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Dynamic stability in parametric resonance of axially accelerating viscoelastic Timoshenko beams

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ABSTRACT

This paper investigates dynamic stability of an axially accelerating viscoelastic beam undergoing parametric resonance. The effects of shear deformation and rotary inertia are taken into account by the Timoshenko thick beam theory. The beam material obeys the Kelvin model in which the material time derivative is used. The axial speed is characterized as a simple harmonic variation about the constant mean speed. The governing partial-differential equations are derived from Newton's second law, Euler's angular momentum principle, and the constitutive relation. The method of multiple scales is applied to the equations to establish the solvability conditions in summation and principal parametric resonances. The sufficient and necessary condition of the stability is derived from the Routh–Hurvitz criterion. Some numerical examples are presented to demonstrate the effects of related parameters on the stability boundaries.

1. Introduction

Many engineering devices, such as band saws and power transmission belts, can be modeled as an axially moving beam, which is a typical gyroscopic continuum. Due to initial, external or parametric excitations, axially moving beams undergo transverse vibration that may limit the devices' applications. Therefore, there have been many analytical and numerical investigations on transverse vibration of axially moving beams, for examples, Chen et al. [1], Gaith and Müftü [2], Hedrih [3], Wang et al. [4], Yang et al. [5].

The time-dependent axial speed may serve as a parametric excitation. Within linear models, dynamic stability of axially accelerating beams is a crucial problem that has been extensively analyzed. Öz and Pakdemirli [6] and Öz [7] applied the method of multiple scales to calculate analytically the stability boundaries of an axially accelerating beam under different boundary conditions. Parker and Lin [8] adopted a 1-term Galerkin discretization and the perturbation method to study dynamic stability of an axially accelerating beam subjected to a tension fluctuation. Özkaya and Öz [9] used an artificial neural network algorithm to determine stability boundary of an axially accelerating beam. Suweken and Horssen [10] applied the method of multiple scales to a discretized system via the Galerkin method to study the dynamic stability of an axially accelerating beam. Pakdemirli and Öz [11] employed the method of multiple scales to analyze the stability in the resonances involved up to four modes. In addition to the investigations on elastic beams, there have been studies focusing on viscoelastic beams, because viscoelascity is an effective approach to model the damping mechanism (Park [12]). Chen et al. [13] applied the averaging method to a discretized system via the Galerkin method to present analytically the stability

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boundaries of axially accelerating viscoelastic beams. Chen and Yang [14] applied the method of multiple scales without discretization to obtain analytically the stability boundaries of axially accelerating viscoelastic beams. In their works, the Kelvin model containing the partial time derivative was used to describe the viscoelastic behavior of beam materials. Mockensturm and Guo [15] convincingly argued that the Kelvin model generalized to axially moving materials should contain the material time derivative to account for the energy dissipation in steady motion. Actually, the material time derivative was also employed in the Kelvin model of axially moving materials by Marrynowski and Kapitaniak [16], Marrynowski [17], and Marrynowski [18], as well as in the three-parameter viscoelastic model by Marynowski and Kapitaniak [19]. Revisiting the problem addressed by Chen and Yang [4] by using the material time derivative in the Kelvin model, Ding and Chen [20] applied the method of multiple scales to demonstrate that the modes not involved in summation resonance have no effects on the stability, and solved the governing equation via the finite difference scheme to validate the analytical results. Chen and Wang [21] developed an asymptotic perturbation approach to analyze dynamic stability of an axially accelerating viscoelastic beams and used a differential quadrature scheme to check the analytical results via solving the governing equation numerically. If the motion amplitude is large, the nonlinearity should be taken into account. In transverse nonlinear vibration of axially accelerating elastic or viscoelastic beams, the straight equilibrium configuration may becomes unstable and bifurcate into periodical steady-state responses that can be predicted by the approximately analytical methods (Parker and Lin [8], Öz et al. [22], Chen and Yang [23]).

All above-mentioned researchers assumed the beams under their consideration to be slender so that the beams can be described by the Euler–Bernoulli model. If a beam is thick, then the effects of shear deformation and rotary inertia, which are neglected in the Euler–Bernoulli model, should be taken into account. Recently, Ghayesh and Balar [24] and Ghayesh and Khadem [25] respectively treated the effects of shear deformation and rotary inertia on nonlinear parametric vibration of axially accelerating viscoelastic beams. The Timoshenko [26] beam theory can account for the effects of both shear deformation and rotary inertia. Although the Timoshenko beam is extensively studied (for example, Challamel [27], Mei et al. [28], Arboleda-Monsalve et al. [29]), the works on axially moving Timoshenko beams are rather limited. Simpson [30] was the first to derive the governing equations for the axially moving thick beam based on the Timoshenko model, but did not consider the axial tension and presented no numerical results. Chonan [31] studied the steady-state response of a moving Timoshenko beam by applying Laplace transform method. Lee et al. [32] used frequency-dependent spectral element matrix to compute natural frequencies, critical speeds and modal functions of axially moving Timoshenko beams. Tang et al. [33] applied the complex modal analysis approach to calculate natural frequencies, modes and critical speeds of axially moving Timoshenko beams. Based on the modal solutions, Tang et al. [34] determined the transverse nonlinear responses of axially moving Timoshenko beams to weak and strong external excitations via the method of multiple scales. To authors' knowledge, there have been no investigations on the axially moving viscoelastic Timoshenko beam, while some researchers, such as Nakao et al. [35], Kocatrk and Simsek [36], Hilton [37], worked on viscoelastic Timoshenko beams without the axial motion.

To address the lacks of research in the aspect, the present investigation focuses on dynamic stability of axially accelerating viscoelastic Timoshenko beams undergoing parametric resonance. The governing equation of transverse motion of axially accelerating viscoelastic Timoshenko beams consists of two partial-differential equations that cannot be decoupled, while the governing equations of axially moving elastic Timoshenko beams (Tang et al. [33]) or stationary viscoelastic Timoshenko beams (Nakao et al. [35]) can be decoupled into two independent equations. The method of multiple scales is applied to determine stability boundary in summation parametric resonance and principal parametric resonance. The approach, as well as the modal solutions to the generating autonomous linear system, can be used to investigate nonlinear vibration of axially moving viscoelastic Timoshenko beams

The present paper is organized as follows. Section 2 derives the mathematical model from the physical laws and the constitutive relation. Section 3 employs the method of multiple scales to analyze the governing equations under the prescribed boundary conditions. Section 4 establishes the stability conditions in the summation parametric resonance and the principal parametric resonance. Section 5 presents some numerical examples to demonstrate the effects of the related parameters on the stability boundaries. Section 6 ends the paper with concluding remarks.

2. Problem formulation

A uniform axially moving Timoshenko beam, with density ρ, cross-sectional area A, area moment of inertia of the cross-section about the neutral axis J, initial axial tension P, travels at the axial transport time-dependent speed $\Gamma(T)$ between two simple supports separated by distance L. When the effects of rotary inertia and shear deformation are considered, the bending vibration can be described by two variables dependent on axial coordinate X and time T, namely, transverse displacement $V(X,T)$ and the slope of the deflection curve due to bending deformation alone $\Phi(X,T)$. The physical model is shown in Fig. 1.

