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Numerical modelling of longitudinal vibrations of a sucker rod string

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A R T I C L E I N F O

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A B S T R A C T

A new technique for analyzing the dynamic behavior of a sucker rod string used in the oil well industry is presented. The main difficulty in the numerical calculation of the examined structure is a multivalued velocity—force relation determined by Coulomb's friction and by loads generated during operation of pump valves. Both the monotonic and nonmonotonic velocity—force relations are considered. A quasi-variational inequality formulation of the problem is proposed. The solution of the inequality amounts to finding the minimum of a convex nonsmooth functional at each time step by means of the Newmark difference time scheme, successive iterations and finite element discretization. The problem of functional minimization is reduced to construction of a sequence of smooth nonlinear programming problems by introducing the auxiliary variables and applying the augmented Lagrangian method. The proposed approach is used to study the longitudinal vibrations of sucker rod strings under near-real conditions. In such systems the most commonly occurring vibration modes are the stick-slip vibrations and the vibrations with natural force excited twice a cycle. The nonmonotonic character of the friction law leads to intensification of these vibrations. In the case of nonmonotonic friction law the stick-slip vibrations can occur even under the action of constant external forces.

1. Introduction

Sucker rod strings are intended to transmit the movement from the rocking machine to the deep well plunger and screw pumps used in various oil recovery processes. The damage of such systems requires extremely expensive underground repair. Calculations that take into account a maximum number of factors influencing the behavior of these strings may essentially reduce the maintenance costs. In the present study, we restrict our consideration to the case of plunger pumps.

Since the 1950s interest in the development of mathematical models for these structures has quickened. In one of the earliest works [1] and in the book by the same author [2], the effect of internal forces is take into consideration by using the modifying factors. These factors are found by solving the wave equation after the phase of initial tension which is assumed to be static. The vibrations excited at the beginning of the moving phase are supposed to die out at the end of the moving phase. These two assumptions are fulfilled if the excitation frequency is small compared to the natural frequency of the sucker rod string, i.e. for short strings and slow regimes.

In works [3–5] the models based on the numerical solution of the wave equation of second order were developed. Here a specified displacement is taken as the boundary condition for the polished rod and the Robin condition (when the linear combination of the displacement and its gradient is specified) is the boundary condition for the pump. In [6,7] the sucker

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rod string is simulated using the systems of first-order partial differential equations. Then these systems are solved by the characteristic method and the finite difference method.

A very interesting technique was proposed in [8,9] to analyze the dynamic behavior of the sucker rod string. Here a complicated system was replaced by a simple one using the adaptive filter matrix method developed by these [10,11].

However, in all aforementioned approaches the strongly nonlinear nature of the forces acting on the sucker rod string have not been studied sufficiently. In the systems with the multivalued force–velocity relation there occurs a transition from the rest to the moving phase and reversal, but the time of these transitions is generally unknown. The multivalued force–velocity dependence arise due to the operation of pump valves and Coulomb's friction. In this case, the zero value of velocity corresponds to a set of force values. The characteristic feature of the systems under such load conditions is the change from one type of boundary conditions (in terms of displacements) to the other type (in terms of forces) and reversal which results in intensive natural frequency vibrations of the structure. Vibrations of this type are well studied for mechanical systems with one degree of freedom or a small number of degrees of freedom [12–16]. Numerical simulation of real systems with a large number of degrees of freedom is still an open problem.

For problems with multivalued nonlinearities, we need special solving techniques, for example, a quasi-variational (for nonmonotonic case—hemivariational) inequality formulation. The solution of this inequality can be reduced to finding the minimum of the convex nonsmooth functional at each time step by means of the Newmark difference time scheme, successive iterations and finite element discretization. This approach is widely used in the analysis of mechanical systems with Coulomb friction as well as in the studies on the dynamical problems of plasticity [17–27]. The forces generated in the pump systems show certain similarity to the Coulomb friction forces. Therefore it seems natural to use the variational inequality approach for these forces. Such an approach is proposed by the authors of the present work in [28].

The friction forces acting both on the pump plunger and on the sucker rod surface is generally dependent on velocity. This dependence can be nonmonotonic. In this case, the variation inequalities are not sufficient for modelling the sucker rod string. The more general hemivariational inequality approach is required [29–34].

In this paper, the quasi-variational and quasi-hemivariational inequality formulations of the problem under study are developed and the algorithm for their numerical realization is constructed.

Some examples of the behavior of the sucker rod string under near-real conditions are given. The first three examples illustrate the operation of the sucker rod string under periodical loads generated by a machine tool-rocking chair. In this case in addition to vibrations with loading frequencies, there occur natural vibrations exited twice a cycle. The nonmonotonicity of the friction law leads to intensification of vibrations. The fourth example shows the self-excited stick-slip vibrations of the structure pulled out at constant velocity.

