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Nonlinear traveling wave vibration of a circular cylindrical shell subjected to a moving concentrated harmonic force

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ABSTRACT

This is a study of nonlinear traveling wave response of a cantilever circular cylindrical shell subjected to a concentrated harmonic force moving in a concentric circular path at a constant velocity. Donnell's shallow-shell theory is used, so that moderately large vibrations are analyzed. The problem is reduced to a system of ordinary differential equations by means of the Galerkin method. Frequency–responses for six different mode expansions are studied and compared with that for single mode to find the more contracted and accurate mode expansion investigating traveling wave vibration. The method of harmonic balance is applied to study the nonlinear dynamic response in forced oscillations of this system. Results obtained with analytical method are compared with numerical simulation, and the agreement between them bespeaks the validity of the method developed in this paper. The stability of the period solutions is also examined in detail.

1. Introduction

A rotating circular cylindrical shell is a fundamental component of many machines and mechanisms, and is often subjected to a stationary transverse force: for example, interference effects in mechanism containing shells, shell turbine design and localized pressure discontinuity. There are two problems of physical importance. One is the problem that the transverse force is moving around the shell in a circular path, and the other is the effect that rotation has on the elastic properties of the shell. For the latter problem has been studied by Bryan [1], Mizoguchi [2], Chen et al. [3], Lam and Li [4] and Li and Lam [5], the purpose of the work described in this paper is to study theoretically the former problem. One of the interesting aspects of the problem is the question of what vibratory characteristic of a shell to a moving concentrated harmonic force shows.

Critical speed of a rotating cylindrical shell to constant axial load was investigated by Ng and Lam [6], then dynamic stability of rotating cylindrical shells subjected to periodic axial loads was studied by Liew et al. [7]. Huang and Hsu [8] examined the resonance of a rotating cylindrical shell due to the action of harmonic moving loads. However, these three articles mainly concentrated the loads synchronous whirl with the rotating shells. Applying the Fourier transform method in conjunction with the contour integral, Huang [9] made a study of the steady-state response of an elastic, infinitely long, cylindrical shell subjected to a ring load traveling at a constant velocity.

In most published works on dynamics of circular cylindrical shells, the nonlinear responses of stationary cylindrical shells to moving concentrated harmonic forces are not found. This paper presents a theoretical analysis of the steady-state

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Nomenclature

c	the coefficient of damping of the shell
D	the flexural rigidity of the shell
E	Young's modulus of the shell
$F(t)$	external excitation
h	the wall thickness of the shell
k	multiples of frequency
L	the length of the shell
m	the number of axial half-waves
n	the number of circumferential waves
R	the middle-surface radius of the shell
t	time

Greek letters

δ	the Dirac delta function
μ	the Poisson ratio of the shell
ρ	the mass density of the shell
ω	radian frequency of external excitation
$\omega_{m,n}$	the linear radian frequency corresponding to the mode (m, n)
ω_n	the angular velocity of the moving concentrated harmonic force

response of this model. By considering a stationary cylindrical shell, the additional complication of the effect that rotation has on the elastic properties of the shell is eliminated. The study is carried out using Donnell's nonlinear shallow-shell theory for thin shells together with the consideration of geometric nonlinearity. In order to reduce a drastic calculating effort, it is important to use only the most significant modes. In this study, a more accurate and simpler mode expansion to describe the vibration property of the shell is found by comparing frequency–response curves for six different mode expansions with that for single mode. The method of harmonic balance is used to present an approximate analytical solution of this system, and the results obtained are compared with numerical simulation. The good agreement between them bespeaks the validity of the method developed in this paper. The stability of the period solutions is also examined in detail.

2. Differential equation of motion

In this study, attention is focused on a cantilever stationary cylindrical shell to a moving concentrated harmonic force, as shown in Fig. 1. The cylindrical shell is considered to be thin, with length L, wall thickness h, and middle-surface radius R. Its material properties are mass density ρ, the Poisson ratio μ, Young's modulus E and the damping coefficient c. A cylindrical coordinate system (x, θ, z) is chosen, with the origin O fixed on the center of one end of the shell, where x is the axial and z is the radial coordinate. The displacements of points of the middle surface of the shell are denoted by u, v and w, in the axial, circumferential and radial directions, respectively; w is taken positive outwards. The harmonic excitation is assumed to be in the neighborhood of the mode (m, n) of the shell having prevalent radial displacement, where m is the number of axial half-waves and n is the number of circumferential waves.

Considering a cell on the neutral surface of the shell with damping and large-amplitude shell motion effects, as shown in Fig. 2, we can obtain the equations of motion in the x, θ and z directions as follows:

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\theta}}{\partial \theta} + q_x - \rho h \frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

$$\frac{1}{R}\frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{x\theta}}{\partial x} + \frac{Q_\theta}{R} + q_\theta - \rho h \frac{\partial^2 v}{\partial t^2} = 0 \tag{2}$$

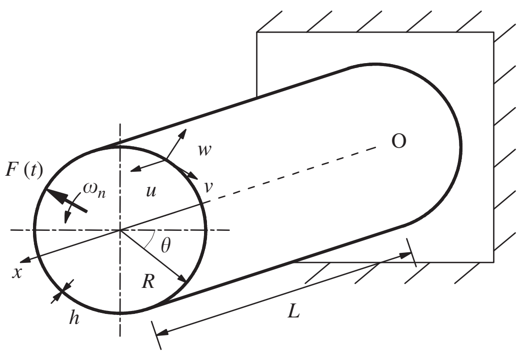

Fig. 1. Coordinate system of a circular cylindrical shell.

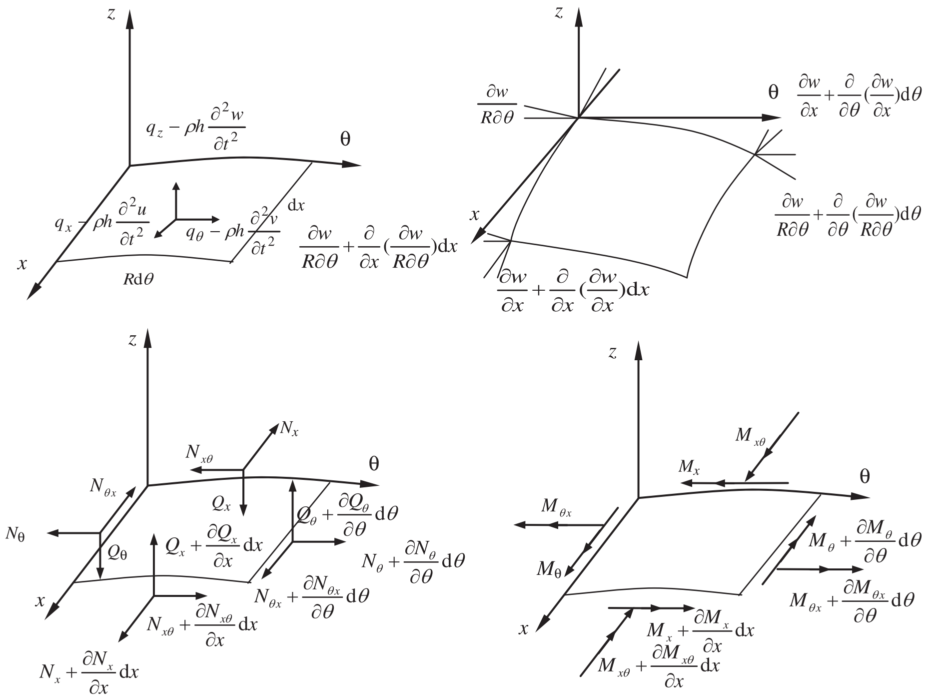

Fig. 2. The distortion of cell and the force on cell.

$$\frac{\partial Q_x}{\partial x}+\frac{1}{R}\frac{\partial Q_\theta}{\partial \theta}-\frac{1}{R}N_\theta+N_x\frac{\partial^2 w}{\partial x^2}+\frac{N_\theta}{R^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{2N_{x\theta}}{R}\frac{\partial^2 w}{\partial x\partial \theta}+q_z-c\frac{\partial w}{\partial t}-\rho h\frac{\partial^2 w}{\partial t^2}=0 \tag{3}$$

where

$$Q_x=\frac{\partial M_x}{\partial x}+\frac{1}{R}\frac{\partial M_{x\theta}}{\partial \theta},\, Q_\theta=\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial M_\theta}{\partial \theta}$$

and q_z is given by $q_z=F(t)$.