Application of the Newton second law in the transverse direction yields

$$\rho A(V_{,TT} + 2\Gamma V_{,XT} + \dot{\Gamma} V_{,X} + \Gamma^2 V_{,XX}) = PV_{,XX} - Q_{,X} \tag{1}$$

where $Q(X,T)$ denotes the shear force. The Euler angular momentum principle yields

$$\rho J \Phi_{,TT} = M_{,X} - Q \tag{2}$$

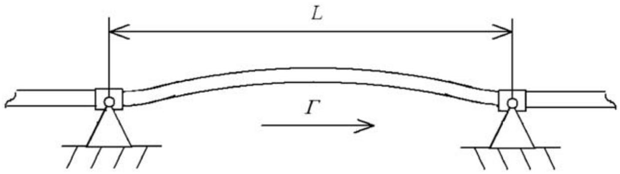

Fig. 1. The physical model of an axially moving Timoshenko beam.

where $M(X,T)$ denotes the bending moment defined by

$$M(X,T)=\int_A Z\sigma(X,Z,T)\,\mathrm{d}A \tag{3}$$

for principal plane of bending ZX-plane and normal stress $\sigma(X,Z,T)$.

The beam is described by the Timoshenko model with shape factor κ. The shear deformation is given by

$$\theta(X,T)=\Phi(X,T)-V_{,X} \tag{4}$$

The viscoelastic material of the beam obeys the Kelvin model with the constitutive relations

$$\sigma=E[e+\alpha(e_{,T}+\Gamma e_{,X})] \tag{5}$$

$$\tau=G[\theta+\alpha(\theta_{,T}+\Gamma\theta_{,X})] \tag{6}$$

where τ is the shear stress, e is the axial strain, E is the modulus of elasticity, G is the shearing modulus, and α is the viscosity coefficient. For small deflections, the geometrical relation is

$$e=Z\Phi_{,X} \tag{7}$$

Eqs. (3), (5) and (7) lead to

$$M=EJ[\Phi_{,x}+\alpha(\Phi_{,XT}+\Gamma\Phi_{,XX})] \tag{8}$$

From Eqs. (4) and (6), Q=$\kappa A\tau$ gives

$$Q=\kappa AG[(\Phi-V_{,X})+\alpha(\Phi_{,T}-V_{,XT}+\Gamma\Phi_{,X}-\Gamma V_{,XX})] \tag{9}$$

Substitution of Eqs. (8) and (9) into Eqs. (1) and (2) yields the governing equations of axially moving viscoelastic Timoshenko beam

$$\rho A[V_{,TT}+2\Gamma V_{,XT}+\dot{\Gamma}V_{,X}-(P-\Gamma^2)V_{,XX}]=-\kappa AG[(\Phi_{,X}-V_{,XX})+\alpha(\Phi_{,TX}-V_{,XXT}+\Gamma\Phi_{,XX}-\Gamma V_{,XXX})] \tag{10}$$

$$\rho J\Phi_{,TT}=EJ[\Phi_{,XX}+\alpha(\Phi_{,XXT}+\Gamma\Phi_{,XXX})]-\kappa AG[(\Phi-V_{,X})+\alpha(\Phi_{,T}-V_{,XT}+\Gamma\Phi_{,X}-\Gamma V_{,XX})] \tag{11}$$

The boundary conditions for the simple supports at both ends are

$$V|_{X=0}=0, V|_{X=L}=0; M|_{X=0}=0, M|_{X=L}=0 \tag{12}$$

Introduce the dimensionless variables, coordinates, and parameters

$$v=\frac{V}{\varepsilon L},\varphi=\frac{\Phi}{\varepsilon},x=\frac{X}{L},t=T\sqrt{\frac{P}{\rho AL^2}},\gamma=\Gamma\sqrt{\frac{\rho A}{P}},$$

$$k_1=\frac{AG}{\kappa P},k_2=\frac{J}{AL^2},k_f^2=\frac{EJ}{PL^2},\eta=\frac{\alpha}{\varepsilon}\sqrt{\frac{P}{\rho AL^2}} \tag{13}$$

where ε is a dimensionless small number accounting for the smallness of the beam bending deformation and the viscosity coefficient. Dimensionless parameter k_1 accounts for the effects of shear deformation, k_2 represents the effects of the rotary inertia, and k_f denotes the stiffness of the beam. Using the dimensionless variables, coordinates, and parameters defined in Eq. (13), Eqs. (10) and (11) can be cast into the dimensionless form

$$v_{,tt}+2\gamma v_{,xt}+(\gamma^2-1)v_{,xx}+\gamma_{,t}v_{,x}+k_1(\varphi_{,x}-v_{,xx})=-\varepsilon k_1\eta(\varphi_{,xt}-v_{,xxt}+\gamma\varphi_{,xx}-\gamma v_{,xxx}) \tag{14}$$

$$k_2\varphi_{,tt}-k_f^2\varphi_{,xx}+k_1(\varphi-v_{,x})=\varepsilon k_f^2\eta(\varphi_{,xxt}+\gamma\varphi_{,xxx})-\varepsilon k_1\eta(\varphi_{,t}-v_{,xt}+\gamma\varphi_{,x}-\gamma v_{,xx}) \tag{15}$$

It should be remarked that Eqs. (14) and (15) cannot decoupled, and the coupling differentiate the axially accelerating viscoelastic Timoshenko beams from axially moving elastic Timoshenko beams (Tang et al. [33]) and stationary viscoelastic Timoshenko beams (Nakao et al. [35]). Eqs. (8), (12) and (13) yield the dimensionless boundary conditions

$$v|_{x=0}=0, v|_{x=1}=0;$$

$$\varphi_{,x}|_{x=0}+\varepsilon\eta(\varphi_{,xt}|_{x=0}+\gamma\varphi_{,xx}|_{x=0})=0, \varphi_{,x}|_{x=1}+\varepsilon\eta(\varphi_{,xt}|_{x=1}+\gamma\varphi_{,xx}|_{x=1})=0 \tag{16}$$

3. Multi-scale analysis

In the present investigation, the axial speed is supposed to be a small simple harmonic variation about the constant mean speed,

$$\gamma(t)=\gamma_0+\varepsilon\gamma_1\sin(\omega t) \tag{17}$$

where γ_0 is the mean axial speed, and $\varepsilon\gamma_1$ and ω are respectively the amplitude and the frequency of the axial speed variation, all in the dimensionless form.

$$\begin{aligned}&v_{,tt}+2\gamma_0 v_{,xt}+(\gamma_0^2-1)v_{,xx}+k_1(\varphi_{,x}-v_{,xx})\\&=-\varepsilon[\gamma_1\omega\cos(\omega t)v_{,x}+2\gamma_1\sin(\omega t)v_{,xt}+2\gamma_0\gamma_1\sin(\omega t)v_{,xx}-k_1\eta(\varphi_{,xt}-v_{,xxt}\\&\quad+\gamma_0\varphi_{,xx}-\gamma_0 v_{,xxx})]-\varepsilon^2[\gamma_1^2\sin^2(\omega t)v_{,xx}+k_1\gamma_1\eta\sin(\omega t)(\varphi_{,xx}-v_{,xxx})]\end{aligned} \tag{18}$$

$$\begin{aligned}k_2\varphi_{,tt}-k_f^2\varphi_{,xx}+k_1(\varphi-v_{,x})&=\varepsilon\eta[k_f^2(\varphi_{,xxt}+\gamma_0\varphi_{,xxx})-k_1(\varphi_{,t}-v_{,xt}+\gamma_0\varphi_{,x}-\gamma_0 v_{,xx})]\\&\quad-\varepsilon^2\gamma_1\eta\sin(\omega t)(k_1\varphi_{,x}-k_1 v_{,xx}-k_f^2\varphi_{,xxx})\end{aligned} \tag{19}$$