2. Problem formulation and solution

The sucker rod string under study is a heavy elastic rod placed in a curved channel with the viscous fluid flowing through it. The scheme of the problem solution is shown in Fig. 1. The rod bounded by the channel walls can move only along its axis. The upper end of the rod moves periodically according to the known kinematic law, and its lower end is under the action of the force, which depends on the direction of the motion of the string end:

$$P_B(\dot{u}_b) = \begin{cases} P^-, & \dot{u}_B < 0, \\ [P^+, P^-], & \dot{u}_B = 0, \\ P^+, & \dot{u}_B > 0, \end{cases} \tag{1}$$

$$P^- > P^+.$$

Knowing inlet p_{in} and outlet pump pressures p_{out} one can calculate the limit forces P^- and P^+ as follows. The force acting on the plunger of the pump and on the lower end of sucker rod, coupled with it, is given by

$$P = p_{\text{out}} \cdot \frac{\pi}{4} \cdot (d_{\text{pl}}^2 - d_{\text{near}}^2) - p_{\text{c}} \cdot \frac{\pi}{4} \cdot d_{\text{pl}}^2 \pm P_{\text{f}}. \tag{2}$$

Here d_{pl}—plunger diameter, d_{near}—pumping rod diameter near the pump, P_{f}—magnitude of the force of friction between the plunger and the cylinder of the pump, p_{C}—the pressure in the pump cylinder.

If friction depends on velocity Eq. (2) is replaced by

$$P = p_{\text{out}} \cdot \frac{\pi}{4} \cdot (d_{\text{pl}}^2 - d_{\text{near}}^2) - p_{\text{c}} \cdot \frac{\pi}{4} \cdot d_{\text{pl}}^2 + P_{\text{f}}(\dot{u}_B). \tag{3}$$

When the plunger is moving upward, the suction valve is opened and the delivery valve is closed. The pressure in the pump cylinder is

$$p_{\text{c}} = p_{\text{in}} - \Delta p_{\text{in}},$$

where Δp_{in} is the pressure loss in the suction valve. In this case, the force acting on the plunger is

$$P = P^- = \frac{\pi}{4} \cdot (p_{\text{out}} \cdot (d_{\text{pl}}^2 - d_{\text{near}}^2) - (p_{\text{in}} - \Delta p_{\text{in}}) \cdot d_{\text{pl}}^2) + P_{\text{f}}.$$

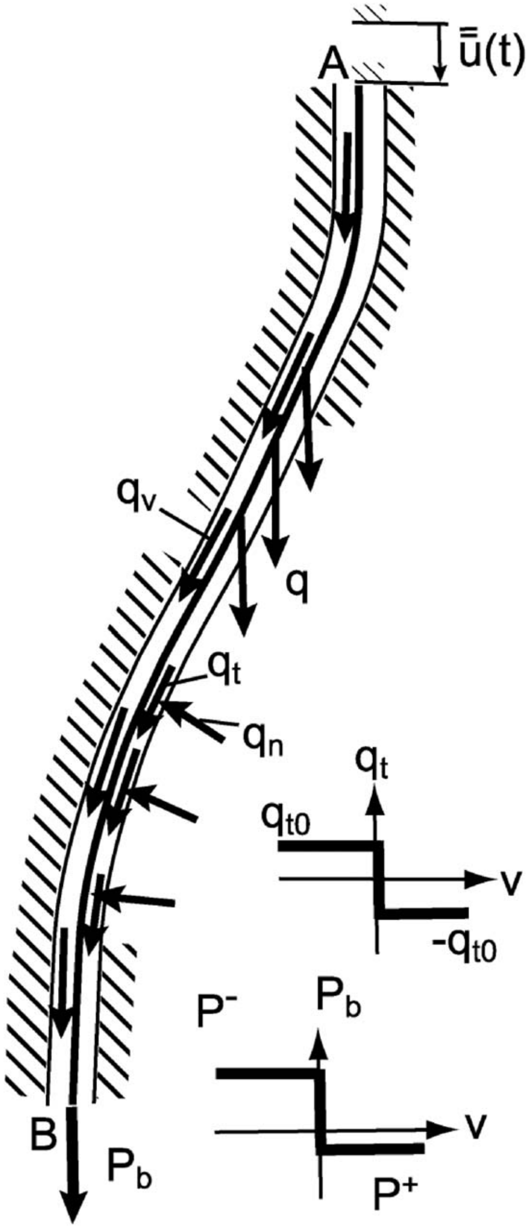

Fig. 1. Scheme of the problem.

When the plunger is moving downward, the suction valve is closed and the delivery valve is opened. The pressure in the pump cylinder is

$$p_c = p_{out} - \Delta p_{out},$$

where Δp_{out} is the pressure loss in the delivery valve. In this case, the force acting on the plunger is

$$P = P^+ = \frac{\pi}{4} \cdot (\Delta p_{out} \cdot d_{pl}^2 + p_{out} \cdot d_{near}^2) - P_f.$$

The values of the pressure losses are small enough compared to p_{in} and p_{out}. It is considered that $p_c = p_{in}$ when the plunger is moving upward, and that $p_c = p_{out}$ when the plunger is moving downward.

When the force acting on the lower end of the sucker rod is between P^- and P^+ the plunger is at rest.

The operation scheme of the pump is shown in Fig. 2.