Taking into account Donnell's nonlinear shallow-shell theory, Eqs. (1)–(3) reduce to

$$\frac{\partial N_x}{\partial x}+\frac{1}{R}\frac{\partial N_{x\theta}}{\partial \theta}=0 \tag{4}$$

$$\frac{1}{R}\frac{\partial N_\theta}{\partial \theta}+\frac{\partial N_{x\theta}}{\partial x}=0 \tag{5}$$

$$D\nabla^2\nabla^2 w+c\frac{\partial w}{\partial t}+\rho h\frac{\partial^2 w}{\partial t^2}+\frac{N_\theta}{R}-N_x\frac{\partial^2 w}{\partial x^2}-\frac{N_\theta}{R^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{2N_{x\theta}}{R}\frac{\partial^2 w}{\partial x\partial \theta}=F(t) \tag{6}$$

Introducing Airy stress function Φ, the forces per unit length in the axial and circumferential directions, as well as the shear force, are given by [10]

$$N_x=h\frac{\partial^2 \Phi}{R^2\partial \theta^2}=h\sigma_x,\, N_\theta=h\frac{\partial^2 \Phi}{\partial x^2}=h\sigma_\theta,\, N_{x\theta}=-h\frac{\partial^2 \Phi}{\partial x\partial \theta}=h\tau_{x\theta} \tag{7}$$

Stress–strain relationships can be written as [10]

$$\sigma_x=\frac{E}{1-\mu^2}(\varepsilon_x+\mu\varepsilon_\theta) \tag{8a}$$

$$\sigma_\theta=\frac{E}{1-\mu^2}(\varepsilon_\theta+\mu\varepsilon_x) \tag{8b}$$

$$\tau_{x\theta}=G\varepsilon_{x\theta} \tag{8c}$$

Nonlinear geometric equations of the system can be written as

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{9a}$$

$$\varepsilon_\theta = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{1}{2}\left(\frac{\partial w}{R\partial \theta}\right)^2 \tag{9b}$$

$$\gamma_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{1}{R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} \tag{9c}$$

For the circular cylindrical shell, the following relationships between transverse and in-plane displacements are used [10]:

$$\frac{\partial u}{\partial x} = 0,\ \frac{\partial v}{\partial \theta} = -w,\ \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta} = 0 \tag{10}$$

Substituting Eqs. (9) and (10) in Eq. (8) and substituting Eqs. (7) and (8) in Eqs. (4)–(6), and replacing all force resultants with displacements variables, Eqs. (4)–(6) reduce to

$$D\nabla^2\nabla^2 w + c\frac{\partial w}{\partial t} + \rho h\frac{\partial^2 w}{\partial t^2} = F(t) + A_{\text{nonlin}} \tag{11}$$

where the harmonic operator is defined as $\nabla^2 = \partial^2/\partial x^2 + \partial^2/(R^2\,\partial\theta^2)$, the flexural rigidity is $D = Eh^3/[12(1-\mu^2)]$, $F(t)$ is an external excitation moving along the shell, having the form

$$F(t) = F_0\cos(\omega t)\delta(x - x_0)\delta(\theta + \omega_n t) \tag{12}$$

where ω_n is the rotating speed of the force, δ the Dirac delta function, ω radian frequency of external excitation, F_0 gives the force amplitude, x_0 give the axial positions of the point of application of the force. Here, the point excitation is located at $x_0 = 0.335\,\text{m}$.

The geometric nonlinearity is given by

$$A_{\text{nonlin}} = \alpha_1\frac{\partial^2 w}{\partial\theta^2}\left(\frac{\partial w}{\partial\theta}\right)^2 + \alpha_2\frac{\partial^2 w}{\partial\theta^2}\left(\frac{\partial w}{\partial x}\right)^2 + \alpha_3\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + \alpha_2\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial\theta}\right)^2 + \alpha_4\frac{\partial w}{\partial x}\frac{\partial w}{\partial\theta}\frac{\partial^2 w}{\partial x\,\partial\theta} \tag{13}$$

where the functions α_1, α_2, α_3 and α_4 are given in Appendix. Note that Donnell's nonlinear shallow-shell equations are accurate only for modes with a large number n of circumferential waves; it is generally assumed that $1/n^2 \ll 1$ is required in order to have fairly good accuracy (i.e. $n \geq 6$). Donnell's nonlinear shallow-shell equations are obtained by neglecting the in-plane inertia, transverse shear deformation and rotary inertia, giving accurate results only for very thin shells. In-plane displacements are assumed to be infinitesimal, whereas w is of the same order as the shell thickness.

3. Responses of different mode expansions

The following mode expansion of the flexural deformation w has been used:

$$w(x,\theta,t) = \sum_{m=1}^{M}\sum_{n=N_0}^{N}\sum_{k=1}^{K} U_m(x)[A_{m,kn}(t)\cos(kn\theta) + B_{m,kn}(t)\sin(kn\theta)] \tag{14}$$

where $A_{m,kn}(t)$ and $B_{m,kn}(t)$ are unknown functions of time t, k is multiples of frequency and $U_m(x)$ is the functions of axial vibrating shape of the shell having the following form:

$$U_m(x) = C_{m,1}e^{P_{m,1}x} + C_{m,2}e^{-P_{m,1}x} + C_{m,3}\cos(P_{m,2}x) + C_{m,4}\sin(P_{m,2}x)$$

in which $C_{m,1}$, $C_{m,2}$, $C_{m,3}$, $C_{m,4}$, $P_{m,1}$ and $P_{m,2}$ are appropriate coefficients obtained by the free vibration equation of the shell.

By using Galerkin method, the ordinary, coupled nonlinear differential equations can be obtained for the variables $A_{m,n}(t)$ and $B_{m,n}(t)$, by successively weighting the single original equation with suitable functions z_s, and integrating over the shell middle surface. The weighting functions z_s are formed from axial and circumferential vibrating shape functions.

The Galerkin projection, in this case, can be defined as

$$\int_0^L\int_0^{2\pi}\left(D\nabla^2\nabla^2 w + c\frac{\partial w}{\partial t} + \rho h\frac{\partial^2 w}{\partial t^2}\right)z_s R\,dx\,d\theta = \int_0^L\int_0^{2\pi}[F(t) + A_{\text{nonlin}}]z_s R\,dx\,d\theta \tag{15}$$

3.1. Single mode

The nonlinear response of the system for mode including six circumferential waves and one longitudinal half-wave ($K = 1$, $N_0 = N = 6$, $M = 1$) is investigated by using the following mode expansion:

$$w(x,\theta,t) = U_1(x)[A_{1,6}(t)\cos(6\theta) + B_{1,6}(t)\sin(6\theta)] \tag{16}$$

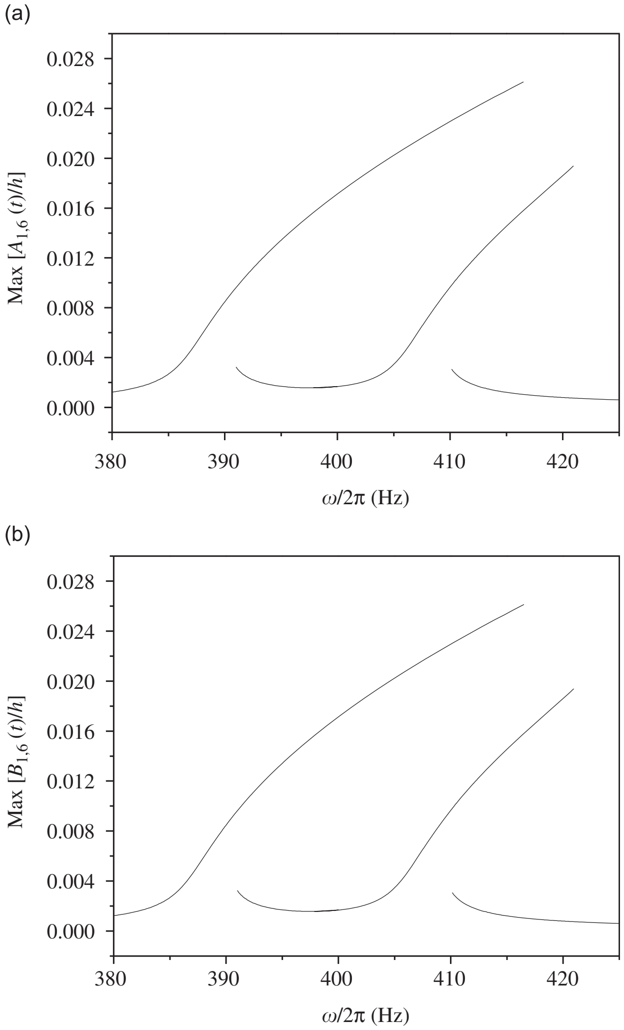

Fig. 3. Frequency–response curves for single mode for $\omega_n = 10\,\mathrm{rad/s}$, $F_0 = 20\,\mathrm{N}$: (a) maximum of $A_{1,6}(t)/h$; and (b) maximum of $B_{1,6}(t)/h$.