The method of multiple scales will be employed to solve coupled Eqs. (18) and (19). Suppose that the uniform approximate solutions to Eqs. (18) and (19) are

$$v(x,t;\varepsilon)=v_0(x,T_0,T_1)+\varepsilon v_1(x,T_0,T_1)+o(\varepsilon) \tag{20}$$

$$\varphi(x,t;\varepsilon)=\varphi_0(x,T_0,T_1)+\varepsilon\varphi_1(x,T_0,T_1)+o(\varepsilon) \tag{21}$$

where T_0=t and T_1=εt are respectively the fast and slow time scales. Substitution of Eqs. (20) and (21) and the following relationship

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+o(\varepsilon),\frac{\partial^2}{\partial t^2}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1}+o(\varepsilon) \tag{22}$$

into Eqs. (18) and (19) and then equalization of coefficients of ε^0 and ε^1 in the resulting equations lead to:

$$v_{0,T_0T_0}+2\gamma_0 v_{0,xT_0}+(\gamma_0^2-1)v_{0,xx}+k_1(\varphi_{0,x}-v_{0,xx})=0 \tag{23}$$

$$k_2\varphi_{0,T_0T_0}-k_f^2\varphi_{0,xx}+k_1(\varphi_0-v_{0,x})=0 \tag{24}$$

$$\begin{aligned}&v_{1,T_0T_0}+2\gamma_0 v_{1,xT_0}+(\gamma_0^2-1)v_{1,xx}+k_1(\varphi_{1,x}-v_{1,xx})\\&=-2v_{0,T_0T_1}-\gamma_1\omega\cos(\omega t)v_{0,x}-2\gamma_0 v_{0,xT_1}-2\gamma_1\sin(\omega t)v_{0,xT_0}\\&\quad-2\gamma_0\gamma_1\sin(\omega t)v_{0,xx}-k_1\eta(\varphi_{0,xT_0}-v_{0,xxT_0}+\gamma_0\varphi_{0,xx}-\gamma_0 v_{0,xxx})\end{aligned} \tag{25}$$

$$\begin{aligned}k_2\varphi_{1,T_0T_0}-k_f^2\varphi_{1,xx}+k_1(\varphi_1-v_{1,x})&=-2k_2\varphi_{0,T_0T_1}+k_f^2\eta(\varphi_{0,xxT_0}+\gamma_0\varphi_{0,xxx})\\&\quad-k_1\eta(\varphi_{0,T_0}-v_{0,xT_0}+\gamma_0\varphi_{0,x}-\gamma_0 v_{0,xx})\end{aligned} \tag{26}$$

3.1. Modal analysis on order ε^0 equation

The two ε^0-order Eqs. (23) and (24) can be decoupled into

$$\begin{aligned}&v_{0,T_0T_0}+2\gamma_0 v_{0,x_0T_0}+(\gamma_0^2-1)v_{0,xx}-\frac{1}{k_1}(k_1k_2+k_2+k_f^2-k_2\gamma_0^2)v_{0,xxT_0T_0}\\&-\frac{k_f^2}{k_1}(1+k_1-\gamma_0^2)v_{0,xxxx}+\frac{k_2}{k_1}(v_{0,T_0T_0T_0T_0}+2\gamma_0 v_{0,xT_0T_0T_0})-2\frac{k_f^2}{k_1}\gamma_0 v_{0,xxxT_0}=0\end{aligned} \tag{27}$$

$$\begin{aligned}&\varphi_{0,T_0T_0}+2\gamma_0\varphi_{0,x_0T_0}+(\gamma_0^2-1)\varphi_{0,xx}-\frac{1}{k_1}(k_1k_2+k_2+k_f^2-k_2\gamma_0^2)\varphi_{0,xxT_0T_0}\\&-\frac{k_f^2}{k_1}(1+k_1-\gamma_0^2)\varphi_{0,xxxx}+\frac{k_2}{k_1}(\varphi_{0,T_0T_0T_0T_0}+2\gamma_0\varphi_{0,xT_0T_0T_0})-2\frac{k_f^2}{k_1}\gamma_0\varphi_{0,xxxT_0}=0\end{aligned} \tag{28}$$

Substitution of Eqs. (20), (21) and (22) into Eq. (16) and then equalization of coefficients of ε^0 in the resulting equations lead to

$$v_0|_{x=0}=0,v_0|_{x=1}=0;\varphi_{0,x}|_{x=0}=0,\varphi_{0,x}|_{x=1}=0 \tag{29}$$

The solutions to Eqs. (27) and (28) can be assumed as

$$v_0(x,t)=\sum_{n=1}^{\infty}\phi_n(x)e^{i\omega_nT_0}+cc\quad\varphi_0(x,t)=\sum_{n=1}^{\infty}\vartheta_n(x)e^{i\omega_nT_0}+cc \tag{30}$$

where ϕ_n and ϑ_n are the nth mode function, ω_n are the nth natural frequency of the generating autonomous linear system and cc stands for complex conjugate of the proceeding terms.

Substitution of Eq. (30) into (27) and (28) respectively yields,

$$\frac{k_2}{k_1}\phi_n\omega_n^4 - 2\mathrm{i}\frac{k_2}{k_1}\gamma_0\phi_n'\,\omega_n^3 + \left[\frac{1}{k_1}(k_2+k_f^2+k_1k_2-k_2\gamma_0^2)\phi_n'' - \phi_n\right]\omega_n^2$$
$$+2\mathrm{i}\gamma_0\frac{1}{k_1}(k_1\phi_n' - k_f^2\phi_n''')\omega_n + (\gamma_0^2-1)\phi_n'' + \frac{k_f^2}{k_1}(1+k_1-\gamma_0^2)\phi_n'''' = 0 \tag{31}$$

$$\frac{k_2}{k_1}\vartheta_n\omega_n^4 - 2\mathrm{i}\frac{k_2}{k_1}\gamma_0\vartheta_n'\,\omega_n^3 + \left[\frac{1}{k_1}(k_2+k_f^2+k_1k_2-k_2\gamma_0^2)\vartheta_n'' - \vartheta_n\right]\omega_n^2$$
$$+2\mathrm{i}\gamma_0\frac{1}{k_1}(k_1\vartheta_n' - k_f^2\vartheta_n''')\omega_n + (\gamma_0^2-1)\vartheta_n'' + \frac{k_f^2}{k_1}(1+k_1-\gamma_0^2)\vartheta_n'''' = 0 \tag{32}$$

where the prime denotes the derivation with respect to dimensionless spatial variable x. Substitution of Eqs. (29) and (30) into Eq. (23) respectively yields

$$\phi_n|_{x=0} = 0, [(\gamma_0^2-k_1-1)\phi_n'' + 2\mathrm{i}\gamma_0\omega_n\phi']|_{x=0} = 0,$$
$$\phi_n|_{x=1} = 0, [(\gamma_0^2-k_1-1)\phi_n'' + 2\mathrm{i}\gamma_0\omega_n\phi_n']|_{x=1} = 0. \tag{33}$$

It should be remarked that the boundary conditions derived by Tang et al. [33] missed the terms resulted from the axially motion.