The rod, being placed in the liquid, is subjected to pressure and viscous forces. Due to the channel curvature, the pressing forces and Coulomb's friction forces arise between the rod and the walls of the channel. To take into account the

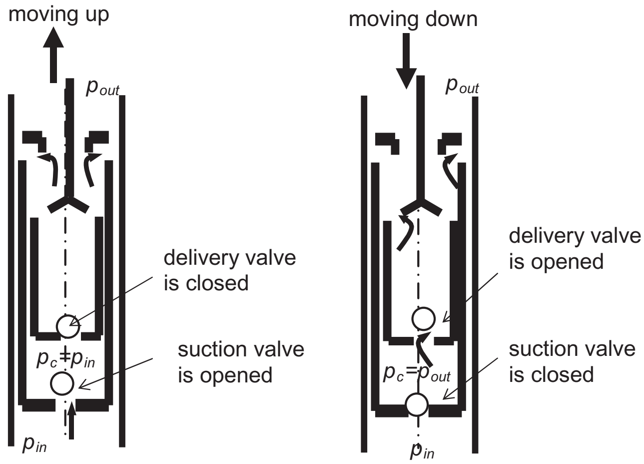

Fig. 2. Operation scheme of the pump.

pressure acting on the rod from the side of the liquid in the channel, we assume that the material strength is independent of the mean stress. Hence, the strength, stability and force of pressing against the channel walls of the rod section, whose lateral surface is subjected to pressure p at the longitudinal force N, are equivalent to those of the rod section experiencing no pressure at the following longitudinal force:

$$N_{\text{eq}} = N + pA, \tag{4}$$

where A is the area of the considered rod section.

In the examined system, the velocity–force relations for such loads as the Coulomb friction force and the force acting on the lower end of the rod are expressed as the multivalued relations. The interval of force values corresponds to the zero value of velocity. These relations can be written in a subdifferential form.

For the force acting on the lower end of the rod, the subdifferential relation can be defined as follows. It is shown [17] that in the one-dimensional case the subdifferential of the function can be represented as

$$\partial f(x) = \{x \in \mathbf{R} | f_-' \le x \le f_+'\}, \tag{5}$$

where f_-' and f_+' denote, respectively, the left- and right-hand derivatives. Hence, relation (1) can be written as

$$-P_{B,\text{eq}} \in \partial j_B(\dot{u}_B), \tag{6}$$

Here $P_{B,\text{eq}} = P_B + p_B A_B$, $j_B(\dot{u}_B)$ is the convex superpotential defined by

$$j_B(\dot{u}_B) = \sup_{u^* \in K_B} (u^*, \dot{u}_B)_{R^k}, \tag{7}$$

where for this case $k = 1$ and

$$K_B = [-P_{\text{eq}}^-, -P_{\text{eq}}^+], \tag{8}$$

$$P_{\text{eq}}^- = P^- + p_B A_B, \quad P_{\text{eq}}^+ = P^+ + p_B A_B.$$

The subdifferential relation for Coulomb's friction acting on the lateral surface of the sucker rod string is

$$-q_t \in \partial j_{t,q_n(u)}(\dot{u}), \tag{9}$$

where $j_{t,q_n(u)}(\dot{u})$ is the convex superpotential defined by

$$j_{t,q_n(u)}(\dot{u}) = \mu |q_n(u)||\dot{u}| = \sup_{u^* \in K_t(u)} (u^*, \dot{u})_{R^1}. \tag{10}$$

Here

$$K_t(u) = [-\mu|q_n(u)|, \mu|q_n(u)|] = [-q_{t0}(u), q_{t0}(u)]. \tag{11}$$

According to the definition of the subgradient, Eqs. (6) and (9) can be written as the variational inequality

$$j_{B,\text{eq}}(v) - j_{B,\text{eq}}(\dot{u}_B) \ge -P_{B,\text{eq}} \cdot (v - \dot{u}_B), \quad \forall v \in R^1 \tag{12}$$

and the quasi-variational inequality

$$j_{t,q_n(u)}(v) - j_{t,q_n(u)}(\dot{u}) \ge -q_t \cdot (v - \dot{u}), \quad \forall v \in R^1, \tag{13}$$

respectively. In the numerical analysis, we use the principle of virtual displacements:

$$\int_L \rho \ddot{u} w A \,\mathrm{d}x + \int_L C_{\tau v}\dot{u}_{\mathrm{eq}} w \,\mathrm{d}x + \int_L N_{\mathrm{eq}}(u_{\mathrm{eq}})\varepsilon(w)\,\mathrm{d}x = \int_L q_t(\dot{u}) w \,\mathrm{d}x + \int_L q_{\tau Q} w \,\mathrm{d}x + \int_L \left(\rho g \cos\alpha - \frac{\partial p_f}{\partial x}\right) w A \,\mathrm{d}x + P_{B,\mathrm{eq}}(\dot{u}_B) w_B - p_A A_{n_s} w_A, \tag{14}$$

where

$$P_{B,\mathrm{eq}} = P_B + p_B A_b.$$

Here ρ is the density of the rod material, $C_\tau v$ is the coefficient of hydrodynamic resistance, α is the angle between the rod axis and the vertical, and w is the trial function equal to zero at the point where the displacement has been prescribed.

With the principle of virtual displacements (14), taking into account the subdifferential boundary conditions (12) and (13) and representing the trial function w as

$$w = v - \dot{u}$$

we can formulate the problem as follows.

Find the displacement field $u(t)$ satisfying the quasi-variational inequality

$$(\rho A \ddot{u}, v - \dot{u}) + (C_{\tau v}\dot{u}, v - \dot{u}) + a(u, v - \dot{u}) + j_B(v_B) - j_B(\dot{u}_B) + \Phi_{t,q_n(u)}(v) - \Phi_{t,q_n(u)}(\dot{u}) \geq l(v - \dot{u}), \quad \forall v \in U, \tag{15}$$

the initial conditions for u and $\dot{u}$ and such as $\dot{u}(t) \in U$.

