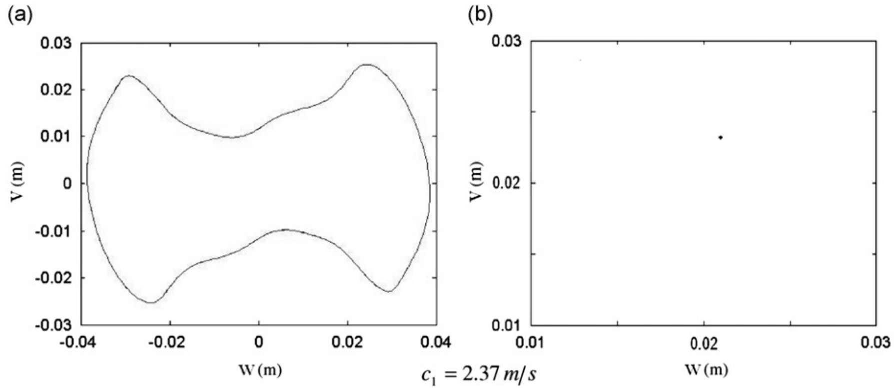

Fig. 16. The periodic response of system (23) is depicted when $c_1 = 2.37$ m/s: (a) phase portrait and (b) Poincare map.

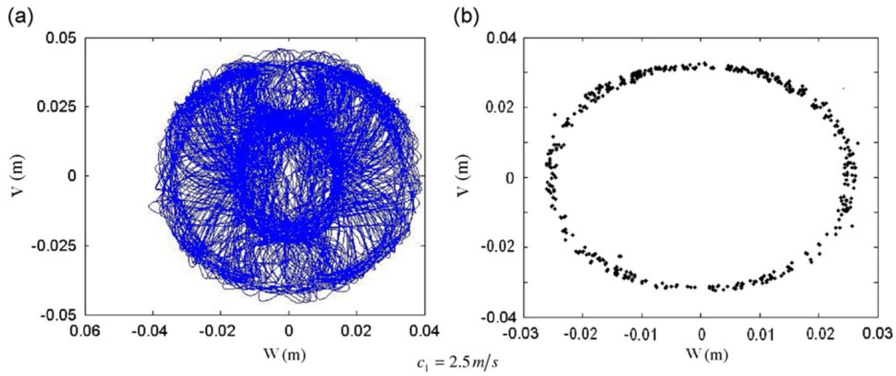

Fig. 17. The chaotic motion of system (23) exists when $c_1 = 2.5$ m/s: (a) phase portrait and (b) Poincare map.

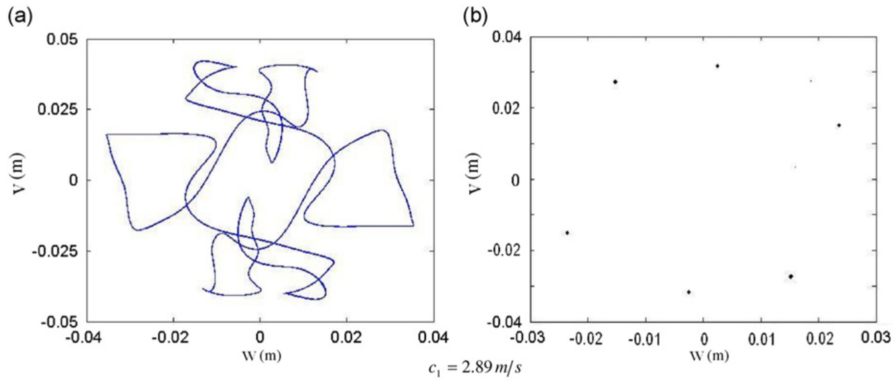

Fig. 18. The periodic-6 response of system (23) is obtained when $c_1 = 2.89$ m/s: (a) phase portrait and (b) Poincare map.

calculated, as shown in Fig. 7. It is seen from Fig. 7 that the largest Lyapunov exponents are positive. When the axial velocity fluctuation changes to $c_1 = 1.6$ m/s, the chaotic motion disappears and a quasi-periodic motion appears, as shown in Fig. 11.

Further increasing the axial velocity fluctuation c_1 gives rise to a window of the periodic-2 motion for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam, as shown in Figs. 6 and 12. Within the interval

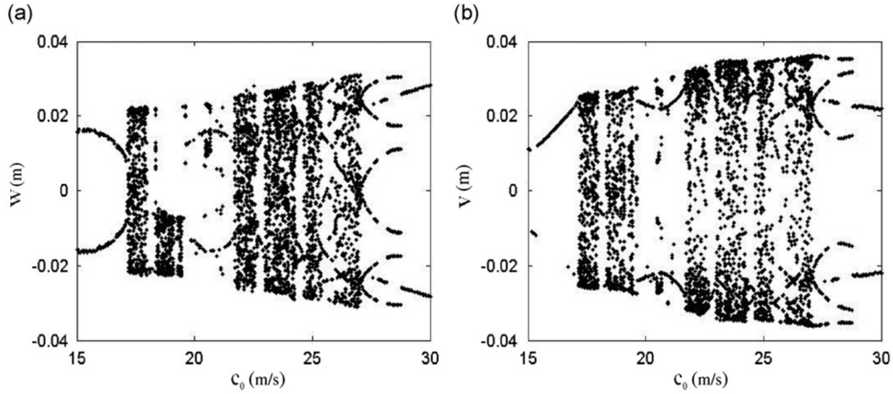

Fig. 19. The bifurcation diagrams for the two displacements v and w via the mean axial speed c_0 are presented when we choose $c_1 = 1$ m/s and ω=15: (a) bifurcation diagrams for the displacement w via c_0 and (b) bifurcation diagrams for the displacement v via c_0.

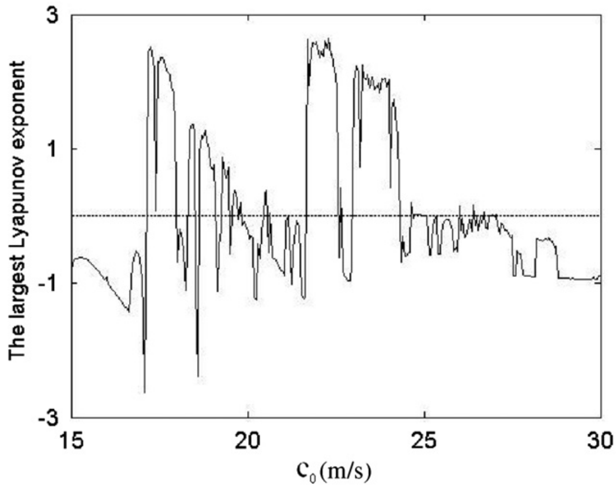

Fig. 20. Largest Lyapunov exponents via the c_0 are calculated when $c_1 = 1$ m/s.