Eqs. (31) and (32), the differential equations for $\phi_n(x)$ and $\vartheta_n(x)$, have the same form. Therefore the solutions of $\phi_n(x)$ and $\vartheta_n(x)$ also have the same form with different constants as

$$\phi_n(x) = C_{1n}\mathrm{e}^{\mathrm{i}\beta_{1n}x} + C_{2n}\mathrm{e}^{\mathrm{i}\beta_{2n}x} + C_{3n}\mathrm{e}^{\mathrm{i}\beta_{3n}x} + C_{4n}\mathrm{e}^{\mathrm{i}\beta_{4n}x}$$
$$\vartheta_n(x) = D_{1n}\mathrm{e}^{\mathrm{i}\beta_{1n}x} + D_{2n}\mathrm{e}^{\mathrm{i}\beta_{2n}x} + D_{3n}\mathrm{e}^{\mathrm{i}\beta_{3n}x} + D_{4n}\mathrm{e}^{\mathrm{i}\beta_{4n}x} \tag{34}$$

Substitution of Eq. (34) into (31) and (33) respectively yields

$$\frac{k_f^2}{k_1}(1+k_1-\gamma_0^2)\beta_{in}^4 - \frac{2k_f^2}{k_1}\gamma_0\omega_n\beta_{in}^3 - \left[\frac{1}{k_1}(k_2+k_f^2+k_1k_2-k_2\gamma_0^2)\omega_n^2 + (\gamma_0^2-1)\right]\beta_{in}^2$$
$$+2\left(\frac{k_2}{k_1}\omega_n^3 - \omega_n\right)\gamma_0\beta_{in} + \frac{k_2}{k_1}\omega_n^4 - \omega_n^2 = 0, \tag{35}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ B_{1n} & B_{2n} & B_{3n} & B_{4n} \\ \mathrm{e}^{\mathrm{i}\beta_{1n}} & \mathrm{e}^{\mathrm{i}\beta_{2n}} & \mathrm{e}^{\mathrm{i}\beta_{3n}} & \mathrm{e}^{\mathrm{i}\beta_{4n}} \\ B_{1n}\mathrm{e}^{\mathrm{i}\beta_{2n}} & B_{2n}\mathrm{e}^{\mathrm{i}\beta_{2n}} & B_{3n}\mathrm{e}^{\mathrm{i}\beta_{2n}} & B_{4n}\mathrm{e}^{\mathrm{i}\beta_{2n}} \end{pmatrix}\begin{pmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n}\end{pmatrix} = 0. \tag{36}$$

where

$$B_{jn} = \beta_{jn}^2(1+k_1-\gamma_0^2) - 2\beta_{jn}\gamma_0\omega_n \quad (j=1,2,3,4) \tag{37}$$

For the non-trivial solution of Eq. (36), the determinant of the coefficient matrix must be zero. That is

$$[\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{2n})} + \mathrm{e}^{\mathrm{i}(\beta_{3n}+\beta_{4n})}](B_{1n}-B_{2n})(B_{3n}-B_{4n}) - [\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{2n}+\beta_{4n})}](B_{1n}-B_{3n})(B_{2n}-B_{4n})$$
$$+[\mathrm{e}^{\mathrm{i}(\beta_{2n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{4n})}](B_{2n}-B_{3n})(B_{1n}-B_{4n}) = 0. \tag{38}$$

Based on Eqs. (37) and (38), the nth values β_{jn} (j=1, 2, 3, 4) and the corresponding natural frequency ω_n can be calculated numerically. Using Eq. (34), one can obtain the modal function of the simply supported beam as follow

$$\phi_n(x) = c_1\left\{\mathrm{e}^{\mathrm{i}\beta_{1n}x} - \frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{2n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{2n}})}\mathrm{e}^{\mathrm{i}\beta_{2n}x} - \frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{3n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{3n}})}\mathrm{e}^{\mathrm{i}\beta_{3n}x}\right.$$
$$\left.-\left(1 - \frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{2n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{2n}})} - \frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{3n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{3n}})}\right)\mathrm{e}^{\mathrm{i}\beta_{4n}x}\right\} \tag{39}$$

Substitution of Eqs. (30) and (34) into (23) yields,

$$D_{jn} = \frac{\mathrm{i}k_1\beta_{jn}}{k_f^2\beta_{jn}^2 + k_1 - k_2\omega_n^2}C_{jn} \quad (j=1,2,3,4) \tag{40}$$

Further substitution of Eqs. (39) and (40) into (33) leads to

$$\vartheta_n(x) = c_1\left\{\frac{\mathrm{i}k_1\beta_{1n}\mathrm{e}^{\mathrm{i}\beta_{1n}x}}{k_f^2\beta_{1n}^2 + k_1 - k_2\omega_n^2} - \frac{\mathrm{i}k_1\beta_{2n}\mathrm{e}^{\mathrm{i}\beta_{2n}x}}{k_f^2\beta_{2n}^2 + k_1 - k_2\omega_n^2}\cdot\frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{2n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{2n}})}\right.$$

$$-\frac{\mathrm{i}k_1\beta_{3n}\mathrm{e}^{\mathrm{i}\beta_{3n}x}}{k_{\mathrm{f}}^2\beta_{3n}^2+k_1-k_2\omega_n^2}\cdot\frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{3n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{3n}})}-\frac{\mathrm{i}k_1\beta_{4n}\mathrm{e}^{\mathrm{i}\beta_{4n}x}}{k_{\mathrm{f}}^2\beta_{4n}^2+k_1-k_2\omega_n^2}$$
$$\cdot\left(1-\frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{2n})(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{2n}})}-\frac{(B_{4n}-B_{1n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(B_{4n}-B_{3n})(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{3n}})}\right)\Bigg\} \quad (41)$$

3.2. Solvability in summation parametric resonance

If the axial speed variation frequency ω approaches the sum of any two natural frequencies of generating autonomous linear system (23) and (24), summation parametric resonance may occur. A detuning parameter μ is introduced to quantify the deviation of ω from $\omega_n+\omega_m$, and ω is described by

$$\omega=\omega_n+\omega_m+\varepsilon\mu \quad (42)$$

where ω_n and ω_m are, respectively, the nth and the mth natural frequencies of the ε^0-order system defined by Eqs. (35) and (38). To investigate summation parametric resonance with the possible contributions of modes not involved in the resonance, the solutions to Eqs. (23) and (24) are assumed to be expressed by

$$v_0(x,T_0,T_1)=\phi_l(x)A_l(T_1)\mathrm{e}^{\mathrm{i}\omega_lT_0}+\phi_n(x)A_n(T_1)\mathrm{e}^{\mathrm{i}\omega_nT_0}+\phi_m(x)A_m(T_1)\mathrm{e}^{\mathrm{i}\omega_mT_0}+cc \quad (43)$$

$$\varphi_0(x,T_0,T_1)=\vartheta_l(x)B_l(T_1)\mathrm{e}^{\mathrm{i}\omega_lT_0}+\vartheta_n(x)B_n(T_1)\mathrm{e}^{\mathrm{i}\omega_nT_0}+\vartheta_m(x)B_m(T_1)\mathrm{e}^{\mathrm{i}\omega_mT_0}+cc \quad (44)$$