The weighting functions z_s are defined as

$$z_s(x,\theta) = \begin{cases} U_1(x)\cos(6\theta), & s=1 \\ U_1(x)\sin(6\theta), & s=2 \end{cases} \tag{17}$$

Substituting the expansion of w, Eq. (16), and Eq. (17) in Eq. (15), two coupled nonlinear ordinary differential equations are obtained for the variables $A_{1,6}(t)$ and $B_{1,6}(t)$:

$$\ddot{A}_{1,6}(t) + 2\zeta_{1,6}\omega_{1,6}\dot{A}_{1,6}(t) + \omega_{1,6}^2 A_{1,6}(t) = \tilde{F}\cos(\omega t)\cos(6\omega_n t) + HA_{1,6}(t)^3 + HA_{1,6}(t)B_{1,6}(t)^2 \tag{18}$$

$$\ddot{B}_{1,6}(t) + 2\zeta_{1,6}\omega_{1,6}\dot{B}_{1,6}(t) + \omega_{1,6}^2 B_{1,6}(t) = -\tilde{F}\cos(\omega t)\sin(6\omega_n t) + HB_{1,6}(t)^3 + HB_{1,6}(t)A_{1,6}(t)^2 \tag{19}$$

where $\zeta_{1,6}$, $\tilde{F}$ and H are appropriate coefficients given in Appendix. The projection of part of Eqs. (18) and (19) is quite tedious and was performed by using the *Mathematica* computer software [11]. The case relates to a circular cylindrical shell, having the following dimensions and properties: $L = 0.335\,\mathrm{m}$, $R = 0.15\,\mathrm{m}$, $h = 0.001\,\mathrm{m}$, $\mu = 0.3$, $c = 20\,\mathrm{N\,s\,m^{-3}}$, $E = 2.06 \times 10^{11}\,\mathrm{Pa}$, $\rho = 7.85 \times 10^3\,\mathrm{kg\,m^{-3}}$, the linear radian frequency is $\omega_{1,6} = 2\pi \times 397.58\,\mathrm{rad\,s^{-1}}$, all the numerical solutions have been obtained by using the software *Fortran 95* [12], based on the Runge–Kutta method. The periodic solutions obtained show the maximum amplitude in a period.

Fig. 3 shows the frequency–response relationships for $A_{1,6}(t)$ and $B_{1,6}(t)$, when the excitation frequency is in the neighborhood of the linear resonance of mode (m=1, n=6). It could be found that there are two traveling waves with different linear resonant frequencies which are symmetric about the natural frequency $\omega_{1,6}$, showing that the vibratory mode of the shell is traveling with respect to the rotary force.

3.2. Multi-modes

It has been known that linear modal base is the simplest choice for discretizing the system, in particular, in order to reduce the number of degrees of freedom, it is important to use only the most significant modes. In this paper, six different mode expansions are chosen to study the nonlinear responses of the shell, respectively. Then the results are compared with that of single mode, aiming to find the proper mode expansion to describe the resonant characteristic of the shell. They are given by

Case 1 (two modes): ($K=1$, $N_0=5$, $N=6$, $M=1$), has the following mode expansion:

$$w(x,\theta,t)=\sum_{n=5}^{6}U_1(x)[A_{1,n}(t)\cos(6\theta)+B_{1,n}(t)\sin(6\theta)] \tag{20}$$

Case 2 (two modes): ($K=1$, $N_0=6$, $N=7$, $M=1$), has the following mode expansion:

$$w(x,\theta,t)=\sum_{n=6}^{7}U_1(x)[A_{1,n}(t)\cos(n\theta)+B_{1,n}(t)\sin(n\theta)] \tag{21}$$

Case 3 (three modes): ($K=1$, $N_0=5$, $N=7$, $M=1$), has the following mode expansion:

$$w(x,\theta,t)=\sum_{n=5}^{7}U_1(x)[A_{1,n}(t)\cos(n\theta)+B_{1,n}(t)\sin(n\theta)] \tag{22}$$

Case 4 (two modes): ($K=2$, $N_0=N=6$, $M=1$), has the following mode expansion:

$$w(x,\theta,t)=\sum_{k=1}^{2}U_1(x)[A_{1,6k}(t)\cos(6k\theta)+B_{1,6k}(t)\sin(6k\theta)] \tag{23}$$

Case 5 (two modes): the mode $A_{1,0}(t)$ is considered for the single mode, with the following mode expansion:

$$w(x,\theta,t)=U_1(x)[A_{1,6}(t)\cos(6\theta)+B_{1,6}(t)\sin(6\theta)]+A_{1,0}(t)U_1(x) \tag{24}$$

Case 6 (two modes): ($K=1$, $N_0=N=6$, $M=2$), has the following mode expansion:

$$w(x,\theta,t)=\sum_{m=1}^{2}U_m(x)[A_{m,6}(t)\cos(6\theta)+B_{m,6}(t)\sin(6\theta)] \tag{25}$$

Here we only take the case 6 as example, to give the numerical solving process. For the case 6, the weighting functions z_s are defined as

$$z_s(x,\theta)=\begin{cases}U_1(x)\cos(6\theta), & s=1\\ U_1(x)\sin(6\theta), & s=2\\ U_2(x)\cos(6\theta), & s=3\\ U_2(x)\sin(6\theta), & s=4\end{cases} \tag{26}$$

The Galerkin projection of the equation of motion (11) has been performed by using the *Mathematica* computer software, and the following system of four equations is obtained for the variables $A_{1,6}(t)$, $B_{1,6}(t)$, $A_{2,6}(t)$ and $B_{2,6}(t)$:

$$\begin{aligned}&\ddot{A}_{1,6}(t)+2\zeta_{1,6}\omega_{1,6}\dot{A}_{1,6}(t)+\omega_{1,6}^2A_{1,6}(t)+l_1A_{2,6}(t)\\&=\tilde{F}_1\cos(\omega t)\cos(6\omega_n t)+H_1A_{1,6}(t)^3+H_2A_{1,6}(t)^2A_{2,6}(t)+H_3A_{1,6}(t)A_{2,6}(t)^2\\&\quad+H_4A_{1,6}(t)B_{1,6}(t)^2+H_5A_{1,6}(t)B_{1,6}(t)B_{2,6}(t)+H_6A_{1,6}(t)B_{2,6}(t)^2\\&\quad+H_7A_{2,6}(t)^3+H_8A_{2,6}(t)B_{1,6}(t)^2+H_9A_{2,6}(t)B_{1,6}(t)B_{2,6}(t)+H_{10}A_{2,6}(t)B_{2,6}(t)^2\end{aligned} \tag{27}$$

$$\begin{aligned}&\ddot{B}_{1,6}(t)+2\zeta_{1,6}\omega_{1,6}\dot{B}_{1,6}(t)+\omega_{1,6}^2B_{1,6}(t)+l_1B_{2,6}(t)\\&=-\tilde{F}_1\cos(\omega t)\sin(6\omega_n t)+H_1B_{1,6}(t)^3+H_2B_{1,6}(t)^2B_{2,6}(t)+H_3B_{1,6}(t)B_{2,6}(t)^2\\&\quad+H_4B_{1,6}(t)A_{1,6}(t)^2+H_5B_{1,6}(t)A_{1,6}(t)B_{2,6}(t)+H_6B_{1,6}(t)A_{2,6}(t)^2\\&\quad+H_7B_{2,6}(t)^3+H_8B_{2,6}(t)A_{1,6}(t)^2+H_9B_{2,6}(t)A_{1,6}(t)A_{2,6}(t)+H_{10}B_{2,6}(t)A_{2,6}(t)^2\end{aligned} \tag{28}$$

$$\begin{aligned}&\ddot{A}_{2,6}(t)+2\zeta_{2,6}\omega_{2,6}\dot{A}_{2,6}(t)+\omega_{2,6}^2A_{2,6}(t)+l_2A_{1,6}(t)\\&=\tilde{F}_2\cos(\omega t)\cos(6\omega_n t)+G_1A_{1,6}(t)^3+G_2A_{1,6}(t)^2A_{2,6}(t)+G_3A_{1,6}(t)A_{2,6}(t)^2\\&\quad+G_4A_{1,6}(t)B_{1,6}(t)^2+G_5A_{1,6}(t)B_{1,6}(t)B_{2,6}(t)+G_6A_{1,6}(t)B_{2,6}(t)^2\\&\quad+G_7A_{2,6}(t)^3+G_8A_{2,6}(t)B_{1,6}(t)^2+G_9A_{2,6}(t)B_{1,6}(t)B_{2,6}(t)+G_{10}A_{2,6}(t)B_{2,6}(t)^2\end{aligned} \tag{29}$$

$$\begin{aligned}&\ddot{B}_{2,6}(t)+2\zeta_{2,6}\omega_{2,6}\dot{B}_{2,6}(t)+\omega_{2,6}^2B_{2,6}(t)+l_2B_{1,6}(t)\\&=-\tilde{F}_2\cos(\omega t)\sin(6\omega_n t)+G_1B_{1,6}(t)^3+G_2B_{1,6}(t)^2B_{2,6}(t)+G_3B_{1,6}(t)B_{2,6}(t)^2\\&\quad+G_4B_{1,6}(t)A_{1,6}(t)^2+G_5B_{1,6}(t)A_{1,6}(t)B_{2,6}(t)+G_6B_{1,6}(t)A_{2,6}(t)^2\\&\quad+G_7B_{2,6}(t)^3+G_8B_{2,6}(t)A_{1,6}(t)^2+G_9B_{2,6}(t)A_{1,6}(t)A_{2,6}(t)+G_{10}B_{2,6}(t)A_{2,6}(t)^2\end{aligned}\tag{30}$$

where $\zeta_{2,6}$, l_1, l_2, $\tilde{F}_1$, $\tilde{F}_2$, H_i (i=1,…,10) and G_j (j=1,…,10) are appropriate coefficients.