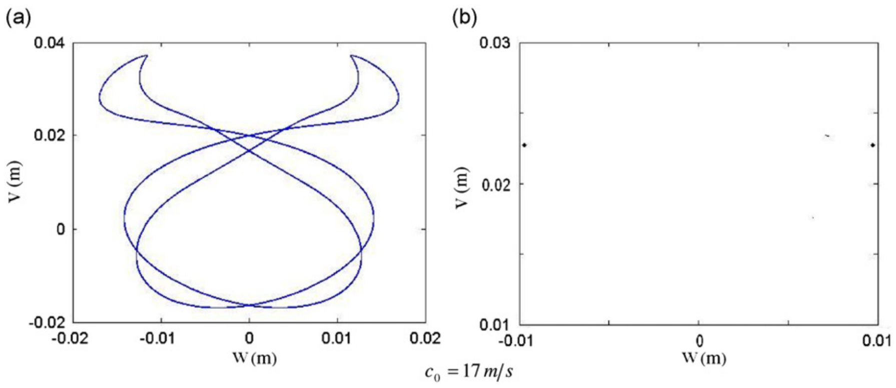

Fig. 21. The periodic-2 motion of system (23) appears when $c_0 = 17$ m/s: (a) phase portrait and (b) Poincare map.

$1.61 \le c_1 \le 2.23$, there exists a very small window of the periodic-6 motion, as shown in Figs. 6 and 13. Fig. 14 illustrates the existence of the periodic-2 motion.

The periodic-2 motion becomes the periodic nonlinear vibration with increasing the axial velocity fluctuation to c_1=2.28 and 2.37 m/s, as shown in Figs. 15 and 16. In this interval of the axial velocity fluctuation c_1, the periodic motion can suddenly become the large amplitude chaotic motion through the explosive bifurcation. It is observed that chaotic motion and periodic motion alternatively appear and disappear, as shown in Figs. 6 and 7. This phenomenon of the nonlinear vibrations is a typical intermittent chaos. When the axial velocity fluctuation c_1 is located in the interval [2.47, 3.5] m/s, a very large window of the chaotic motions appears for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam, as shown in Fig. 17. In addition, a very small window of the periodic-6 motion also exists in this window of the chaotic motions, as shown in Fig. 18.

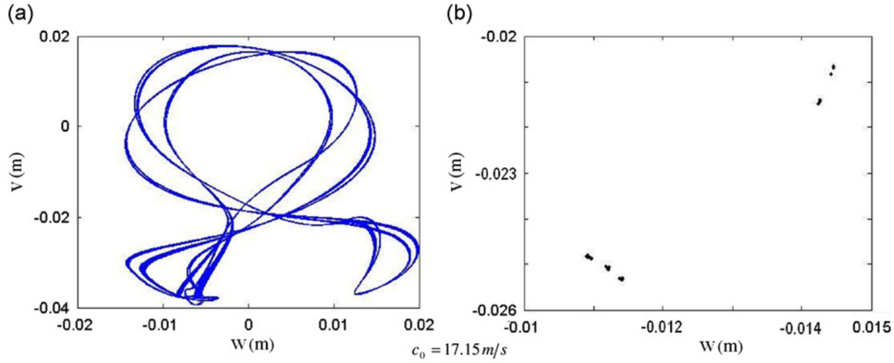

Fig. 22. The quasi-periodic motion of system (23) is depicted when c_0=17.15 m/s: (a) phase portrait and (b) Poincare map.

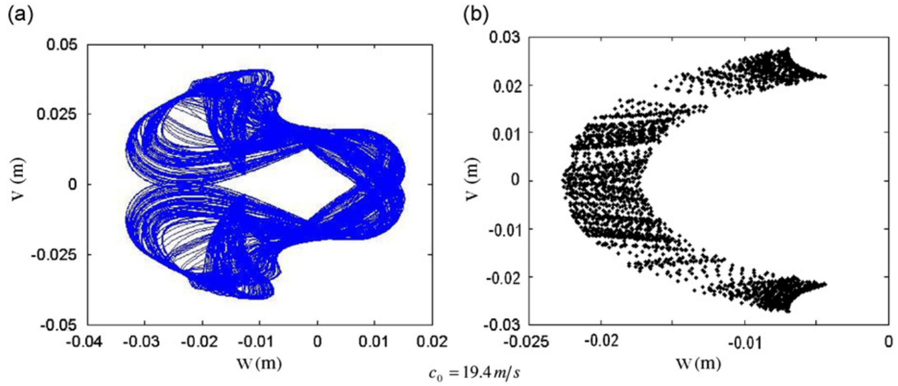

Fig. 23. The chaotic motion of system (23) is depicted when $c_0 = 19.4\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

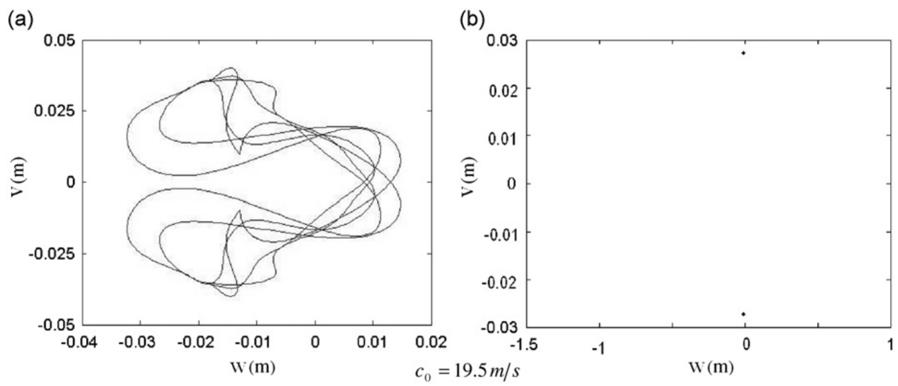

Fig. 24. The periodic-2 response of system (23) is depicted when c_0=19.5 m/s: (a) phase portrait and (b) Poincare map.

4.2.2. *Effect of mean axial velocity*

In the following analysis, we investigate the influence of the mean axial velocity on the nonlinear dynamic behaviors of the axially accelerating moving viscoelastic beam. Fig. 19 presents the bifurcation diagrams for the two displacements v and w via the mean axial speed c_0 when we choose $c_1 = 1\,\mathrm{m/s}$ and ω=15 Hz. The Lyapunov exponents are calculated for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam, as shown in Fig. 20. The Poincare maps and the phase portraits are depicted to indicate the nonplanar nonlinear responses of the axially accelerating moving viscoelastic beam. The observation obtained from Figs. 21–32 gives qualitative explanation of the various bifurcation phenomena.