Substitution of Eqs. (42), (43) and (44) into (25) and (26) yields

$$\begin{aligned}
&v_{1,T_0T_0}+2\gamma_0v_{1,xT_0}+(\gamma_0^2-1)v_{1,xx}+k_1(\varphi_{1,x}-v_{1,xx})\\
&=-[\eta k_1(\mathrm{i}\omega_l\vartheta_l'+\gamma_0\vartheta_l'')B_l-\eta k_1(\mathrm{i}\omega_l\phi_l''+\gamma_0\phi_l''')A_l+2(\mathrm{i}\omega_l\phi_l+\gamma_0\phi_l')\dot{A}_l]\mathrm{e}^{\mathrm{i}\omega_lT_0}\\
&\quad-\Big\{\eta k_1(\mathrm{i}\omega_m\vartheta_m'+\gamma_0\vartheta_m'')B_m-\eta k_1(\mathrm{i}\omega_m\phi_m''+\gamma_0\phi_m''')A_m+2(\mathrm{i}\omega_m\phi_m+\gamma_0\phi_m')\dot{A}_m\\
&\quad+[\tfrac{1}{2}\gamma_1(\omega_m-\omega_n)\overline{\phi}_n'-i\gamma_0\gamma_1\overline{\phi}_n'']\overline{A}_n\mathrm{e}^{\mathrm{i}\mu T_1}\Big\}\mathrm{e}^{\mathrm{i}\omega_mT_0}\\
&\quad-\Big\{\eta k_1(\mathrm{i}\omega_n\vartheta_n'+\gamma_0\vartheta_n'')B_n-\eta k_1(\mathrm{i}\omega_n\phi_n''+\gamma_0\phi_n''')A_n+2(\mathrm{i}\omega_n\phi_n+\gamma_0\phi_n')\dot{A}_n\\
&\quad+[\tfrac{1}{2}\gamma_1(\omega_n-\omega_m)\overline{\phi}_m'-i\gamma_0\gamma_1\overline{\phi}_m'']\overline{A}_m\mathrm{e}^{\mathrm{i}\mu T_1}\Big\}\mathrm{e}^{\mathrm{i}\omega_nT_0}+cc+NST
\end{aligned} \quad (45)$$

$$\begin{aligned}
&k_2\varphi_{1,T_0T_0}-k_{\mathrm{f}}^2\varphi_{1,xx}+k_1(\varphi_1-v_{1,x})\\
&=-\{[\mathrm{i}\eta\omega_l(k_1\vartheta_l-k_{\mathrm{f}}^2\vartheta_l'')+\gamma_0\eta(k_1\vartheta_l'-k_{\mathrm{f}}^2\vartheta_l''')]B_l-\eta k_1(\mathrm{i}\omega_l\phi_l'+\gamma_0\phi_l'')A_l\\
&\quad+2\mathrm{i}\omega_lk_2\vartheta_l\dot{B}_l\}\mathrm{e}^{\mathrm{i}\omega_lT_0}-\{[\mathrm{i}\eta\omega_m(k_1\vartheta_m-k_{\mathrm{f}}^2\vartheta_m'')+\gamma_0\eta(k_1\vartheta_m'-k_{\mathrm{f}}^2\vartheta_m''')]B_m\\
&\quad-\eta k_1(\mathrm{i}\omega_m\phi_m'+\gamma_0\phi_m'')A_m+2\mathrm{i}\omega_mk_2\vartheta_m\dot{B}_m\}\mathrm{e}^{\mathrm{i}\omega_mT_0}-\{[\mathrm{i}\eta\omega_n(k_1\vartheta_n-k_{\mathrm{f}}^2\vartheta_n'')\\
&\quad+\gamma_0\eta(k_1\vartheta_n'-k_{\mathrm{f}}^2\vartheta_n''')]B_n-\eta k_1(\mathrm{i}\omega_n\phi_n'+\gamma_0\phi_n'')A_n+2\mathrm{i}\omega_nk_2\vartheta_n\dot{B}_n\}\mathrm{e}^{\mathrm{i}\omega_nT_0}+cc+NST
\end{aligned} \quad (46)$$

where the dot denotes the derivation with respect to slow time T_1, and NST stands for non-secular terms.

It can be checked that the linear part of the mass and stiffness operators in governing Eqs. (27) and (28) are symmetric and the gyroscopic operator is skew symmetric under the corresponding boundary conditions (29) and (34). The solvability condition presented by Chen and Zu [38] demands the orthogonal relationships

$$\langle\eta k_1(\mathrm{i}\omega_l\vartheta_l'+\gamma_0\vartheta_l'')B_l-\eta k_1(\mathrm{i}\omega_l\phi_l''+\gamma_0\phi_l''')A_l+2(\mathrm{i}\omega_l\phi_l+\gamma_0\phi_l')\dot{A}_l,\phi_l\rangle=0 \quad (47a)$$

$$\begin{aligned}
&\langle\eta k_1(\mathrm{i}\omega_m\vartheta_m'+\gamma_0\vartheta_m'')B_m-\eta k_1(\mathrm{i}\omega_m\phi_m''+\gamma_0\phi_m''')A_m+2(\mathrm{i}\omega_m\phi_m+\gamma_0\phi_m')\dot{A}_m\\
&\quad+[\tfrac{1}{2}\gamma_1(\omega_m-\omega_n)\overline{\phi}_n'-\mathrm{i}\gamma_0\gamma_1\overline{\phi}_n'']\overline{A}_n\mathrm{e}^{\mathrm{i}\mu T_1},\phi_m\rangle=0
\end{aligned} \quad (47b)$$

$$\begin{aligned}
&\langle\eta k_1(\mathrm{i}\omega_n\vartheta_n'+\gamma_0\vartheta_n'')B_n-\eta k_1(\mathrm{i}\omega_n\phi_n''+\gamma_0\phi_n''')A_n+2(\mathrm{i}\omega_n\phi_n+\gamma_0\phi_n')\dot{A}_n\\
&\quad+[\tfrac{1}{2}\gamma_1(\omega_n-\omega_m)\overline{\phi}_m'-\mathrm{i}\gamma_0\gamma_1\overline{\phi}_m'']\overline{A}_m\mathrm{e}^{\mathrm{i}\mu T_1},\phi_n\rangle=0
\end{aligned} \quad (47c)$$

$$\langle[\mathrm{i}\eta\omega_l(k_1\vartheta_l-k_{\mathrm{f}}^2\vartheta_l'')+\gamma_0\eta(k_1\vartheta_l'-k_{\mathrm{f}}^2\vartheta_l''')]B_l-\eta k_1(\mathrm{i}\omega_l\phi_l'+\gamma_0\phi_l'')A_l+2\mathrm{i}\omega_lk_2\vartheta_l\dot{B}_l,\phi_l\rangle=0 \quad (47d)$$