The other five cases can be dealt with in the same way as case 6, and are omitted here. Numerical computations have been carried out for the six cases discussed above. The dimensions and properties of the shell is the same as that in the single mode analyses.

Fig. 4(a)–(f) show the frequency–response comparisons of the six different mode expansions with that of single mode for backward waves when the excitation frequency is in the neighborhood of the linear resonance of principal mode (m=1, n=6). It can be found, the mode expansions ($K=1$, $N_0=5$, $N=6$, $M=1$), ($K=1$, $N_0=6$, $N=7$, $M=1$), ($K=1$, $N_0=5$, $N=7$, $M=1$), ($K=2$, $N_0=N=6$, $M=1$), and single mode with mode $A_{1,0}(t)$ participation do not

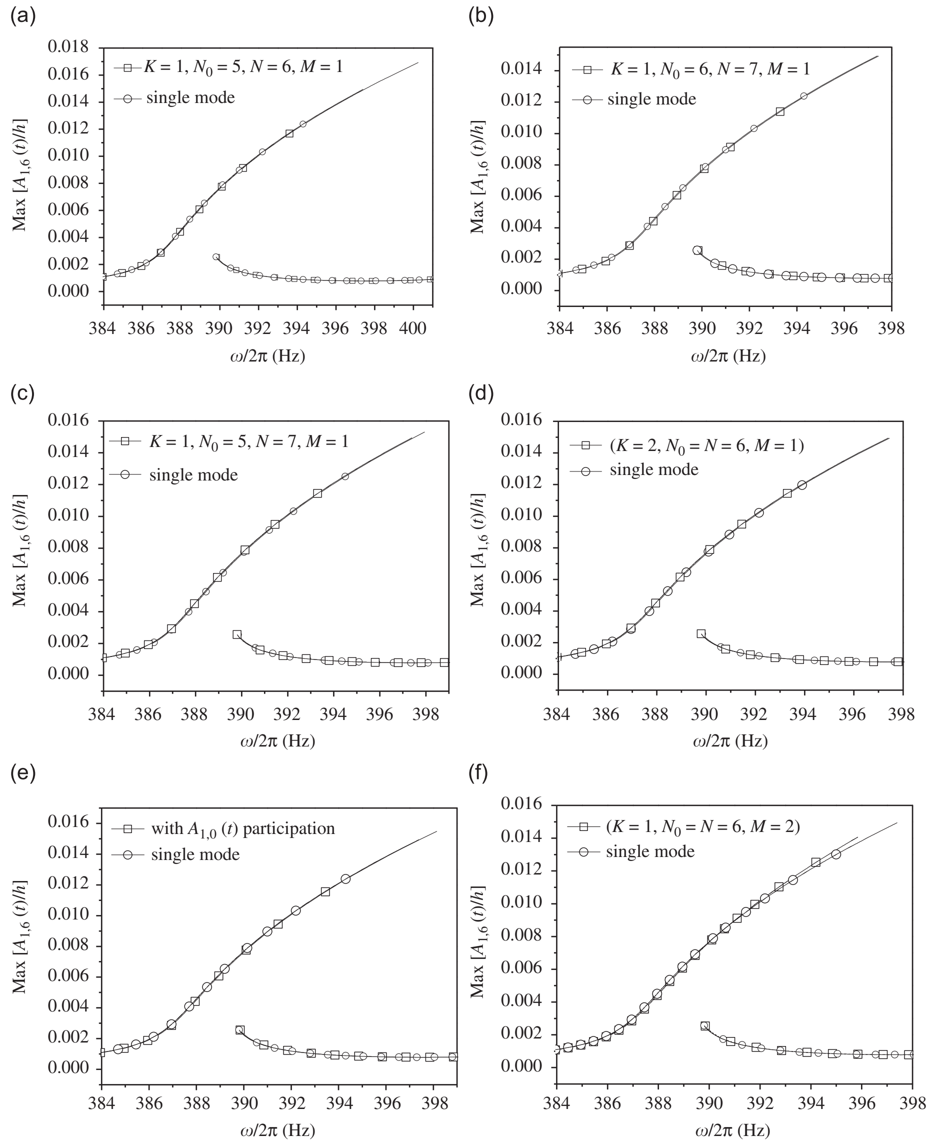

Fig. 4. Frequency–response comparisons of different mode expansions with single mode (m=1, n=6) for $\omega_n=10$ rad/s, $F_0=10$ N: (a) mode expansion ($K=1$, $N_0=5$, $N=6$, $M=1$); (b) mode expansion ($K=1$, $N_0=6$, $N=7$, $M=1$); (c) mode expansion ($K=1$, $N_0=5$, $N=7$, $M=1$); (d) mode expansion ($K=2$, $N_0=N=6$, $M=1$); (e) single mode with mode $A_{1,0}(t)$ participation; and (f) mode expansion ($K=1$, $N_0=N=6$, $M=2$).

significantly change the single-mode ($K=1,\ N_0=N=6,\ M$=1) result, whereas, the response for mode expansion ($K=1,\ N_0=N=6,\ M=2$) is apparently different from that for single mode, especially in the neighborhood of nonlinear region, as shown in Fig. 4(f).

As a consequence of the insensitivity of the response to additional mode expansions of n and k, it is reasonably believed that further increases in the number of n and k would not significantly change the single-mode response.

The result shows that the effects of both additional circumferential waves n and multiples of frequency k are absolutely insignificant but that of additional axial half-waves m is significant on principal mode (m=1, n=6) resonant response. Thus adopting one principal circumferential mode (n=6) is adequate to study the response of the shell, but additional longitudinal mode (m=2) should be considered for more accurate solutions in the neighborhood of the principal mode.

4. Analytical solution

Adopting double modes ($K=1, N_0=N=6, M=2$), and introducing the non-dimensional variables and parameters in Appendix, Eqs. (27)–(30) may be written as a dimensionless form