When the mean axial velocity c_0 is located in the interval [15 ,17.1]m/s, it is found from Fig. 19 that the in-plane motion is the nonlinear periodic response and the out-of-plane motion is the periodic-2 response, as shown in Fig. 21. Fig. 22 illustrates the existence of the quasi-periodic motion when the mean axial velocity changes to c_0=17.15 m/s. Increasing the

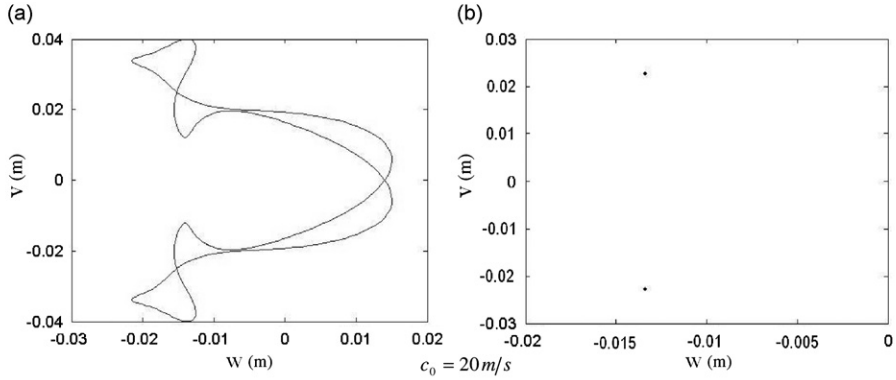

Fig. 25. The periodic-2 motion of system (23) is depicted when $c_0 = 20\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

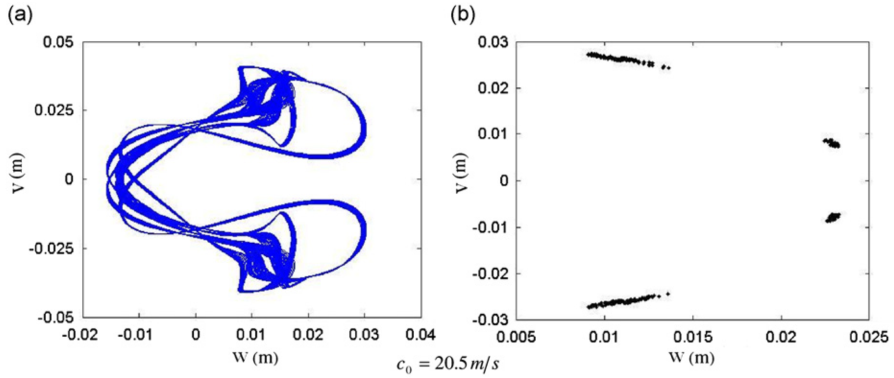

Fig. 26. The chaotic motion of system (23) appears when c_0=20.5 m/s: (a) phase portrait and (b) Poincare map.

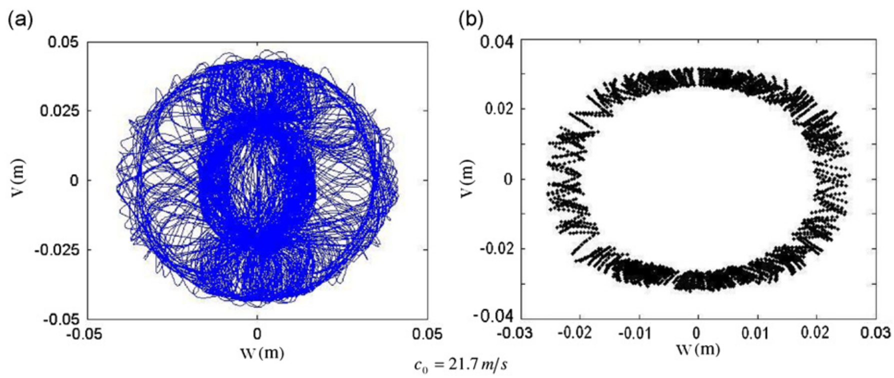

Fig. 27. The chaotic motion of system (23) is depicted when $c_0 = 21.7\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

mean axial velocity c_0, the quasi-periodic bifurcation results in the chaotic motion of the axially accelerating moving viscoelastic beam. Fig. 23 demonstrates that there exists the chaotic motion when c_0=19.4 m/s. The largest Lyapunov exponents of the chaotic motion are calculated and are positive. We also observe that the chaotic motion is interrupted by a small window of the periodic-2 motion, as shown in Fig. 24, where c_0=19.5 m/s.

Continuously increasing the mean axial velocity c_0, it is found that the nonplanar nonlinear responses of the axially accelerating moving viscoelastic beam become the periodic-2 motions. Fig. 25 indicates the existence of the periodic-2 motion when c_0=20 m/s. In the interval 20 m/s $< c_0 \leq$ 21.65 m/s of the mean axial velocity, there is a small window of the chaotic motion, as shown in Fig. 26, where c_0=20.5 m/s. As we further increase the mean axial velocity c_0, it is observed that the periodic motion suddenly becomes the large amplitude chaotic motion through the explosive bifurcation, as shown in Figs. 19 and 20. Furthermore, the chaotic, periodic and quasi-periodic motions alternatively appear and disappear, as shown in Figs. 27–29. When we continuously increase the mean axial velocity c_0, a large window of the

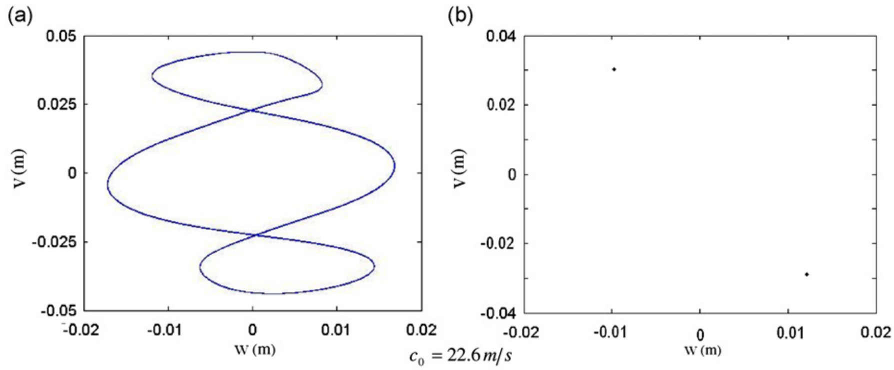

Fig. 28. The periodic-2 response of system (23) is depicted when $c_0 = 22.6\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

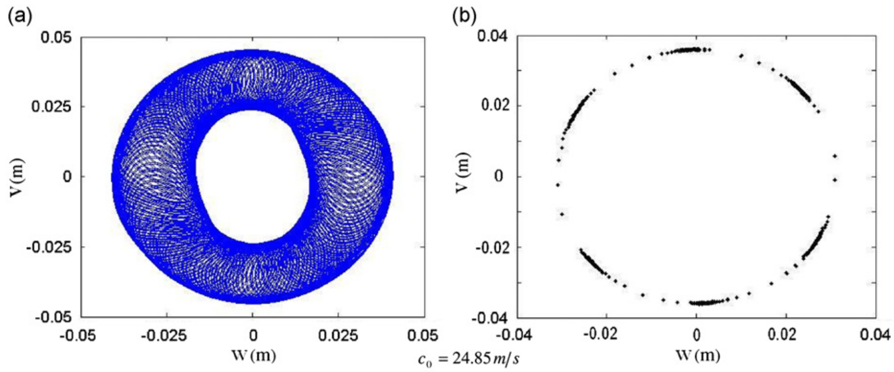

Fig. 29. The quasi-periodic motion of system (23) is depicted when $c_0 = 24.85\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