$$\langle[\mathrm{i}\eta\omega_m(k_1\vartheta_m-k_{\mathrm{f}}^2\vartheta_m'')+\gamma_0\eta(k_1\vartheta_m'-k_{\mathrm{f}}^2\vartheta_m''')]B_m-\eta k_1(\mathrm{i}\omega_m\phi_m'+\gamma_0\phi_m'')A_m+2\mathrm{i}\omega_mk_2\vartheta_m\dot{B}_m,\phi_m\rangle=0 \quad (47e)$$

$$\langle[\mathrm{i}\eta\omega_n(k_1\vartheta_n-k_{\mathrm{f}}^2\vartheta_n'')+\gamma_0\eta(k_1\vartheta_n'-k_f^2\vartheta_n''')]B_n-\eta k_1(\mathrm{i}\omega_n\phi_n'+\gamma_0\phi_n'')A_n+2\mathrm{i}\omega_nk_2\vartheta_n\dot{B}_n,\phi_n\rangle=0 \quad (47f)$$

where the inner product is defined for complex functions f and g on [0,1] as

$$\langle f,g\rangle=\int_0^1 f\overline{g}\,\mathrm{d}x \quad (48)$$

Application of the distributive law of the inner product to Eq. (47) leads to

$$\dot{A}_l + \eta\mu_l B_l + \eta\kappa_l A_l = 0 \tag{49a}$$

$$\dot{A}_m + \eta\mu_m B_m + \eta\kappa_m A_m + \gamma_1\chi_{mn}\overline{A}_n e^{i\mu T_1} = 0 \tag{49b}$$

$$\dot{A}_n + \eta\mu_n B_n + \eta\kappa_n A_n + \gamma_1\chi_{nm}\overline{A}_m e^{i\mu T_1} = 0 \tag{49c}$$

$$\dot{B}_l + \eta\xi_l B_l + \eta\zeta_l A_l = 0 \tag{49d}$$

$$\dot{B}_m + \eta\xi_m B_m + \eta\zeta_m A_m = 0 \tag{49e}$$

$$\dot{B}_n + \eta\xi_n B_n + \eta\zeta_n A_n = 0 \tag{49f}$$

where

$$\mu_k = \frac{k_1(i\omega_k\int_0^1 \vartheta_k'\overline{\phi}_k\,dx + \gamma_0\int_0^1 \vartheta_k''\overline{\phi}_k\,dx)}{2(i\omega_k\int_0^1 \phi_k\overline{\phi}_k\,dx + \gamma_0\int_0^1 \phi_k'\overline{\phi}_k\,dx)} \quad (k = l, m, n) \tag{50a}$$

$$\kappa_k = -\frac{k_1(i\omega_k\int_0^1 \phi_k''\overline{\phi}_k\,dx + \gamma_0\int_0^1 \phi_k'''\overline{\phi}_k\,dx)}{2(i\omega_k\int_0^1 \phi_k\overline{\phi}_k\,dx + \gamma_0\int_0^1 \phi_k'\overline{\phi}_k\,dx)} \quad (k = l, m, n) \tag{50b}$$

$$\xi_k = \frac{i\omega_k(k_1\int_0^1 \vartheta_k\overline{\phi}_k\,dx - k_f^2\int_0^1 \vartheta_k''\overline{\phi}_k\,dx) + \gamma_0(k_1\int_0^1 \vartheta_k'\overline{\phi}_k\,dx - k_f^2\int_0^1 \vartheta_k'''\overline{\phi}_k\,dx)}{2i\omega_k k_2\int_0^1 \vartheta_k\overline{\phi}_k\,dx} \quad (k = l, m, n) \tag{50c}$$

$$\zeta_k = -\frac{k_1(i\omega_k\int_0^1 \phi_k'\overline{\phi}_k\,dx + \gamma_0\int_0^1 \phi_k''\overline{\phi}_k\,dx)}{2i\omega_k k_2\int_0^1 \vartheta_k\overline{\phi}_k\,dx} \quad (k = l, m, n) \tag{50d}$$

$$\chi_{kj} = \frac{(\omega_k - \omega_j)\int_0^1 \overline{\phi}_j'\overline{\phi}_k\,dx - 2i\gamma_0\int_0^1 \overline{\phi}_j''\overline{\phi}_k\,dx}{4(i\omega_k\int_0^1 \phi_k\overline{\phi}_k\,dx + \gamma_0\int_0^1 \phi_k'\overline{\phi}_k\,dx)} \quad (k = n, m;\ j = m, n) \tag{50e}$$

It can be examined numerically that κ_k, ξ_k are positive real numbers, and μ_k, ζ_k are negative real numbers, while χ_{kj} is complex number.

3.3. Solvability in principal parametric resonance

In addition to the summation parametric resonance, the principal parametric resonance may occur if the variation frequency ω approaches two times of a natural frequency of generating autonomous linear system (23) and (24). In this case, denote

$$\omega = 2\omega_n + \varepsilon\mu \tag{51}$$

where ω_n is the nth natural frequency of Eqs. (23) and (24). To investigate the principal parametric resonance with the possible contributions of modes not involved the resonance, the solutions to Eqs. (23) and (24) can be expressed as

$$v_0(x, T_0, T_1) = \phi_l(x)A_l(T_1)e^{i\omega_l T_0} + \phi_n(x)A_n(T_1)e^{i\omega_n T_0} + cc \tag{52}$$

$$\varphi_0(x, T_0, T_1) = \vartheta_l(x)B_l(T_1)e^{i\omega_l T_0} + \vartheta_n(x)B_n(T_1)e^{i\omega_n T_0} + cc \tag{53}$$

Substitution of Eqs. (51), (52), (53) into (25) and (26), application of the solvability condition to the resulting equations, and simplification the outcomes via the properties of the inner product yield

$$\dot{A}_l + \eta\mu_l B_l + \eta\kappa_l A_l = 0 \tag{54a}$$

$$\dot{A}_n + \eta\mu_n B_n + \eta\kappa_n A_n + \gamma_1\chi_n\overline{A}_n e^{i\mu T_1} = 0 \tag{54b}$$

$$\dot{B}_l + \eta\xi_l B_l + \eta\zeta_l A_l = 0 \tag{54c}$$

$$\dot{B}_n + \eta\xi_n B_n + \eta\zeta_n A_n = 0 \tag{54d}$$

where

$$\mu_k=\frac{k_1(\mathrm{i}\omega_k\int_0^1\vartheta_k'\overline{\phi}_k\,\mathrm{d}x+\gamma_0\int_0^1\vartheta_k''\overline{\phi}_k\,\mathrm{d}x)}{2(\mathrm{i}\omega_k\int_0^1\phi_k\overline{\phi}_k\,\mathrm{d}x+\gamma_0\int_0^1\phi_k'\overline{\phi}_k\,\mathrm{d}x)}\quad(k=l,n) \tag{55a}$$