$$
\begin{aligned}
&\ddot{\tilde{A}}_{1,6}(\tau)+2\zeta_{1,6}\dot{\tilde{A}}_{1,6}(\tau)+\tilde{A}_{1,6}(\tau)+\tilde{l}_1\tilde{A}_{2,6}(\tau)\\
&=\bar{\tilde{F}}_1\cos(\Omega_1\tau)+\bar{\tilde{F}}_1\cos(\Omega_2\tau)+\tilde{H}_1\tilde{A}_{1,6}(\tau)^3+\tilde{H}_2\tilde{A}_{1,6}(\tau)^2\tilde{A}_{2,6}(\tau)+\tilde{H}_3\tilde{A}_{1,6}(\tau)\tilde{A}_{2,6}(\tau)^2\\
&+\tilde{H}_4\tilde{A}_{1,6}(\tau)\tilde{B}_{1,6}(\tau)^2+\tilde{H}_5\tilde{A}_{1,6}(\tau)\tilde{B}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)+\tilde{H}_6\tilde{A}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)^2\\
&+\tilde{H}_7\tilde{A}_{2,6}(\tau)^3+\tilde{H}_8\tilde{A}_{2,6}(\tau)\tilde{B}_{1,6}(\tau)^2+\tilde{H}_9\tilde{A}_{2,6}(\tau)\tilde{B}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)+\tilde{H}_{10}\tilde{A}_{2,6}(\tau)\tilde{B}_{2,6}(\tau)^2
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\ddot{\tilde{B}}_{1,6}(\tau)+2\zeta_{1,6}\dot{\tilde{B}}_{1,6}(\tau)+\tilde{B}_{1,6}(\tau)+\tilde{l}_1B_{2,6}(\tau)\\
&=\bar{\tilde{F}}_1\sin(\Omega_1\tau)-\bar{\tilde{F}}_1\sin(\Omega_2\tau)+\tilde{H}_1\tilde{B}_{1,6}(\tau)^3+\tilde{H}_2\tilde{B}_{1,6}(\tau)^2\tilde{B}_{2,6}(\tau)+\tilde{H}_3\tilde{B}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)^2\\
&+\tilde{H}_4\tilde{B}_{1,6}(\tau)A_{1,6}(\tau)^2+\tilde{H}_5\tilde{B}_{1,6}(\tau)A_{1,6}(\tau)\tilde{B}_{2,6}(\tau)+\tilde{H}_6\tilde{B}_{1,6}(\tau)A_{2,6}(\tau)^2\\
&+\tilde{H}_7\tilde{B}_{2,6}(\tau)^3+\tilde{H}_8\tilde{B}_{2,6}(\tau)A_{1,6}(\tau)^2+\tilde{H}_9\tilde{B}_{2,6}(\tau)A_{1,6}(\tau)A_{2,6}(\tau)+\tilde{H}_{10}\tilde{B}_{2,6}(\tau)A_{2,6}(\tau)^2
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&\ddot{\tilde{A}}_{2,6}(\tau)+2\zeta_{2,6}\frac{\omega_{2,6}}{\omega_{1,6}}\dot{\tilde{A}}_{2,6}(\tau)+\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2\tilde{A}_{2,6}(\tau)+\tilde{l}_2\tilde{A}_{1,6}(\tau)\\
&=\bar{\tilde{F}}_2\cos(\Omega_1\tau)+\bar{\tilde{F}}_2\cos(\Omega_2\tau)+\tilde{G}_1\tilde{A}_{1,6}(\tau)^3+\tilde{G}_2\tilde{A}_{1,6}(\tau)^2\tilde{A}_{2,6}(\tau)+\tilde{G}_3\tilde{A}_{1,6}(\tau)\tilde{A}_{2,6}(\tau)^2\\
&+\tilde{G}_4\tilde{A}_{1,6}(\tau)\tilde{B}_{1,6}(\tau)^2+\tilde{G}_5\tilde{A}_{1,6}(\tau)\tilde{B}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)+\tilde{G}_6\tilde{A}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)^2\\
&+\tilde{G}_7\tilde{A}_{2,6}(\tau)^3+\tilde{G}_8\tilde{A}_{2,6}(\tau)\tilde{B}_{1,6}(\tau)^2+\tilde{G}_9\tilde{A}_{2,6}(\tau)\tilde{B}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)+\tilde{G}_{10}\tilde{A}_{2,6}(\tau)\tilde{B}_{2,6}(\tau)^2
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&\ddot{\tilde{B}}_{2,6}(\tau)+2\zeta_{2,6}\frac{\omega_{2,6}}{\omega_{1,6}}\dot{\tilde{B}}_{2,6}(\tau)+\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2\tilde{B}_{2,6}(\tau)+\tilde{l}_2\tilde{B}_{1,6}(\tau)\\
&=\bar{\tilde{F}}_2\sin(\Omega_1\tau)-\bar{\tilde{F}}_2\sin(\Omega_2\tau)+\tilde{G}_1\tilde{B}_{1,6}(\tau)^3+\tilde{G}_2\tilde{B}_{1,6}(\tau)^2\tilde{B}_{2,6}(\tau)+\tilde{G}_3\tilde{B}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)^2\\
&+\tilde{G}_4\tilde{B}_{1,6}(\tau)\tilde{A}_{1,6}(\tau)^2+\tilde{G}_5\tilde{B}_{1,6}(\tau)\tilde{A}_{1,6}(\tau)\tilde{B}_{2,6}(\tau)+\tilde{G}_6\tilde{B}_{1,6}(\tau)\tilde{A}_{2,6}(\tau)^2\\
&+\tilde{G}_7\tilde{B}_{2,6}(\tau)^3+\tilde{G}_8\tilde{B}_{2,6}(\tau)\tilde{A}_{1,6}(\tau)^2+\tilde{G}_9\tilde{B}_{2,6}(\tau)\tilde{A}_{1,6}(\tau)\tilde{A}_{2,6}(\tau)+\tilde{G}_{10}\tilde{B}_{2,6}(\tau)\tilde{A}_{2,6}(\tau)^2
\end{aligned}
\tag{34}
$$

The solutions of Eqs. (31)–(34) can be assumed as

$$
\begin{cases}
\tilde{A}_{1,6}(\tau)=P_1\cos(\Omega_1\tau+\alpha_1)+Q_1\cos(\Omega_2\tau+\beta_1)\\
\tilde{B}_{1,6}(\tau)=P_1\sin(\Omega_1\tau+\alpha_2)+Q_1\sin(\Omega_2\tau+\beta_2)\\
\tilde{A}_{2,6}(\tau)=P_2\cos(\Omega_1\tau+\alpha_3)+Q_2\cos(\Omega_2\tau+\beta_3)\\
\tilde{B}_{2,6}(\tau)=P_2\sin(\Omega_1\tau+\alpha_4)+Q_2\sin(\Omega_2\tau+\beta_4)
\end{cases}
\tag{35}
$$

In these expressions for $\tilde{A}_{1,6}$, $\tilde{B}_{1,6}$, $\tilde{A}_{2,6}$ and $\tilde{B}_{2,6}$ the two terms are harmonic oscillations with the forcing frequencies Ω_1 and Ω_2. Here $\Omega_1/\Omega_2=(\omega+6\omega_n)/(\omega-6\omega_n)$.

Substituting (35) into (31)–(34), using some trigonometric identities and equating the coefficients of the terms in $\cos(\Omega_1\tau+\alpha_1)$, $\sin(\Omega_1\tau+\alpha_1)$, $\cos(\Omega_2\tau+\beta_1)$, $\sin(\Omega_2\tau+\beta_1)$, $\cos(\Omega_1\tau+\alpha_2)$, $\sin(\Omega_1\tau+\alpha_2)$, $\cos(\Omega_2\tau+\beta_2)$, $\sin(\Omega_2\tau+\beta_2)$, $\cos(\Omega_1\tau+\alpha_3)$, $\sin(\Omega_1\tau+\alpha_3)$, $\cos(\Omega_2\tau+\beta_3)$, $\sin(\Omega_2\tau+\beta_3)$, $\cos(\Omega_1\tau+\alpha_4)$, $\sin(\Omega_1\tau+\alpha_4)$, $\cos(\Omega_2\tau+\beta_4)$, $\sin(\Omega_2\tau+\beta_4)$ of

Eqs. (31)–(34), we obtain the following 16 equations in the 12 unknowns P_1, P_2, Q_1, Q_2, α_1, α_2, α_3, α_4, β_1, β_2, β_3, β_4:

$$\begin{cases} P_1(1-\Omega_1^2)=\bar{\tilde{F}}_1\cos(\alpha_1)+\left(\frac{3}{4}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)P_1^3+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)P_1P_2^2 \\ \quad+\left(\frac{3}{2}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)P_1Q_1^2+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)P_1Q_2^2 \\ \quad-2P_1\zeta_1\Omega_1=\bar{\tilde{F}}_1\sin(\alpha_1) \\ Q_1(1-\Omega_2^2)=\bar{\tilde{F}}_1\cos(\beta_1)+\left(\frac{3}{4}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)Q_1^3+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)Q_1Q_2^2 \\ \quad+\left(\frac{3}{2}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)Q_1P_1^2+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)Q_1P_2^2 \\ \quad-2Q_1\zeta_1\Omega_2=\bar{\tilde{F}}_1\sin(\beta_1) \\ P_1(1-\Omega_1^2)=\bar{\tilde{F}}_1\cos(\alpha_2)+\left(\frac{3}{4}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)P_1^3+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)P_1P_2^2 \\ \quad+\left(\frac{3}{2}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)P_1Q_1^2+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)P_1Q_2^2 \\ \quad-2P_1\zeta_1\Omega_1=\bar{\tilde{F}}_1\sin(\alpha_2) \\ Q_1(1-\Omega_2^2)=\bar{\tilde{F}}_1\cos(\beta_2)+\left(\frac{3}{4}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)Q_1^3+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)Q_1Q_2^2 \\ \quad+\left(\frac{3}{2}\tilde{H}_1+\frac{1}{2}\tilde{H}_4\right)Q_1P_1^2+\left(\frac{1}{2}\tilde{H}_3+\frac{1}{2}\tilde{H}_6\right)Q_1P_2^2 \\ \quad-2Q_1\zeta_1\Omega_2=\bar{\tilde{F}}_1\sin(\beta_2) \\ P_2\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2-\Omega_1^2\right]=\bar{\tilde{F}}_2\cos(\alpha_3)+\left(\frac{3}{4}\tilde{G}_7+\frac{1}{2}\tilde{G}_{10}\right)P_2^3+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)P_2P_1^2 \\ \quad+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)P_2Q_1^2+\left(\frac{3}{2}\tilde{G}_3+\frac{1}{2}\tilde{G}_{10}\right)P_2Q_2^2 \\ \quad-2P_2\zeta_2\Omega_1\frac{\omega_{2,6}}{\omega_{1,6}}=\bar{\tilde{F}}_2\sin(\alpha_3) \\ Q_2\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2-\Omega_2^2\right]=\bar{\tilde{F}}_2\cos(\beta_3)+\left(\frac{3}{4}\tilde{G}_7+\frac{1}{2}\tilde{G}_{10}\right)Q_2^3+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)Q_2Q_1^2 \\ \quad+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)Q_2P_1^2+\left(\frac{3}{2}\tilde{G}_3+\frac{1}{2}\tilde{G}_{10}\right)Q_2P_2^2 \\ \quad-2Q_2\zeta_2\Omega_2\frac{\omega_{2,6}}{\omega_{1,6}}=\bar{\tilde{F}}_2\sin(\beta_3) \\ P_2\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2-\Omega_1^2\right]=\bar{\tilde{F}}_2\cos(\alpha_4)+\left(\frac{3}{4}\tilde{G}_7+\frac{1}{2}\tilde{G}_{10}\right)P_2^3+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)P_2P_1^2 \\ \quad+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)P_2Q_1^2+\left(\frac{3}{2}\tilde{G}_3+\frac{1}{2}\tilde{G}_{10}\right)P_2Q_2^2 \\ \quad-2P_2\zeta_2\Omega_1\frac{\omega_{2,6}}{\omega_{1,6}}=\bar{\tilde{F}}_2\sin(\alpha_4) \\ Q_2\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2-\Omega_2^2\right]=\bar{\tilde{F}}_2\cos(\beta_4)+\left(\frac{3}{4}\tilde{G}_7+\frac{1}{2}\tilde{G}_{10}\right)Q_2^3+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)Q_2Q_1^2 \\ \quad+\left(\frac{1}{2}\tilde{G}_2+\frac{1}{2}\tilde{G}_8\right)Q_2P_1^2+\left(\frac{3}{2}\tilde{G}_3+\frac{1}{2}\tilde{G}_{10}\right)Q_2P_2^2 \\ \quad-2Q_2\zeta_2\Omega_2\frac{\omega_{2,6}}{\omega_{1,6}}=\bar{\tilde{F}}_2\sin(\beta_4) \end{cases} \tag{36}$$

from Eq. (36) we get

$$\begin{cases}\alpha_1=\alpha_2\\ \beta_1=\beta_2\end{cases}\begin{cases}\alpha_3=\alpha_4\\ \beta_3=\beta_4\end{cases} \tag{37}$$

It can be found the fifth, sixth, seventh, eighth, thirteenth, fourteenth, fifteenth and sixteenth equations in (36) have the same forms with the first, second, third, fourth, ninth, tenth, eleventh and twelfth equations, respectively (i.e., the fifth, sixth, seventh, eighth, thirteenth, fourteenth, fifteenth and sixteenth equations can be omitted). The eight equations retained yield a process to find P_1, P_2, Q_1, Q_2, α_1, α_3, β_1, β_3, the amplitudes and the phase angles of the harmonic oscillations with the forcing frequencies Ω_1 and Ω_2.

The contributions of backward wave Q_1 of the principal mode (m=1, n=6) and the forward and backward waves (P_2,Q_2) of additional longitudinal mode (m=2) should be linearized when one considers the resonance of forward wave P_1 of the principal mode. In this first approximation we have the following expressions:

$$Q_1 = \frac{\bar{\tilde{F}}_1}{\sqrt{(1-\Omega_2^2)^2 + (2\zeta_1\Omega_2)^2}} \tag{38}$$

$$\beta_1 = \arctan\left(\frac{-2\zeta_1\Omega_2}{1-\Omega_2^2}\right) \tag{39}$$

$$P_2 = \bar{\tilde{F}}_2 \Bigg/ \sqrt{\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2 - \Omega_1^2\right]^2 + \left(2\zeta_2\Omega_1\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2} \tag{40}$$

$$\alpha_3 = \arctan\left\{-2\zeta_2\Omega_1\frac{\omega_{2,6}}{\omega_{1,6}} \Bigg/ \left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2 - \Omega_1^2\right]\right\} \tag{41}$$

$$Q_2 = \bar{\tilde{F}}_2 \Bigg/ \sqrt{\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2 - \Omega_2^2\right]^2 + \left(2\zeta_2\Omega_2\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2} \tag{42}$$

$$\beta_3 = \arctan\left\{-2\zeta_2\Omega_1\frac{\omega_{2,6}}{\omega_{1,6}} \Bigg/ \left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2 - \Omega_2^2\right]\right\} \tag{43}$$

where Eqs. (38) and (39) are derived from the third and the fourth, Eqs. (40) and (41) from the ninth and the tenth, and Eqs. (42) and (43) from the eleventh and the twelfth equations of (36), respectively.

Substituting Eqs. (38), (40) and (42) in the first and second equations of (36), we obtain the equations of the phase angles and the response curves for the forward wave of the principal mode α_1, P_1

$$\alpha_1 = \arctan[-2\zeta_1\Omega_1/(1-\Omega_1^2) - (\tfrac{3}{4}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)P_1^2 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)P_2^2 - (\tfrac{3}{2}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)Q_1^2 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)Q_2^2] \tag{44}$$

$$[P_1(1-\Omega_1^2) - (\tfrac{3}{4}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)P_1^3 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)P_1P_2^2 - (\tfrac{3}{2}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)P_1Q_1^2 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)P_1Q_2^2]^2 + (2P_1\zeta_1\Omega_1)^2 = \bar{\tilde{F}}_1^2 \tag{45}$$

The Eq. (45) is of the third degree in P_1^2. Thus for a given value of Ω_1 or Ω_2 there are one or three real solutions for P_1^2.

Similar to the approximation above, when one considers the resonance of backward wave Q_1 of the principal mode, the contributions of forward wave P_1 of the principal mode and that of forward and backward waves (P_2, Q_2) of the additional longitudinal mode (m=2) should be linearized, this gives

$$P_1 = \frac{\bar{\tilde{F}}_1}{\sqrt{(1-\Omega_1^2)^2 + (2\zeta_1\Omega_1)^2}} \tag{46}$$

$$\alpha_1 = \arctan\left(\frac{-2\zeta_1\Omega_1}{1-\Omega_1^2}\right) \tag{47}$$

$$P_2 = \bar{\tilde{F}}_2 \Bigg/ \sqrt{\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2 - \Omega_1^2\right]^2 + \left(2\zeta_2\Omega_1\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2} \tag{48}$$

$$Q_2 = \bar{\tilde{F}}_2 \Bigg/ \sqrt{\left[\left(\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2 - \Omega_2^2\right]^2 + \left(2\zeta_2\Omega_2\frac{\omega_{2,6}}{\omega_{1,6}}\right)^2} \tag{49}$$

where Eqs. (46) and (47) are derived from the first and the second, Eq. (48) from the ninth and the tenth, and Eq. (49) from the eleventh and the twelfth equations of (36), respectively.

Substituting Eqs. (46), (48) and (49) in the third and forth equations of (36), we obtain the equations of the phase angles and the response curves for the backward wave of the principal mode β_1, Q_1

$$\beta_1 = \arctan[-2\zeta_1\Omega_2/(1-\Omega_2^2) - (\tfrac{3}{4}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)Q_1^2 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)Q_2^2 - (\tfrac{3}{2}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)P_1^2 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)P_2^2] \tag{50}$$

$$[Q_1(1-\Omega_2^2) - (\tfrac{3}{4}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)Q_1^3 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)Q_1Q_2^2 - (\tfrac{3}{2}\tilde{H}_1 + \tfrac{1}{2}\tilde{H}_4)Q_1P_1^2 - (\tfrac{1}{2}\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_6)Q_1P_2^2)]^2 + (2Q_1\zeta_1\Omega_2)^2 = \bar{\tilde{F}}_1^2 \tag{51}$$

For the nonlinear responses of mode (m=2), it would be dealt with similar to the principal mode and are passed over here.

Substituting the appropriate expressions of P_1, P_2, Q_1, Q_2, α_1, α_2, α_3, α_4, β_1, β_2, β_3, β_4 discussed above into Eq. (35), we obtain the equations of the response curves for $\tilde{A}_{1,6}$, $\tilde{B}_{1,6}$, $\tilde{A}_{2,6}$ and $\tilde{B}_{2,6}$.

5. Analytical results

The dimensions and properties of the shell here are the same as those in the single mode case. Fig. 5(a) shows the frequency–response relationship for the principal mode (m=1, n=6), with mode (m=2) participation, and Fig. 5(b) shows the frequency–response relationship for mode (m=2, n=6).