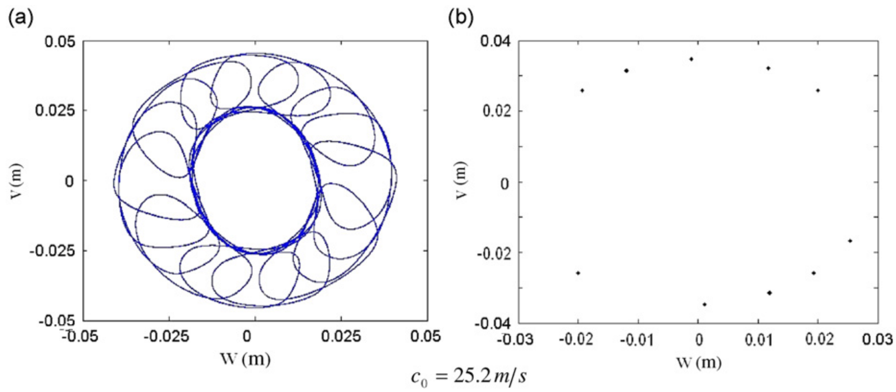

Fig. 30. The periodic-10 motion of system (23) appears when $c_0 = 25.2\,\mathrm{m/s}$: (a) phase portrait and (b) Poincare map.

periodic motion appears, as shown in Figs. 19 and 20. Fig. 30 demonstrates the existence of periodic-10 motion for the axially accelerating moving viscoelastic beam when c_0=25.2 m/s. The periodic-6 motion exists when c_0=27.1 m/s, as shown in Fig. 31. Changing the mean axial velocity to c_0=29.5 m/s, the periodic-2 motion appears, as shown in Fig. 32.

4.2.3. Effect of frequency of axial velocity fluctuation

We analyze the influence of the frequency of the axial velocity fluctuation on the nonlinear behaviors of the axially accelerating moving viscoelastic beam. Fig. 33 gives the bifurcation diagrams for the two displacements v and w via the

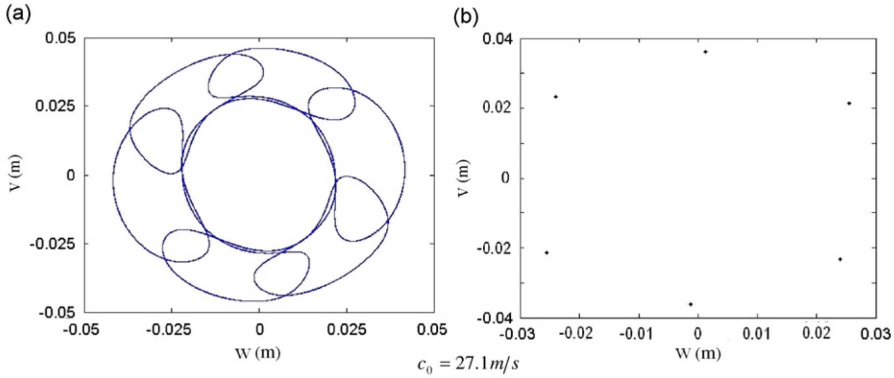

Fig. 31. The periodic-6 response of system (23) is obtained when c_0=27.1 m/s: (a) phase portrait and (b) Poincare map.

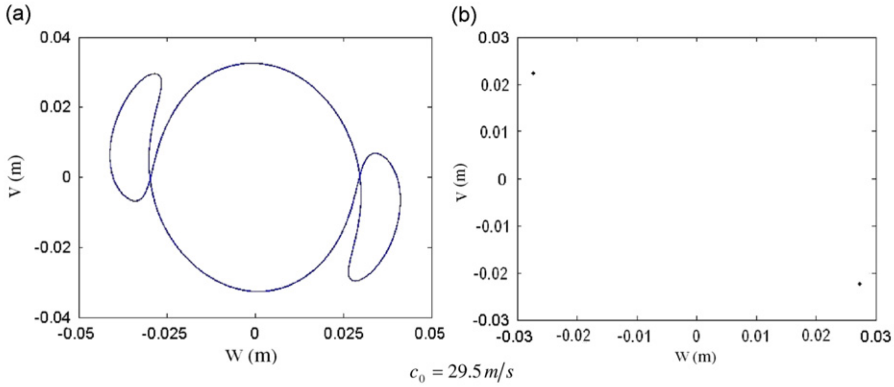

Fig. 32. The periodic-2 response of system (23) exists when $c_0 = 29.5\,\text{m/s}$: (a) phase portrait and (b) Poincare map.

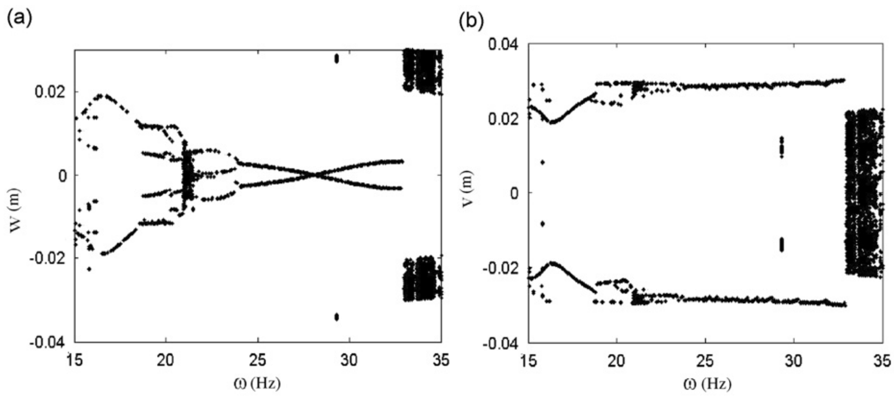

Fig. 33. The bifurcation diagrams for the two displacements v and w via the frequency ω of the axial velocity fluctuation are presented when $c_1 = 1\,\text{m/s}$ and $c_0 = 20\,\text{m/s}$: (a) bifurcation diagrams for the displacement w via ω and (b) bifurcation diagrams for the displacement v via ω.

frequency ω of the axial velocity fluctuation, where c_1=1 m/s and c_0=20 m/s. The Lyapunov exponents are calculated for the nonplanar nonlinear vibrations of the axially accelerating moving viscoelastic beam, as shown in Fig. 34. The Poincare maps and the phase portraits are depicted to illustrate the nonlinear dynamic behaviors of the axially accelerating moving viscoelastic beam, as shown in Figs. 35–45.