$$\kappa_k=-\frac{k_1(\mathrm{i}\omega_k\int_0^1\phi_k''\overline{\phi}_k\,\mathrm{d}x+\gamma_0\int_0^1\phi_k'''\overline{\phi}_k\,\mathrm{d}x)}{2(\mathrm{i}\omega_k\int_0^1\phi_k\overline{\phi}_k\,\mathrm{d}x+\gamma_0\int_0^1\phi_k'\overline{\phi}_k\,\mathrm{d}x)}\quad(k=l,n) \tag{55b}$$

$$\xi_k=\frac{\mathrm{i}\omega_k(k_1\int_0^1\vartheta_k\overline{\phi}_k\,\mathrm{d}x-k_f^2\int_0^1\vartheta_k''\overline{\phi}_k\,\mathrm{d}x)+\gamma_0(k_1\int_0^1\vartheta_k'\overline{\phi}_k\,\mathrm{d}x-k_f^2\int_0^1\vartheta_k'''\overline{\phi}_k\,\mathrm{d}x)}{2\mathrm{i}\omega_k k_2\int_0^1\vartheta_k\overline{\phi}_k\,\mathrm{d}x}\quad(k=l,n) \tag{55c}$$

$$\zeta_k=-\frac{k_1(\mathrm{i}\omega_k\int_0^1\phi_k'\overline{\phi}_k\,\mathrm{d}x+\gamma_0\int_0^1\phi_k''\overline{\phi}_k\,\mathrm{d}x)}{2\mathrm{i}\omega_k k_2\int_0^1\vartheta_k\overline{\phi}_k\,\mathrm{d}x}\quad(k=l,n) \tag{55d}$$

$$\chi_n=-\frac{\mathrm{i}\gamma_0\int_0^1\overline{\phi}_n''\overline{\phi}_n\,\mathrm{d}x}{2(\mathrm{i}\omega_n\int_0^1\phi_n\overline{\phi}_n\,\mathrm{d}x+\gamma_0\int_0^1\phi_n'\overline{\phi}_n\,\mathrm{d}x)} \tag{55e}$$

It can be numerically demonstrated that κ_k, ξ_k are positive real numbers, and μ_k, ζ_k are negative real numbers, while χ_n is a complex number.

3.4. Effects of the mode not involved in resonance

Eq. (49) indicates that the mode not involved in resonance is not coupled with the modes involved in the resonance. The characteristic equation of Eqs. (49a) and (49d) is

$$\begin{vmatrix}\lambda+\eta\kappa_l & -\eta\mu_l\\ -\eta\zeta_l & \lambda+\eta\xi_l\end{vmatrix}=0 \tag{56}$$

that is

$$\lambda^2+\eta(\kappa_l+\xi_l)\lambda+\eta^2\kappa_l\xi_l-\eta^2\mu_l\zeta_l=0 \tag{57}$$

It can be found numerically that λ is negative real number. Thus, the solutions to Eqs. (49a) and (49d) decays to zero exponentially. Therefore, the lth mode has actually no effects on the dynamic stability in summation parametric resonance of the mth and nth modes. Similarly, it can be demonstrated that the lth mode has no effects on the dynamic stability in principal parametric resonance of the mth mode.

4. Stability conditions

4.1. Summation parametric resonance

To cast Eqs. (49b), (49c), (49e) and (49f) into an autonomous system, introduce the transformation

$$A_n(T_1)=a_n(T_1)\mathrm{e}^{\mathrm{i}\mu T_1/2},B_n(T_1)=b_n(T_1)\mathrm{e}^{\mathrm{i}\mu T_1/2};$$

$$A_m(T_1)=a_m(T_1)\mathrm{e}^{\mathrm{i}\mu T_1/2},B_m(T_1)=b_m(T_1)\mathrm{e}^{\mathrm{i}\mu T_1/2} \tag{58}$$

Substitution of Eq. (58) into (49b), (49c), (49e) and (49f) yields

$$\dot a_n+\frac{\mathrm{i}\mu}{2}a_n+\eta\mu_n b_n+\eta\kappa_n a_n+\gamma_1\chi_{nm}\overline{a}_m=0$$

$$\dot b_n+\frac{\mathrm{i}\mu}{2}b_n+\eta\xi_n b_n+\eta\zeta_n a_n=0$$

$$\dot a_m+\frac{\mathrm{i}\mu}{2}a_m+\eta\mu_m b_m+\eta\kappa_m a_m+\gamma_1\chi_{mn}\overline{a}_n=0$$

$$\dot b_m+\frac{\mathrm{i}\mu}{2}b_m+\eta\xi_m b_m+\eta\zeta_m a_m=0 \tag{59}$$

Obviously, Eq. (59) has a zero solution. To investigate the stability of the non-zero solutions of Eq. (59), separation of those solutions into real and imaginary parts as

$$a_n(T_1)=p_1(T_1)+\mathrm{i}q_1(T_1),b_n(T_1)=p_2(T_1)+\mathrm{i}q_2(T_1);$$

$$a_m(T_1)=p_3(T_1)+\mathrm{i}q_3(T_1),b_m(T_1)=p_4(T_1)+\mathrm{i}q_4(T_1) \tag{60}$$

where $p_k(T_1)$ and $q_k(T_1)$ (k=1,2,3,4) are real functions with respect to T_1. Substituting Eq. (60) into (59) and separating the resulting equations into real and imaginary parts lead to

$$\dot{p}_1 = -\eta\kappa_n p_1 + \frac{\mu}{2}q_1 - \eta\mu_n p_2 - \gamma_1 \mathrm{Re}(\chi_{nm})p_3 - \gamma_1 \mathrm{Im}(\chi_{nm})q_3$$

$$\dot{q}_1 = -\frac{\mu}{2}p_1 - \eta\kappa_n q_1 - \eta\mu_n q_2 - \gamma_1 \mathrm{Im}(\chi_{nm})p_3 + \gamma_1 \mathrm{Re}(\chi_{nm})q_3$$

$$\dot{p}_2 = -\eta\zeta_n p_1 - \eta\xi_n p_2 + \frac{\mu}{2}q_2$$

$$\dot{q}_2 = -\eta\zeta_n q_1 - \frac{\mu}{2}p_2 - \eta\xi_n q_2 \tag{61a}$$

$$\dot{p}_3 = -\gamma_1 \mathrm{Re}(\chi_{mn})p_1 - \gamma_1 \mathrm{Im}(\chi_{mn})q_1 - \eta\kappa_m p_3 + \frac{\mu}{2}q_3 - \eta\mu_m p_4$$

$$\dot{q}_3 = -\gamma_1 \mathrm{Im}(\chi_{mn})p_1 + \gamma_1 \mathrm{Re}(\chi_{mn})q_1 - \frac{\mu}{2}p_3 - \eta\kappa_m q_3 - \eta\mu_m q_4$$