Here

$$\Phi_{t,q_n(u)}(v) = \int_L j_{t,q_n(u)}(v)\,\mathrm{d}x,$$

U is a set of admissible velocities, $(u, v) = \int_L uv \,\mathrm{d}x$ is the inner product, $a(u, v)$ is the bilinear form, and $l(v)$ is the linear functional.

To perform time discretization, we use the Newmark scheme. By expressing the acceleration and displacements at t_{n+1} in terms of velocities at this time step and $u^{(n)}$, $\dot{u}^{(n)}$, $\ddot{u}^{(n)}$ calculated at the previous time step and taking into account the properties of the bilinear form $a(\cdot,\cdot)$, the scalar product and linearity of the functional $l(\cdot)$, we obtain the following formulation of the problem. Find $\dot{u}^{(n+1)} \in U_{n+1}$ satisfying the quasi-variational inequality

$$\hat{a}(\dot{u}^{(n+1)}, v - \dot{u}^{(n+1)}) + j_B(v_B) - j_B(\dot{u}_B^{(n+1)}) + \Phi_{t,q_n(u^{(n+1)})}(v) - \Phi_{t,q_n(u^{(n+1)})}(\dot{u}^{(n+1)}) \geq l(v - \dot{u}^{(n+1)}), \quad \forall v \in U_{n+1}, \tag{16}$$

where

$$\hat{a}(u, v) = \frac{\rho}{\gamma \Delta t} \cdot (Au, v) + (C_{\tau v} u, v) + \frac{\beta \Delta t}{\gamma} \cdot a(u, v), \tag{17}$$

$$\hat{l}(v) = l(v) + \rho \cdot (\tilde{w}, v) - a(\tilde{u}, v). \tag{18}$$

Here $\tilde{u}$ and $\tilde{w}$ are the quantities dependent on the values of displacements, velocities and accelerations calculated at the previous time step.

As it is shown in works [18,20,21], the solution to the quasi-variational inequality can be found by solving successively the following variational inequalities:

$$\hat{a}(\dot{u}^{[k+1]}, v - \dot{u}^{[k+1]}) + j_B(v_B) - j_B(\dot{u}_B^{[k+1]}) + \Phi_{t,q_n(u^{[k]})}(v) - \Phi_{t,q_n(u^{[k]})}(\dot{u}^{[k+1]}) \geq l(v - \dot{u}^{[k+1]}), \quad \forall v \in U_{n+1}, \tag{19}$$

where $\dot{u}^{[k]}$ is the k-th approximation of the velocity field $u^{(n+1)}$, and $u^{[k]}$ is the corresponding approximation of the displacement field. As the initial approximation, we take the velocity field at the previous time step.

Since the time step is rather small then we need to use only one iteration for getting the acceptable accuracy. In this case, the quasi-variational inequality can be replaced by the variational one. Thus, we have

$$\hat{a}(\dot{u}^{(n+1)}, v - \dot{u}^{(n+1)}) + j_B(v_B) - j_B(\dot{u}_B^{(n+1)}) + \Phi_{t,q_n(u^{(n)})}(v) - \Phi_{t,q_n(u^{(n)})}(\dot{u}^{(n+1)}) \geq l(v - \dot{u}^{(n+1)}), \quad \forall v \in U_{n+1}. \tag{20}$$

Finding solution to the variational inequality (20) is equivalent to a search for the minimum of the nonsmooth functional

$$J(\dot{u}^{(n+1)}) = \tfrac{1}{2}\hat{a}(\dot{u}^{(n+1)}, \dot{u}^{(n+1)}) + j_B(\dot{u}_B^{(n+1)}) + \Phi_t(\dot{u}^{(n+1)}) - \hat{l}(\dot{u}^{(n+1)}) \tag{21}$$

subject to $\dot{u}^{(n+1)} \in U_{n+1}$.

Using the Newmark scheme, a set of acceptable velocities U can be expressed as

$$U_{n+1} = \{\dot{u} \,|\, \dot{u} = \hat{u} \text{ in the point } A\}, \tag{22}$$

where

$$\hat{u} = \frac{\gamma}{\beta \Delta t}(\overline{\overline{u}}(t_{n+1}) - \tilde{u}).$$

Here $\overline{\overline{u}}(t)$ is the prescribed displacement of the upper point A at time t.

For practical implementation of the algorithm, the rod is decomposed into n finite elements linear in displacements. The functionals can be represented in the matrix form as

$$\hat{a}(u,v)=\mathbf{u}^{\mathbf{T}}\mathbf{K}_*\mathbf{v},\quad \hat{l}(v)=\mathbf{b}_*^{\mathbf{T}}\mathbf{v},$$

where

$$\mathbf{K}_*=\frac{\beta\Delta t}{\gamma}\mathbf{K}+\frac{1}{\Delta t\gamma}\mathbf{M}+\mathbf{C},\quad \mathbf{b}_*=\mathbf{b}+\mathbf{M}\tilde{\mathbf{w}}-\mathbf{K}\tilde{\mathbf{u}}.$$

Here, **K**, **M**, **C** are, respectively, the stiffness, mass and damping matrices, **b** is the external force vector, and $\tilde{\mathbf{u}}$,$\tilde{\mathbf{w}}$ are the vectors composed of $\tilde{u}$ and $\tilde{w}$ for discrete nodes.