When the frequency of the axial velocity fluctuation is located in the interval 15 Hz $\leq \omega \leq$ 32.9 Hz, there is a very large window of the periodic motions including periodic-4 and periodic-2 motions, as shown in Figs. 35, 36 and 39, where the

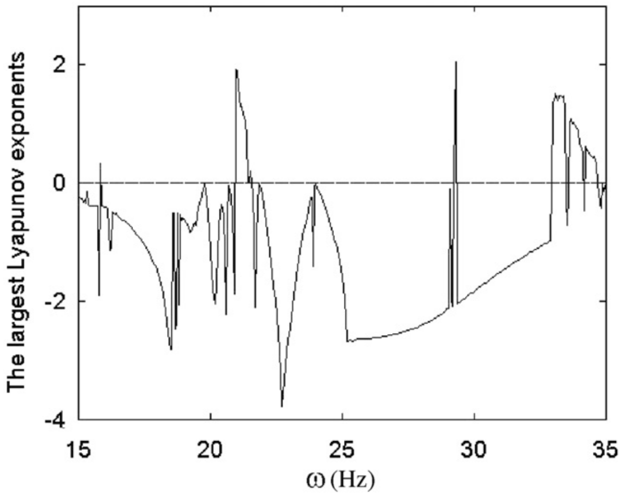

Fig. 34. The largest Lyapunov exponents via the ω are calculated when $c_1 = 1\,\mathrm{m/s}$ and $c_0 = 20\,\mathrm{m/s}$.

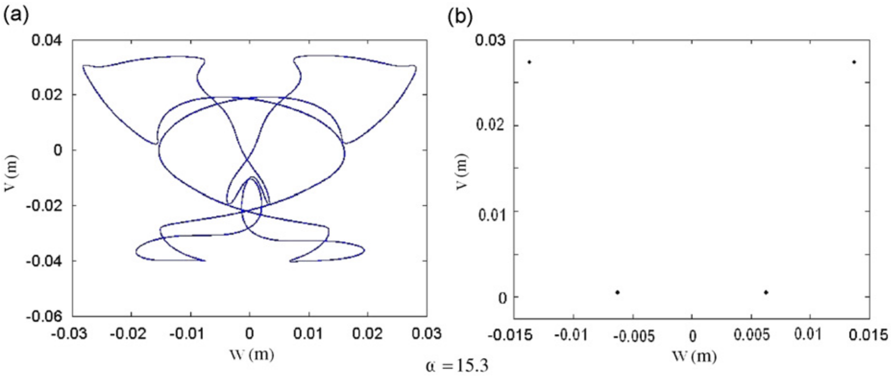

Fig. 35. The periodic-4 response of system (23) is obtained when ω=15.3 Hz: (a) phase portrait and (b) Poincare map.

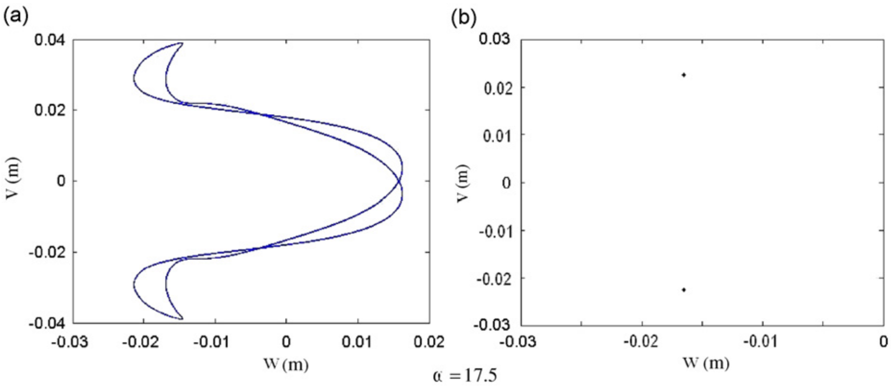

Fig. 36. The periodic-2 response of system (23) is depicted when ω=17.5 Hz: (a) phase portrait and (b) Poincare map.

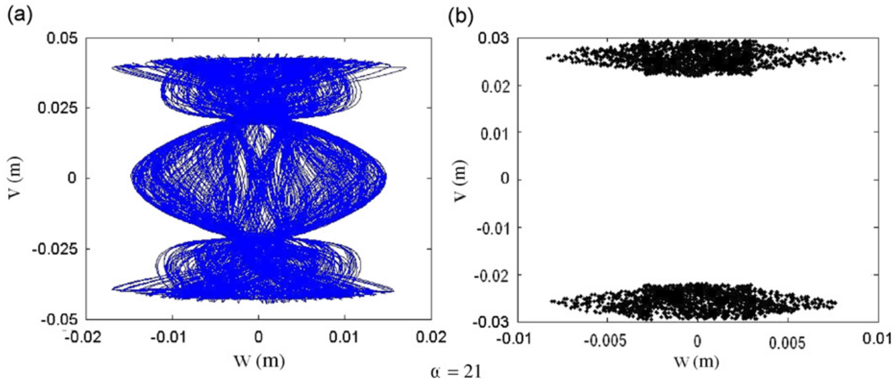

Fig. 37. The chaotic motion of system (23) appears when ω=21 Hz: (a) phase portrait and (b) Poincare map.

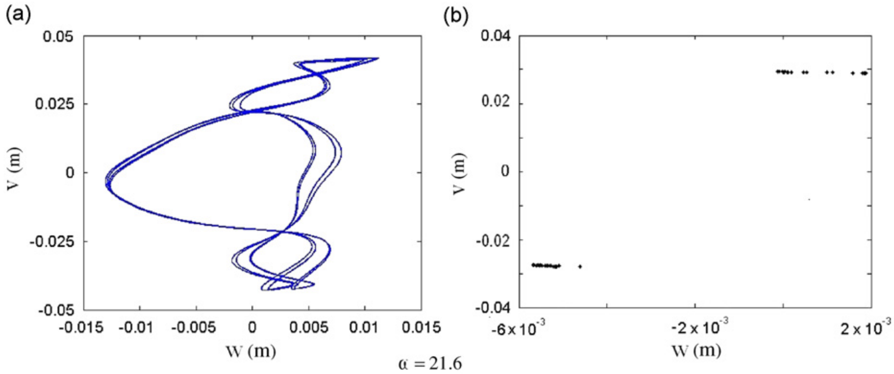

Fig. 38. The quasi-periodic motion of system (23) exists when ω=21.6 Hz: (a) phase portrait and (b) Poincare map.