It can be observed in Fig. 5(b), the response curves for mode (m=2, n=6) present four peaks. Two of them appear in the neighborhood of linear resonance of the principal mode (m=1, n=6), and the other appear in the neighborhood of linear resonance of mode (m=2, n=6), showing resonance of the principal mode is significantly affected by mode (m=2, n=6). In this paper, the contribution of additional mode (m=2, n=6) on the response of the principal mode are considered in the approximation discussed above.

The approximate analytical solutions have been plotted in Fig. 6 together with numerical results in the neighborhood of linear resonance of the principal mode (m=1, n=6). The agreement between them is very good; in particular, it is excellent for the lowest curve. Overall, Fig. 6 bespeaks of the good accuracy and efficiency of the method of harmonic balance developed in the present paper.

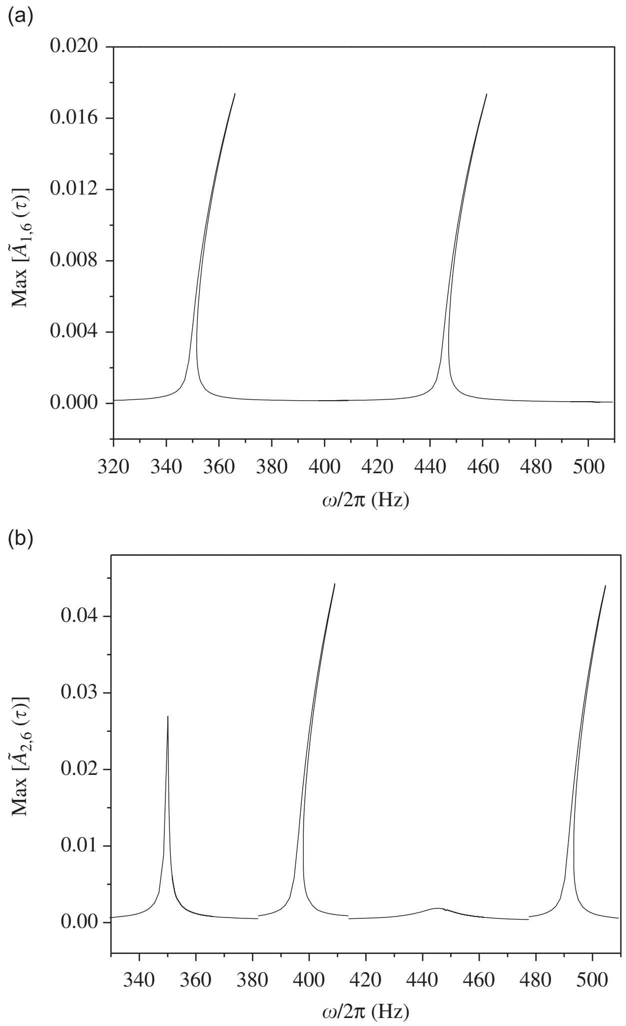

Fig. 5. Frequency–response curves for $\omega_n = 50\,\text{rad/s}$, $F_0 = 10\,\text{N}$: (a) maximum of $\tilde{A}_{1,6}(\tau)$ of the principle mode (m=1, n=6) with mode (m=2) participation; and (b) maximum of $\tilde{A}_{2,6}(\tau)$ of mode (m=2, n=6).

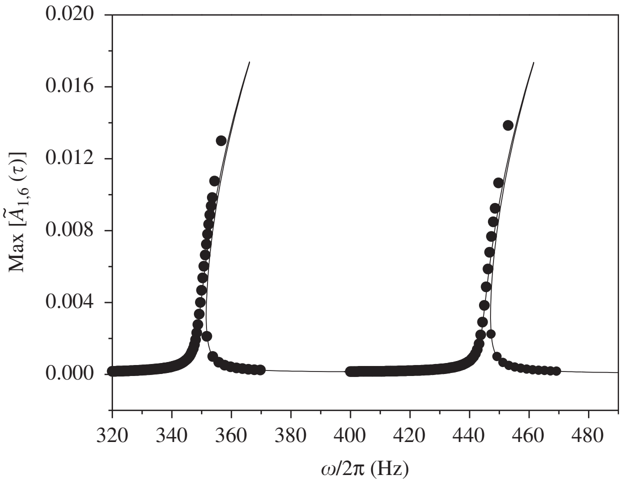

Fig. 6. Frequency–response curves for principle mode (m=1, n=6) with mode (m=2) participation for $\omega_n = 50\,\text{rad/s}$, $F_0 = 10\,\text{N}$: •, numerical solutions; —, approximate analytical solutions.

6. Stability of the period solutions

The equations of the boundary curves of stable regions coincide with the equations of the locus of the vertical tangents to the response curves determined by the conditions [13]

$$\frac{\partial \Omega_1}{\partial P_1} = 0 \text{ and } \frac{\partial \Omega_2}{\partial Q_1} = 0 \tag{52}$$

Defining in accordance with (45) and (51)

$$\begin{cases} g_1(\Omega_1, P_1, Q_1, P_2, Q_2) \equiv \\ \left[P_1(1-\Omega_1^2) - \left(\frac{3}{4}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)P_1^3 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)P_1P_2^2 \right. \\ \left. -\left(\frac{3}{2}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)P_1Q_1^2 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)P_1Q_2^2\right]^2 + (2P_1\zeta_1\Omega_1)^2 - \bar{\tilde{F}}_1^2 = 0 \\ g_2(\Omega_2, P_1, Q_1, P_2, Q_2) \\ \equiv \left[Q_1(1-\Omega_2^2) - \left(\frac{3}{4}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)Q_1^3 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)Q_1Q_2^2 - \right. \\ \left. \left(\frac{3}{2}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)Q_1P_1^2 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)Q_1P_2^2\right]^2 + (2Q_1\zeta_1\Omega_2)^2 - \bar{\tilde{F}}_1^2 = 0 \end{cases} \tag{53}$$

and differentiating these equations with respect to Ω_1 and Ω_2, respectively, we have

$$\frac{\partial g_1}{\partial \Omega_1} + \frac{\partial g_1}{\partial P_1}\frac{\partial P_1}{\partial \Omega_1} + \frac{\partial g_1}{\partial Q_1}\frac{\partial Q_1}{\partial \Omega_1} + \frac{\partial g_1}{\partial P_2}\frac{\partial P_2}{\partial \Omega_1} + \frac{\partial g_1}{\partial Q_2}\frac{\partial Q_2}{\partial \Omega_1} = 0 \tag{54}$$

$$\frac{\partial g_2}{\partial \Omega_2} + \frac{\partial g_2}{\partial P_1}\frac{\partial P_1}{\partial \Omega_2} + \frac{\partial g_2}{\partial Q_1}\frac{\partial Q_1}{\partial \Omega_2} + \frac{\partial g_2}{\partial P_2}\frac{\partial P_2}{\partial \Omega_2} + \frac{\partial g_2}{\partial Q_2}\frac{\partial Q_2}{\partial \Omega_2} = 0 \tag{55}$$

Calculating the terms in (54) from Eqs. (38), (40), (42) and (53) we find that the terms in $\partial g_1/\partial Q_i$ with i=1,2 and $\partial g_1/\partial P_2$, are negligible with respect to $\partial g_1/\partial \Omega_1$, and for (55) vice versa, so that approximately (54) and (55) reduce to

$$\begin{cases} \frac{\partial g_1}{\partial \Omega_1} + \frac{\partial g_1}{\partial P_1}\frac{\partial P_1}{\partial \Omega_1} = 0 \\ \frac{\partial g_2}{\partial \Omega_2} + \frac{\partial g_2}{\partial Q_1}\frac{\partial Q_1}{\partial \Omega_2} = 0 \end{cases} \tag{56}$$

from which

$$\begin{cases} \frac{\partial \Omega_1}{\partial P_1} = -\frac{\partial g_1/\partial P_1}{\partial g_1/\partial \Omega_1} \\ \frac{\partial \Omega_2}{\partial Q_1} = -\frac{\partial g_2/\partial Q_1}{\partial g_2/\partial \Omega_2} \end{cases} \tag{57}$$

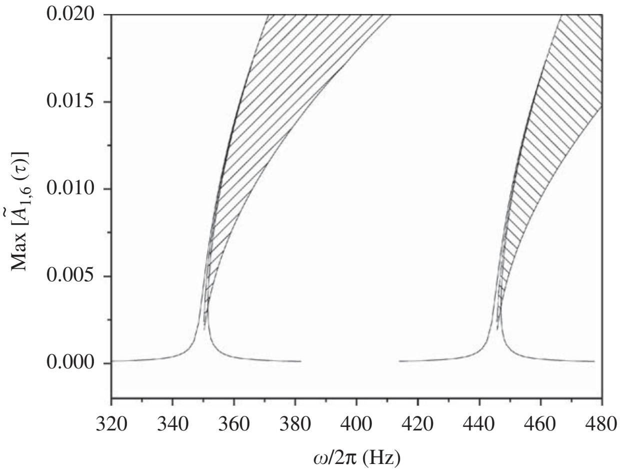

Fig. 7. The frequency–response curves and the boundaries of the stable regions for $\tilde{A}_{1,6}(\tau)$ of the principle mode (m=1, n=6) for $\omega_n = 50\,\mathrm{rad/s}$, $F_0 = 10\,\mathrm{N}$.