Thus, the problem is reduced to minimization of the finite-dimensional functional

$$J^h(\mathbf{v})=\tfrac{1}{2}\hat{a}^h(\mathbf{v},\mathbf{v})+\Phi(\mathbf{v})-\hat{l}^h(\mathbf{v}) \tag{23}$$

subject to $\mathbf{v}\in\mathbf{U}$.

The minimization of functional (23) can be reduced to a series of smooth problems of nonlinear programming using the augmented Lagrangian method [35] or the Udzava method [22].

The details of the finite element realization, solution of the problem by the augmented Lagrangian method and testing of the algorithm in the framework of the one-degree-of-freedom system having accurate analytical solution can be found in work [28].

3. Nonmonotonic friction modelling

In the previous section, the force in the sliding phase is considered to be constant. However, under actual friction conditions, the tangential force depends on the sliding velocity. This velocity–force relation may be very complicated and generally nonmonotonic. The nonmonotonicity of the friction law leads to additional computational difficulties. In this case, the inequality to be solved is hemivariational, and the functional to be minimized at each time step is nonsmooth and nonconvex [17,29].

The variational formulation of the problems with the nonmonotonic multivalued force–velocity relation is as follows.

Find the displacement field $u(t)$ satisfying the dynamic hemivariational inequality

$$(\rho A\ddot{u},v-\dot{u})+(C\dot{u},v-\dot{u})+a(u,v-\dot{u})+\Phi^0_{t,q_n}(\dot{u},v-\dot{u})\geq l(v-\dot{u}),\quad \forall v\in U, \tag{24}$$

the initial conditions for u and $\dot{u}$ and such as $\dot{u}(t)\in U$.

Here

$$\Phi^0_{t,q_n}(v,g)=\int_L j^0_{t,q_n}(v,g)\,\mathrm{d}L+j^0_B(v_B,g),$$

where

$$j^0(x,g)=\limsup_{\mu\to 0,y\to x}\frac{j(y+\mu g)-j(y)}{\mu}$$

is the generalized Clarke derivative of the nonsmooth nonmonotone functional j in the direction g.

In the present work, we consider a sufficiently general class of the nonmonotonic friction laws, which can be represented as a difference of two monotonic multivalued functions

$$P_f=P_{f1}-P_{f2}, \tag{25}$$

$$q_t=q_{t1}-q_{t2}, \tag{26}$$

where

$$-P_{f1}\in\partial j_{f1}(\dot{u}_B),\quad -P_{f2}\in\partial j_{f2}(\dot{u}_B), \tag{27}$$

$$-q_{t1}\in\partial j_{t1,q_n(u)}(\dot{u}),\quad -q_{t2}\in\partial j_{t2,q_n(u)}(\dot{u}). \tag{28}$$

Such relations are called quasi-differential ones [32].

From (3) and (29) we can write

$$P_{B,\mathrm{eq}}=P_{B1,\mathrm{eq}}-P_{f2}. \tag{29}$$

In this case the nonconvex functionals j_{t,q_n} and j_B can be represented as a difference of the convex functionals:

$$j(v)=j_1(v)-j_2(v).$$

According to the Newmark scheme, this dynamic inequality can be reduced to solving at each time step the following problem:

Find $\dot{u}^{(n+1)}(t) \in U_{n+1}$, satisfying the static hemivariational inequality

$$a_1(u^{(n+1)}, v - \dot{u}^{(n+1)}) + \Phi^0_{t,q_n}(u^{(n+1)}, v - \dot{u}^{(n+1)}) \geq l(v - \dot{u}^{(n+1)}), \quad \forall v \in U_{n+1}, \tag{30}$$

which, in turn, can be treated by sequential solution of variational inequalities [29,30,33]

$$a_1(u^{(n+1)}, v - \dot{u}^{(n+1)}) + \Phi^a_{t,q_n}(v) - \Phi^a_{t,q_n}(\dot{u}^{(n+1)}) \geq l(v - \dot{u}^{(n+1)}), \quad \forall v \in U_{n+1}. \tag{31}$$

Here $\Phi^a_{t,q_n}(v)$ is the convex approximation of the nonconvex functional obtained when replacing

$$j(v) = j_1(v) - j_2(v)$$

by

$$j(v) = j_1(v) - j_2(v_a) - \partial j_2(v_a) \cdot (v - v_a),$$

where v_a is the value of velocity obtained at the previous iteration.

The variational inequality (30) is similar to inequality (20) and can be treated by the same way.

4. Applications

The displacement–force curve for the upper section of the sucker rod string (dynamogramme) obtained as a result of numerical modelling is compared with the real one. The column under consideration is composed of two parts (bottom up): 1—diameter 19 mm, length 440 m; 2—diameter 22 mm, length 624 m. The magnitude of the force at the lower end of the sucker rod string as it moves upwards is 9.5 kN, and is −2.2 kN as it moves downwards. The excitation frequency is 6 strokes per minute. The double amplitude of the upper end motion is 2.4 m. These curves are shown in Fig. 3. Both computed and real curves consist of four segments. The upper part corresponds to the phase when the pump plunger is moving up. The lower part corresponds to the downward movement of the plunger. Here we can see a considerable displacement of the upper section of the rod and rather intensive force oscillations. These two segments are separated by steeper ones which correspond to the phases when the pump plunger is at rest. On the last two segments one can observe a significant growth (falling) of the force acting on the upper rod section. The displacement of the upper end of the rod is far smaller than in the first two phases and is due to the rod deformation only. It is seen that there is rather good agreement between the numerical and experimental results.