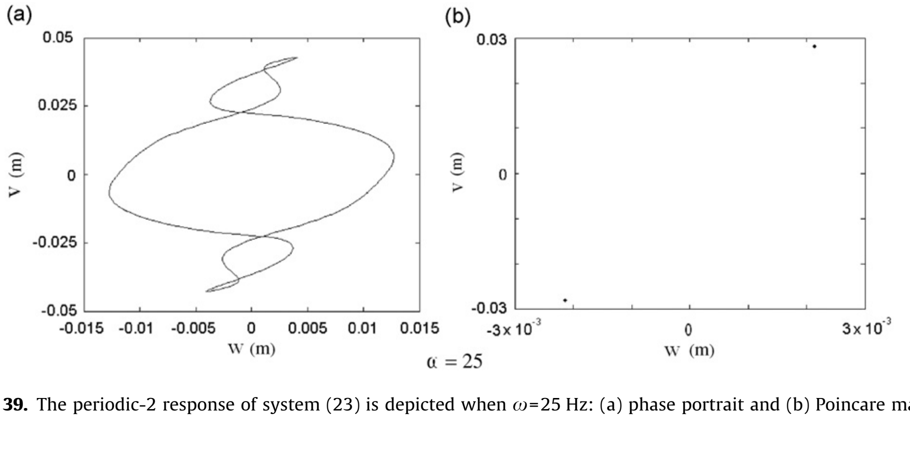

Fig. 39. The periodic-2 response of system (23) is depicted when ω=25 Hz: (a) phase portrait and (b) Poincare map.

frequencies of axial velocity fluctuation, respectively, are ω=15.3, 17.5 and 25 Hz. In addition, there also exist some very small windows of the chaotic motions, as shown in Figs. 37 and 40, where ω=21 and 29.3 Hz. We also observe that there exists a small window of the quasi-periodic motion for the axially accelerating moving viscoelastic beam, as shown in Fig. 38, where ω=21.6 Hz.

Continuously increasing the frequency ω of the axial velocity fluctuation, a quasi-periodic bifurcation results in the chaotic motion of the axially accelerating moving viscoelastic beam, as shown in Figs. 33 and 34. Figs. 41 and 42 illustrate the existence of the quasi-periodic and chaotic motions when ω=32.95 and 33.4 Hz, respectively. The largest Lyapunov exponents of the chaotic motion are calculated and are positive. We also observe that the chaotic motion is interrupted by some small windows of the periodic motion, as shown in Figs. 43–45.

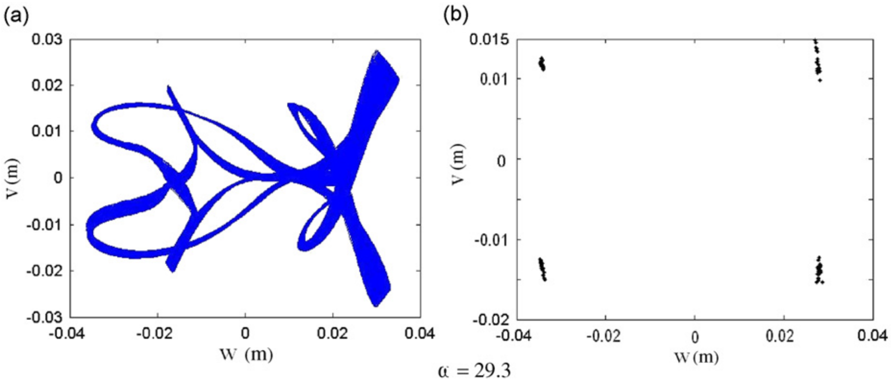

Fig. 40. The chaotic motion of system (23) exists when ω=29.3 Hz: (a) phase portrait and (b) Poincare map.

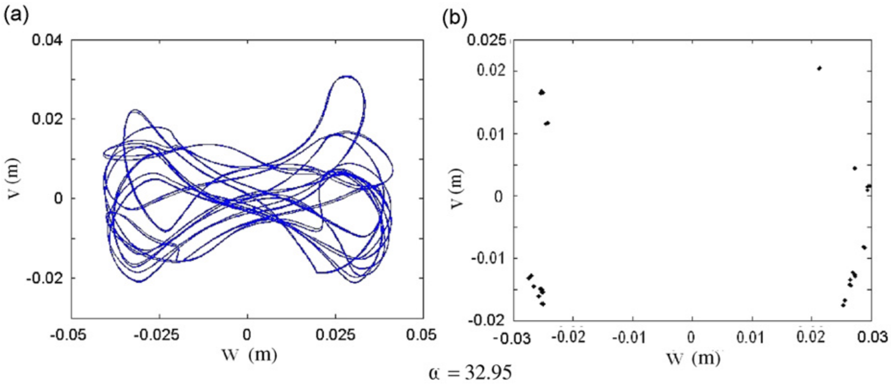

Fig. 41. The quasi-periodic motion of system (23) is depicted when ω=32.95 Hz: (a) phase portrait and (b) Poincare map.

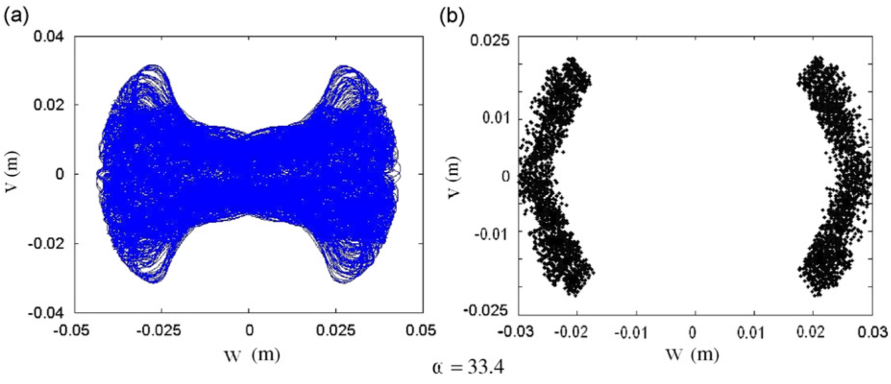

Fig. 42. The chaotic motion of system (23) appears when ω=33.4 Hz: (a) phase portrait and (b) Poincare map.

4.3. Contrast to in-plane transverse vibrations

In this subsection, we numerically investigate Eq. (24) to give a comparison of the dynamic responses of six-degrees-of-freedom nonlinear system and those of three-degrees-of-freedom nonlinear system. To compare the difference between the nonplanar nonlinear transverse vibrations and the in-plane nonlinear transverse vibrations of the axially accelerating moving viscoelastic beam, we give the two bifurcation diagrams for the displacement v via the amplitude c_1 of the axial velocity fluctuation and the mean axial velocity c_0, respectively, as shown in Figs. 46 and 47. In Figs. 46 and 47, we, respectively, choose $c_0 = 20$ m/s and $c_1 = 1$ m/s. The other chosen parameters are same as those of the nonplanar transverse nonlinear vibrations. The displacement on the midpoint of the beam, namely x=0.5 m, can be numerically obtained from Eq. (24). Comparing Fig. 6 with Fig. 46 and Fig. 19 with Fig. 47, it is observed that the nonlinear dynamic

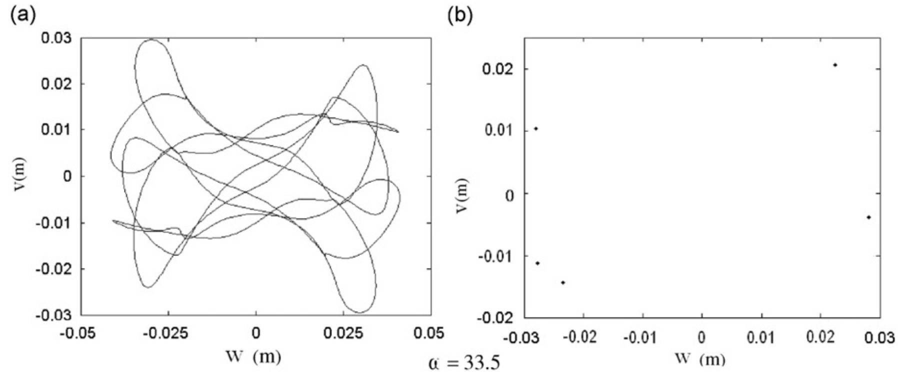

Fig. 43. The periodic-5 response of system (23) is depicted when ω=33.5 Hz: (a) phase portrait and (b) Poincare map.