The conditions (52) are approximately satisfied when

$$\frac{\partial g_1}{\partial P_1} = 0 \text{ and } \frac{\partial g_2}{\partial Q_1} = 0 \tag{58}$$

which from Eq. (53) immediately leads to the equations of boundary curves (59), consequently the locus of the vertical tangents to the response curves yields the boundaries of the stable regions for P_1 and Q_1.

$$\begin{cases} 2\Big[P_1(1-\Omega_1^2) - \left(\frac{3}{4}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)P_1^3 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)P_1P_2^2 - \left(\frac{3}{2}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)P_1Q_1^2 \\ -\left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)P_1Q_2^2\Big]\Big[(1-\Omega_1^2) - 3\left(\frac{3}{4}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)P_1^2 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)P_2^2 \\ -\left(\frac{3}{2}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)Q_1^2 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)Q_1^2\Big] + 8P_1(\zeta_1 Q_1)^2 = 0 \\ 2\Big[Q_1(1-\Omega_2^2) - \left(\frac{3}{4}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)Q_1^3 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)Q_1Q_2^2 - \left(\frac{3}{2}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)Q_1P_1^2 \\ -\left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)Q_1P_2^2\Big]\Big[(1-\Omega_2^2) - 3\left(\frac{3}{4}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)Q_1^2 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)Q_2^2 \\ -\left(\frac{3}{2}\tilde{H}_1 + \frac{1}{2}\tilde{H}_4\right)P_1^2 - \left(\frac{1}{2}\tilde{H}_3 + \frac{1}{2}\tilde{H}_6\right)P_2^2\Big] + 8Q_1(\zeta_1 Q_1)^2 = 0 \end{cases} \tag{59}$$

If, for a given value of ω, the values of the amplitudes P_1 and/or Q_1 belong to a region of instability, then the corresponding vibration of $\tilde{A}_{1,6}$ is unstable. Stable oscillations of $\tilde{A}_{1,6}$ only occur when both the values of the amplitudes P_1 and Q_1 belong to a region of stability. In Fig. 7 we have represented the frequency–response curves and the boundaries of stable regions for the principal mode (m=1, n=6) resonance. The instable regions of the curves are cross-hatched.

7. Conclusions

In this paper the dynamic response of a circular cylindrical shell subjected to a concentrated harmonic force, in the spectral neighborhood of one of the lowest natural frequencies, and moving in a concentric circular path at a constant velocity, is investigated. The following conclusions are drawn.

Additional circumferential waves n and multiples of frequency k in the mode expansions in the analysis of forced vibrations of the shell have but a small effect on principal mode (m=1, n=6) resonant response compared with additional axial half-waves m. This is particularly evident by comparing the nonlinear frequency–response curves for different mode expansions (Fig. 4). The present results allow us to state that it is proper to adopt two neighboring axial modes ($K = 1$, $N_0 = N = 6$, $M = 2$) to study the dynamics of circular cylindrical shells in the neighborhood of one of the lowest natural frequencies corresponding to mode (m=1, n=6).

Adopting double modes ($K = 1$, $N_0 = N = 6$, $M = 2$), the analytical solution has been carried out by the method of harmonic balance for dynamic response of the model analyzed in this study. The accuracy of the method has been validated via comparisons with numerical results, which shows that the present approach is efficient for the dynamic analysis of the circular cylindrical shell problem.

Due to the moving load, it exist two peaks on the frequency–response curves for the principal mode (m=1, n=6), namely forward and backward waves. The linear resonant frequencies of them are $\omega=\omega_{1,6}+6\omega_n$ and $\omega=\omega_{1,6}-6\omega_n$, respectively, symmetrical about one of the lowest nature frequency $\omega_{1,6}$, and the nonlinear resonant frequencies of them are close to the linear ones. The stabilities of period solutions of the system is investigated in detail, and results show that for the three solutions of forward or backward wave in the nonlinear regions, the highest and the lowest values are stable and the other one is instable.

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Appendix

The functions in Eq. (13) are given by

$$\alpha_1=\frac{Eh}{2R^4(1-\mu^2)},\ \alpha_2=\frac{\mu Eh}{2R^2(1-\mu^2)},\ \alpha_3=\frac{Eh}{2(1-\mu^2)},\ \alpha_4=\frac{Eh}{R^2(1+\mu)}$$

The functions in Eqs. (18) and (19) are given by

$$\zeta_{1,6}=\frac{c}{2\rho h\omega_{1,6}}$$

$$\tilde{F}=\frac{F_0U_1(x_0)}{\pi\rho h\int_0^L U_1^2(x)\,dx}$$

$$H=\frac{Eh}{\rho h(1-\mu^2)}\left\{-\frac{162}{R^4}\frac{\int_0^L U_1^4(x)\,dx}{\int_0^L U_1^2(x)\,dx}+\frac{9\int_0^L U_1(x)[\dot{U}_1(x)]^2\,dx}{R^2\int_0^L U_1^2(x)\,dx}-\frac{45\mu\int_0^L U_1(x)[\dot{U}_1(x)]^2\,dx}{2R^2\int_0^L U_1^2(x)\,dx}\right.$$
$$\left.+\frac{9\mu\int_0^L U_1^3(x)\ddot{U}_1(x)\,dx}{2R^2\int_0^L U_1^2(x)\,dx}+\frac{3\int_0^L U_1(x)\ddot{U}_1(x)[\dot{U}_1(x)]^2\,dx}{8\int_0^L U_1^2(x)\,dx}\right\}$$

The non-dimensional variables and parameters in Eqs. (31)–(34) are given by

$$\tau=\omega_{1,6}t,\tilde{A}_{1,6}(\tau)=\frac{A_{1,6}(t)}{h},\ \tilde{B}_{1,6}(\tau)=\frac{B_{1,6}(t)}{h},\ \tilde{A}_{2,6}(\tau)=\frac{A_{2,6}(t)}{h},\ \tilde{B}_{2,6}(\tau)=\frac{B_{2,6}(t)}{h}$$

$$\Omega_1=\frac{\omega+6\omega_n}{\omega_{1,6}},\ \Omega_2=\frac{\omega-6\omega_n}{\omega_{1,6}},\ \tilde{l}_1=\frac{l_1}{\omega_{1,6}^2},\ \tilde{l}_2=\frac{l_2}{\omega_{1,6}^2},\ \tilde{\tilde{F}}_1=\frac{\tilde{F}_1}{2h\omega_{1,6}^2},\ \tilde{\tilde{F}}_2=\frac{\tilde{F}_2}{2h\omega_{1,6}^2}$$

$$\tilde{H}_1=\frac{H_1h^2}{\omega_{1,6}^2},\ \tilde{H}_2=\frac{H_2h^2}{\omega_{1,6}^2},\ \tilde{H}_3=\frac{H_3h^2}{\omega_{1,6}^2},\ \tilde{H}_4=\frac{H_4h^2}{\omega_{1,6}^2},\ \tilde{H}_5=\frac{H_5h^2}{\omega_{1,6}^2},\ \tilde{H}_6=\frac{H_6h^2}{\omega_{1,6}^2}$$

$$\tilde{H}_7=\frac{H_7h^2}{\omega_{1,6}^2},\ \tilde{H}_8=\frac{H_8h^2}{\omega_{1,6}^2},\ \tilde{H}_9=\frac{H_9h^2}{\omega_{1,6}^2},\ \tilde{H}_{10}=\frac{H_{10}h^2}{\omega_{1,6}^2},\ \tilde{G}_1=\frac{G_1h^2}{\omega_{1,6}^2},\ \tilde{G}_2=\frac{G_2h^2}{\omega_{1,6}^2}$$

$$\tilde{G}_3=\frac{G_3h^2}{\omega_{1,6}^2},\ \tilde{G}_4=\frac{G_4h^2}{\omega_{1,6}^2},\ \tilde{G}_5=\frac{G_5h^2}{\omega_{1,6}^2},\ \tilde{G}_6=\frac{G_6h^2}{\omega_{1,6}^2},\ \tilde{G}_7=\frac{G_7h^2}{\omega_{1,6}^2},\ \tilde{G}_8=\frac{G_8h^2}{\omega_{1,6}^2}$$

$$\tilde{G}_9=\frac{G_9h^2}{\omega_{1,6}^2},\ \tilde{G}_{10}=\frac{G_{10}h^2}{\omega_{1,6}^2}$$

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