The obtained algorithm was employed to calculate the rod columns under near-real conditions. The column is composed of three parts (bottom up): 1—diameter 19 mm, length 272 m; 2—diameter 22 mm, length 392 m; 3—diameter 25 mm, length 328 m. The maximal zenith angle describing the channel geometry is 20° at a depth of 336 m. The coefficient of friction of the rod against the tube walls f is 0.1. The Young module $E = 2 \times 10^5$ MPa. The density of the rod material $\rho = 7800\,\mathrm{kg/m^3}$. The magnitude of the force at the lower end of the sucker rod string moving upwards is 11.09 kN, and is 3.55 kN for its downward movement. The excitation frequency is 6 strokes per minute. The double amplitude of the upper end motion is 1.6 m.

Fig. 4 shows the displacement and velocity of the lower end of the pumping rod string and the axial forces acting on the upper (solid line) and lower (dashed line) sections of the rod. The lower end of the rod column is in the stick-slip motion. During stops, the force acting on the lower end changes from one limiting value to the other. Apart from vibrations with the frequency of the excitation load, there occur natural vibrations excited twice a cycle as the direction of the lower end motion changes.

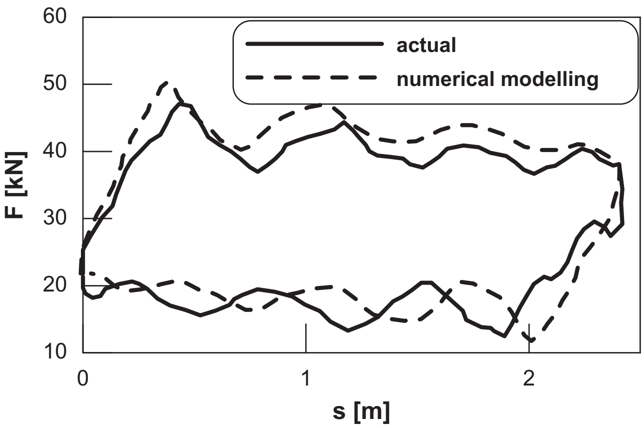

Fig. 3. Comparison of the numerical solution with real force–displacement curve.

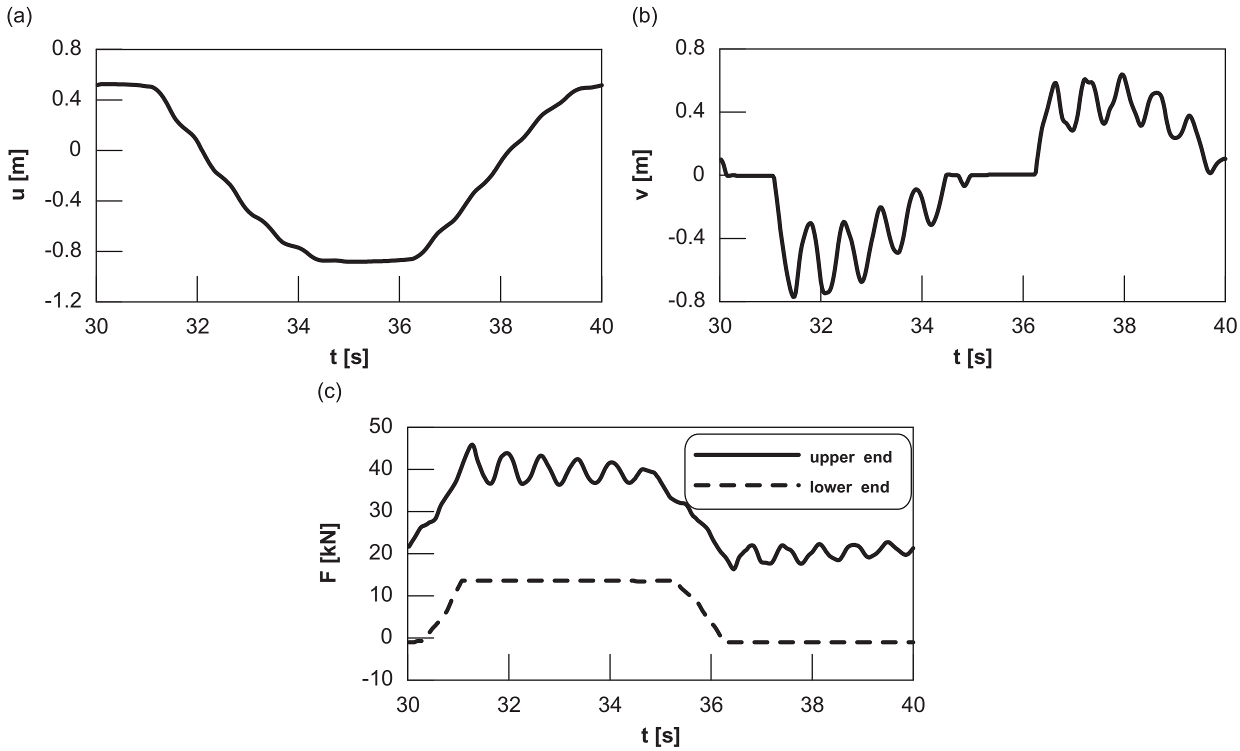

Fig. 4. Vibrations excited in the sucker rod string: (a) displacement of the lower end, (b) velocity of the lower end, (c) axial forces.