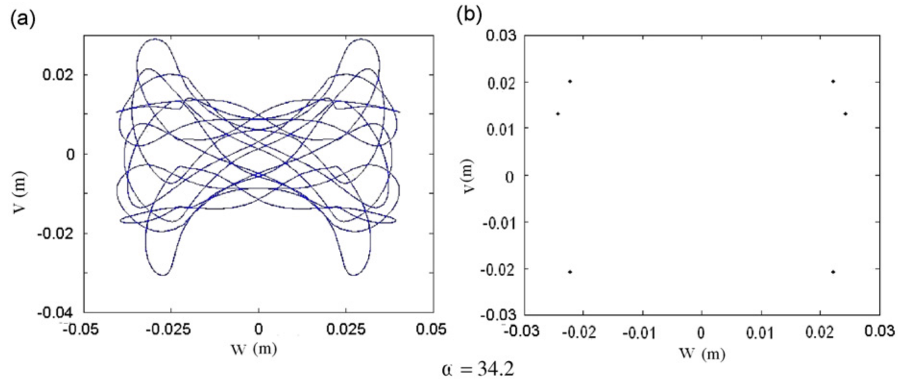

Fig. 44. The periodic-6 motion of system (23) exist when ω=34.2 Hz: (a) phase portrait and (b) Poincare map.

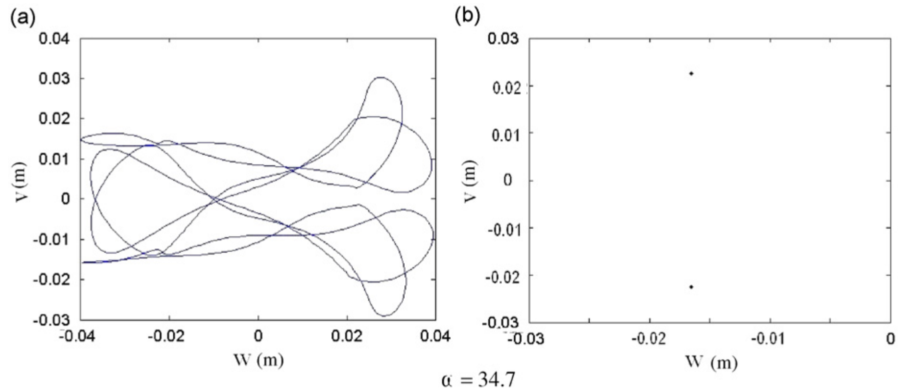

Fig. 45. The periodic-2 response of system (23) is depicted when ω=34.7 Hz: (a) phase portrait and (b) Poincare map.

behaviors of the axially accelerating moving viscoelastic beam for the nonplanar nonlinear transverse vibrations are very different from those of the in-plane nonlinear transverse vibrations.

5. Conclusion

In this paper, we study the bifurcations and chaotic motions for the nonplanar nonlinear transverse vibrations of an axially accelerating moving viscoelastic beam which consists of viscoelastic materials and with the Kelvin–Voigt constitutive relation. The generalized Hamilton's principle is employed to establish the nonlinear governing equations of the nonplanar transverse motion. The three-term Galerkin method is applied to truncating the governing equations to a

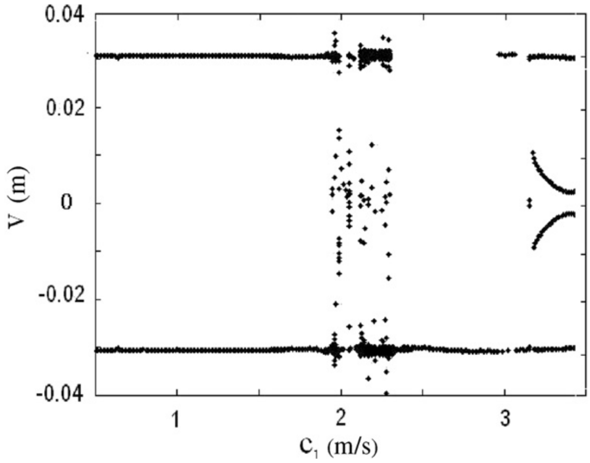

Fig. 46. The bifurcation diagram of system (24) for the displacement v via the amplitude of the axial velocity fluctuation c_1 are presented when c_0=20 m/s and ω=15 Hz.

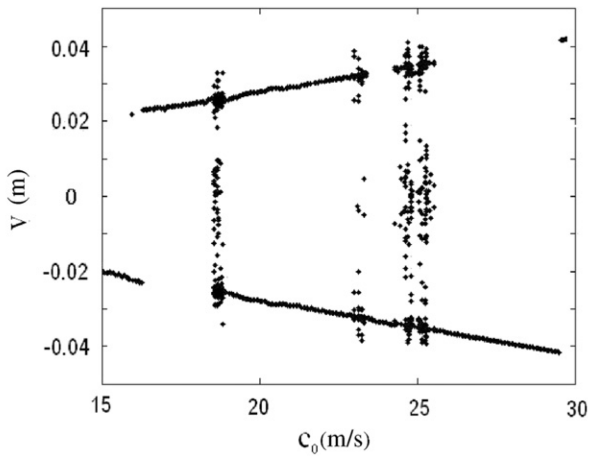

Fig. 47. The bifurcation diagram of system (24) for the displacement v via the mean axial speed c_0 are presented when we choose $c_1 = 1$ m/s and ω=15 Hz.

six-degrees-of-freedom nonlinear system and a three-degrees-of-freedom nonlinear system, which describe the nonplanar transverse and the in-plane nonlinear vibrations of the axially accelerating moving viscoelastic beam, respectively. The nonlinear dynamic behaviors are numerically investigated by means of the Poincare maps, the phase portraits and the largest Lyapunov exponents. The bifurcation diagrams are depicted to demonstrate the changing trend for the two displacements v and w via the amplitude c_1 of the axial velocity fluctuation, the mean axial velocity c_0 and the frequency ω of the axial velocity fluctuation, respectively.

For the axially accelerating moving viscoelastic beam with the square cross-sectional area or circular cross-sectional area, two flexural stiffness coefficients of the in-plane and out-of-plane are same. It is observed that the nonlinear dynamic responses for the in-plane and out-of-plane of the axially accelerating moving viscoelastic beam are completely uniform when the motion of the system is the nonlinear periodic vibrations. In addition, when the motion of the system is the quasi-periodic or chaotic vibrations, two nonlinear responses for the in-plane and out-of-plane also are same besides the phase angle. The aforementioned phenomenon depends on the symmetry of the governing equations for the nonplanar nonlinear transverse vibrations of six-degrees-of-freedom system.

When the axially accelerating moving viscoelastic beam with the rectangular cross-sectional area is considered, the flexural stiffness coefficients of the in-plane and out-of-plane are not equal. It is seen that the motion of the in-plane is different from that of the out-of-plane. In this case, the periodic, quasi-periodic and chaotic motions occur for the nonplanar nonlinear transverse vibrations of the axially accelerating moving viscoelastic beam. Furthermore, it is observed that with the increase of the mean axial velocity, the amplitudes of the nonlinear vibrations for the axially accelerating moving viscoelastic beam also increase. In this case, the trajectory of the transverse motion for any point on the axially accelerating moving viscoelastic beam is not in a plane.

Comparing the bifurcation diagrams of the nonplanar and the in-plane nonlinear transverse vibrations, we can conclude that the nonlinear dynamic responses of two types of systems with six-degrees-of-freedom and three-degrees-of-freedom are very different although the structural and material parameters as well as the initial conditions are same.

Acknowledgements

The authors gratefully acknowledge the support of the National Science Foundation for Distinguished Young Scholars of China (NSFDYSC) through Grant no. 10425209, the National Natural Science Foundation of China (NNSFC) through Grant nos. 10732020 and 10872009, the Funding Project for Academic Human Resources Development in Institutions of Higher Learning under the Jurisdiction of Beijing Municipality (PHRIHLB).

References

[1] M. Pakdemirli, A.G Ulsoy, A. Ceranoglu, Transverse vibration of an axially accelerating string, *Journal of Sound and Vibration* 169 (1994) 179–196.
[2] F. Pellicano, F. Zirilli, Boundary layers and non-linear vibrations in an axially moving beam, *International Journal of Non-linear Mechanics* 33 (1997) 691–711.
[3] M. Pakdemirli, A.G. Ulsoy, Stability analysis of an axially accelerating string, *Journal of Sound and Vibration* 203 (1997) 815–832.
[4] H.R. Oz, M. Pakdemirli, E. Ozkaya, Transition behaviour from string to beam for an axially accelerating material, *Journal of Sound and Vibration* 215 (1998) 571–576.
[5] B. Ravindra, W.D. Zhu, Low dimensional chaotic response of axially accelerating continuum in the supercritical regime, *Archieves of Applied Mechanics* 68 (1998) 195–205.
[6] E. Ozkaya, M. Pakdemirli, Vibrations of an axially accelerating beam with small flexural stiffness, *Journal of Sound and Vibration* 234 (2000) 521–535.
[7] H.R. Oz, M. Pakdemirli, Vibrations of an axially moving beam with time dependent velocity, *Journal of Sound and Vibration* 227 (1999) 239–257.
[8] H.R. Oz, On the vibrations of an axially traveling beam on fixed supports with variable velocity, *Journal of Sound and Vibration* 239 (2001) 556–564.
[9] R.G. Parker, Y. Lin, Parametric instability of axially moving media subjected to multi-frequency tension and speed fluctuations, *ASME Journal of Applied Mechanics* 68 (2001) 49–57.
[10] H.R. Oz, M. Pakdemirli, H. Boyaci, Non-linear vibrations and stability of an axially moving beam with time-dependent velocity, *International Journal of Non-Linear Mechanics* 36 (2001) 107–115.
[11] E. Ozkaya, H.R. Oz, Determination of natural frequencies and stability regions of axially moving beams using artificial neural networks method, *Journal of Sound and Vibration* 254 (2002) 782–789.
[12] K. Marynowski, Non-linear dynamic analysis of an axially moving viscoelastic beam, *Journal of Theoretical and Applied Mechanics* 40 (2002) 465–482.
[13] K. Marynowski, T. Kapitaniak, Kelvin–Voigt versus Bugers internal damping in modeling of axially moving viscoelastic web, *International Journal of Non-linear Mechanics* 37 (2002) 1147–1161.
[14] K. Marynowski, Non-linear vibrations of an axially moving viscoelastic web with time-dependent tension, *Chaos, Solitons, and Fractals* 21 (2004) 481–490.
[15] L.Q. Chen, X.D. Yang, C.J. Cheng, Dynamic stability of an axially accelerating viscoelastic beam, *European Journal of Mechanics A/Solids* 23 (2004) 659–666.
[16] X.D. Yang, L.Q. Chen, Bifurcation and chaos of an axially accelerating viscoelastic beam, *Chaos, Solitons and Fractals* 23 (2005) 249–258.
[17] L.Q. Chen, X.D. Yang, Steady-state response of axially moving viscoelastic beams with pulsating speed: comparison of two nonlinear models, *International Journal of Solids and Structures* 42 (2005) 37–50.
[18] E.M. Mockensturm, J.P. Guo, Nonlinear vibration of parametrically excited, viscoelastic, axially moving strings, *ASME Journal of Applied Mechanics* 72 (2005) 374–380.
[19] L.H. Chen, W. Zhang, Y.Q. Liu, Modeling of nonlinear oscillations for viscoelastic moving belt using Hamilton's principle, *ASME Journal of Vibration and Acoustics* 129 (2007) 128–132.
[20] Y.Q. Liu, L.H. Chen, W. Zhang, Periodic and chaotic oscillations of an axially moving viscoelastic belt with one-to-one internal resonance in three-dimensional space, *International Journal of Nonlinear Sciences and Numerical Simulation* 8 (2007) 589–600.
[21] W. Zhang, C.Z. Song, Research on higher-dimensional periodic and chaotic oscillations for a parametrically excited viscoelastic moving belt with multiple internal resonances, *International Journal of Bifurcation and Chaos* 17 (2007) 1637–1660.
[22] K. Marynowski, T. Kapitaniak, Zener internal damping in modelling of axially moving viscoelastic beam with time-dependent tension, *International Journal of Non-Linear Mechanics* 42 (2007) 118–131.
[23] L.Q. Chen, W. Zhang, J.W. Zu, Nonlinear dynamics in transverse motion of axially moving strings, *Chaos, Solitons and Fractals* 40 (2009) 78–90.
[24] M.J. Gao, W. Zhang, M. H. Yao, Global analysis and chaotic dynamics of six-dimensional nonlinear system for an axially moving viscoelastic belt, *International Journal of Modern Physics B*, accepted.
[25] L .Q. Chen, H. Ding, Steady-state transverse response in coupled planar vibration of axially moving viscoelastic beams, *ASME Journal of Vibration and Acoustics* 132 (2010) 011009.
[26] L. Meirovitch, in: *Fundamentals of Vibrations*, McGraw-Hill, New York, 2001.