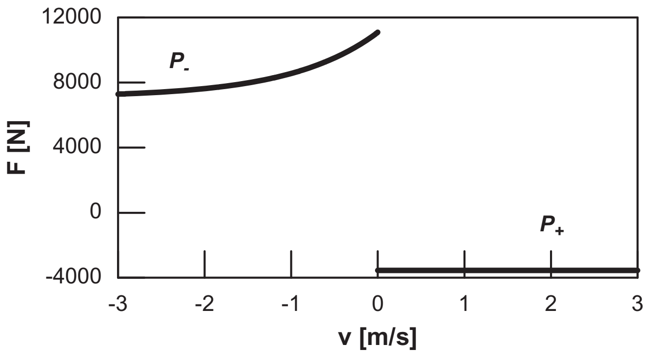

Fig. 5. Multivalued nonmonotone velocity–force dependence.

The algorithm developed is employed to calculate the rod columns in the case when the friction force depends nonmonotonically on the velocity. The geometric parameters, the excitation frequency, the double amplitude of the upper end motion and the force on the lower end of the pumping rod string moving downwards are similar to those considered in the previous case. The force on the lower end of the string moving upwards depends on its velocity v:

$$P^{-} = P_{\infty} + (P_0 - P_{\infty}) \cdot e^{-v/v_0}. \tag{32}$$

Here, P_0 is the value of P^- at $v = 0$; P_∞ is the value of P^- when $v \to \infty$; v_0 is the factor with the velocity dimension. The following values of parameters are taken: $P_0 = 11.09\,\text{kN}$; $P_\infty = 7.09\,\text{kN}$; $v_0 = 1.0\,\text{m/s}$.

The velocity–force relation for the lower end of the string is shown in Fig. 5.

The results of calculation for the upper end of the pumping rod string are shown in Fig. 6 as a displacement–force curve for the upper section of the rod (dynamogramme). The bold line corresponds to the nonmonotonic velocity–force relation (32). Two thin lines correspond to the cases when the lower end of the string (upward motion) is under the action of constant force $P^- = P_0$ and P_∞, respectively.

The results show that the nonmonotonicity of the friction law leads to intensification of vibrations.

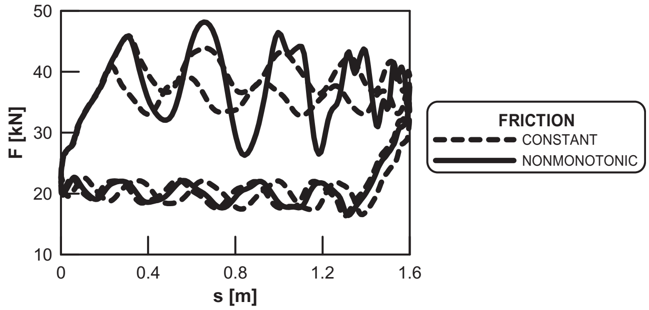

Fig. 6. Displacement–force curve in the upper section of the sucker rod string (dynamogramme) for monotonic and nonmonotonic cases.

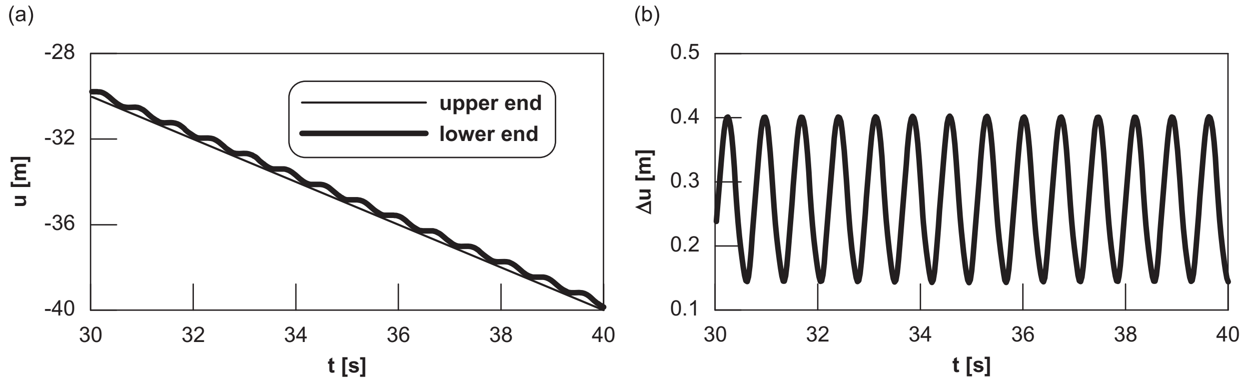

Fig. 7. Pulling out of the sucker rod string. Stick-slip vibrations: (a) displacement of the lower end, (b) difference of lower and upper end displacements.

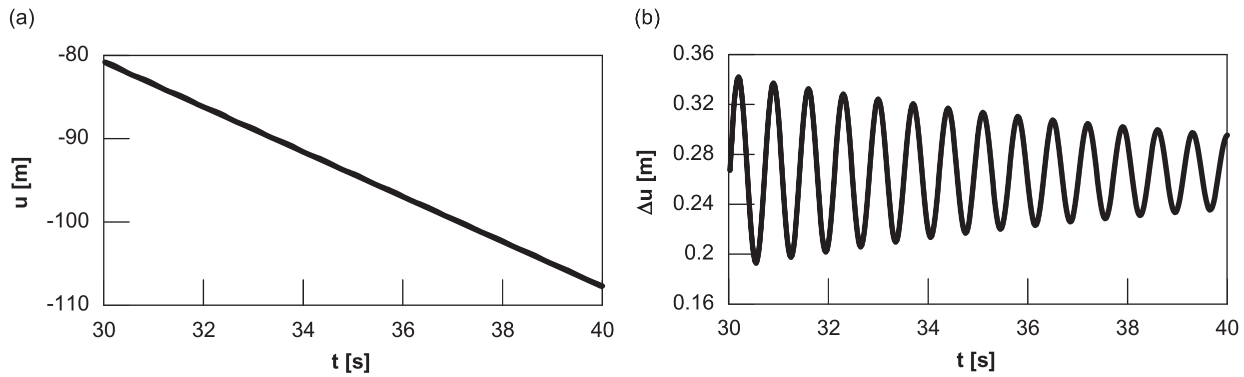

Fig. 8. Pulling out of the sucker rod string. Interruption of the stick-slip vibrations: (a) displacement of the lower end, (b) difference of lower and upper end displacements.

The event when the system is pulled out at constant velocity is also considered. In this case the nonmonotonicity of the friction law leads to the self-excited stick-slip vibrations of the lower end of the string at the low pulling velocity (Fig. 7). There is a critical value of the pulling velocity, at which vibrations disappear (Fig. 8).

Here

$$\Delta u = u_b - u_a,$$

u_a and u_b are the displacements of the upper and lower ends of the string, respectively.

In addition, the dependence of the stick-slip vibration amplitude on the pulling velocity is obtained for different values of the parameters v_0, P_0 and P_∞ (Fig 9).

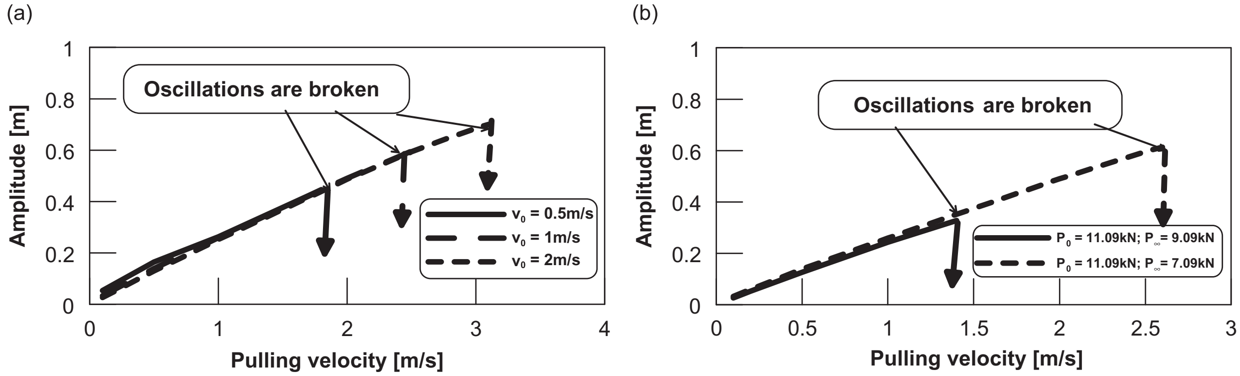

Fig. 9. Dependence of stick-slip vibration amplitude on the pulling velocity.

It has been found that the values of the aforementioned parameters produce no significant effect on the stick-slip vibration amplitude, but the very existence of vibrations depends on these parameters.

5. Conclusions

The mathematical model of longitudinal vibrations of the sucker rod string and its numerical implementation have been developed. This model can take into account strongly nonlinear forces resulting from the action of the pump valves and Coulomb's friction. Both the monotonic and nonmonotonic force–velocity relations can be treated by this model.

Comparison of the displacement–force curve for the upper section of the sucker rod string (dynamogramme) obtained as a result of numerical modelling with the real curve shows rather good agreement between the numerical and experimental results.

The lower end of the column is in stick-slip motion. The vibration with natural frequency of the construction is excited twice a cycle during transition of the lower end from the rest to the movement phase. In the case of nonmonotonicity of the friction law the lower end vibrations are intensified.

When the friction law is nonmonotonic the stick-slip vibration can take place even under the action of constant exciting force. The example of pulling out of the sucker rod string at constant velocity is considered. In this case, the nonmonotonicity of the friction law leads to the self-excited stick-slip vibrations of the lower end of the string at a low value of the pulling velocity. There is a critical value of the pulling velocity at which the oscillations disappear.

The dependence of the stick-slip vibration amplitude on the pulling velocity has been obtained for different values of the parameters describing the friction law. The values of these parameters do not affect significantly the amplitude of stick-slip vibrations, but the existence of these vibrations strongly depends on these parameters.

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