$$\dot{p}_4 = -\eta\zeta_m p_3 - \eta\xi_m p_4 + \frac{\mu}{2}q_4$$

$$\dot{q}_4 = -\eta\zeta_m q_3 - \frac{\mu}{2}p_4 - \eta\xi_m q_4 \tag{61b}$$

The characteristic equation of Eq. (61) is

$$\begin{vmatrix} -\eta\kappa_n-\lambda & \frac{\mu}{2} & -\eta\mu_n & 0 & -\gamma_1\mathrm{R}(\chi_{nm}) & -\gamma_1\mathrm{I}(\chi_{nm}) & 0 & 0 \\ -\frac{\mu}{2} & -\eta\kappa_n-\lambda & 0 & -\eta\mu_n & -\gamma_1\mathrm{I}(\chi_{nm}) & \gamma_1\mathrm{R}(\chi_{nm}) & 0 & 0 \\ -\eta\zeta_n & 0 & -\eta\xi_n-\lambda & \frac{\mu}{2} & 0 & 0 & 0 & 0 \\ 0 & -\eta\zeta_n & -\frac{\mu}{2} & -\eta\xi_n-\lambda & 0 & 0 & 0 & 0 \\ -\gamma_1\mathrm{R}(\chi_{mn}) & -\gamma_1\mathrm{I}(\chi_{mn}) & 0 & 0 & -\eta\kappa_m-\lambda & \frac{\mu}{2} & -\eta\mu_m & 0 \\ -\gamma_1\mathrm{I}(\chi_{mn}) & \gamma_1\mathrm{R}(\chi_{mn}) & 0 & 0 & -\frac{\mu}{2} & -\eta\kappa_m-\lambda & 0 & -\eta\mu_m \\ 0 & 0 & 0 & 0 & -\eta\zeta_m & 0 & -\eta\xi_m-\lambda & \frac{\mu}{2} \\ 0 & 0 & 0 & 0 & 0 & -\eta\zeta_m & -\frac{\mu}{2} & -\eta\xi_m-\lambda \end{vmatrix} = 0 \tag{62}$$

Direct calculation of the determinant leads to the characteristic equation as

$$\alpha_0\lambda^8 + \alpha_1\lambda^7 + \alpha_2\lambda^6 + \alpha_3\lambda^5 + \alpha_4\lambda^4 + \alpha_5\lambda^3 + \alpha_6\lambda^2 + \alpha_7\lambda + \alpha_8 = 0 \tag{63}$$

with

$$\alpha_0 = 1 \tag{64a}$$

$$\alpha_1 = 2\eta(\kappa_m + \kappa_n + \xi_m + \xi_n) \tag{64b}$$

$$\begin{aligned}\alpha_2 = {} & \mu^2 + \eta^2[2(\kappa_m + \kappa_n + \xi_m + \xi_n)^2 + 2(\kappa_m\kappa_n - \zeta_m\mu_m - \zeta_n\mu_n + \xi_m\xi_n) \\ & +2(\kappa_m + \kappa_n)(\xi_m + \xi_n)] - 2\gamma_1^2[\mathrm{Re}(\chi_{mm})\mathrm{Re}(\chi_{nm}) + \mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]\end{aligned} \tag{64c}$$

$$\begin{aligned}\alpha_3 = {} & \tfrac{3}{2}\eta\mu^2(\kappa_m + \kappa_n + \xi_m + \xi_n) + \eta^3[2(\kappa_m + \kappa_n + \xi_m + \xi_n)(\kappa_m\kappa_n - \zeta_m\mu_m - \zeta_n\mu_n + \xi_m\xi_n) \\ & +2(\kappa_m + \kappa_n)(\xi_m + \xi_n)^2 + 2(\kappa_m + \kappa_n)^2(\xi_m + \xi_n) - 2\zeta_m\mu_m(\kappa_n + \xi_n) - 2\zeta_n\mu_n(\kappa_m + \xi_m) \\ & +2(\kappa_m + \kappa_n)\xi_m\xi_n + 2(\xi_m + \xi_n)\kappa_m\kappa_n] - 2\eta\gamma_1^2[\kappa_m + \kappa_n + 2(\xi_m + \xi_n)] \\ & \cdot[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm}) + \mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]\end{aligned} \tag{64d}$$

$$\begin{aligned}\alpha_4 = {} & \frac{3\mu^4}{8} + \eta^2\mu^2\left[\frac{3}{4}(\kappa_m^2 + \kappa_n^2 + \xi_m^2 + \xi_n^2) + 2(\kappa_m\kappa_n + \xi_m\xi_n) + 2(\kappa_m + \kappa_n)(\xi_m + \xi_n) - \frac{1}{2}(\zeta_m\mu_m + \zeta_n\mu_n)\right] \\ & +\eta^4[(\kappa_m\kappa_n + \xi_m\xi_n)^2 + (\kappa_m\xi_n + \kappa_n\xi_m)^2 + (\zeta_m\mu_m + \zeta_n\mu_n - \kappa_m\xi_m - \kappa_n\xi_n)^2 \\ & -4(\zeta_m\mu_m + \zeta_n\mu_n)(\kappa_m + \xi_m)(\kappa_n + \xi_n) - 2(\zeta_m\mu_m - \kappa_m\xi_m)(\kappa_n^2 - \zeta_n\mu_n + 3\kappa_n\xi_n + \xi_n^2)\end{aligned}$$

$$-2\zeta_n\mu_n(\kappa_m+\xi_m)^2+2\kappa_m\xi_m(\kappa_n+\xi_n)^2+4\kappa_n\xi_m(\kappa_m^2+\xi_n^2)+4\kappa_n\xi_n(\kappa_m^2+\xi_n^2)+4\kappa_m\xi_n(\kappa_n^2+\xi_m^2)$$
$$+4\kappa_m\kappa_n(\xi_m^2+\xi_n^2)+4\xi_m\xi_n(\kappa_m^2+\kappa_n^2)]+\eta\mu\gamma_1^2(\kappa_m-\kappa_n)[\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-\frac{3}{2}\mu^2\gamma_1^2[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]+2\eta^2\gamma_1^2[\zeta_m\mu_m+\zeta_n\mu_n-\kappa_m\kappa_n-2\xi_m\xi_n$$
$$-(\xi_m+\xi_n)^2-2(\kappa_m+\kappa_n)(\xi_m+\xi_n)][\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]+\gamma_1^4|\chi_{mn}|^2|\chi_{nm}|^2 \tag{64e}$$

$$\alpha_5=\frac{3}{8}\eta\mu^4(\kappa_m+\kappa_n+\xi_m+\xi_n)+\eta^3\mu^2\{(\kappa_m+\xi_m)(\kappa_m\kappa_n-\zeta_m\mu_m+\kappa_n\xi_m)+(\kappa_n+\xi_n)(\kappa_n\xi_m$$
$$+\xi_m\xi_n-\zeta_n\mu_n)+\xi_m[(\kappa_m+\kappa_n)^2+\xi_m^2+\kappa_n\xi_n]+\kappa_m[(\xi_m+\xi_n)^2+\kappa_m^2+\kappa_m\xi_m]\}$$
$$+\eta^5\{2(\kappa_n+\xi_n)[(\zeta_m\mu_m+\kappa_m\xi_m)^2-\zeta_n\mu_n(\kappa_m^2+4\kappa_m\xi_m+\xi_m^2)+\kappa_n\xi_n(\kappa_m^2-2\zeta_m\mu_m+\xi_m^2)]$$
$$+2(\kappa_m+\xi_m)\cdot[(\zeta_n\mu_n+\kappa_n\xi_n)^2-\zeta_m\mu_m(\kappa_n^2+4\kappa_n\xi_n+\xi_n^2)+\kappa_m\xi_m(\kappa_n^2-2\zeta_n\mu_n+\xi_n^2)]$$
$$+4(\kappa_m+\kappa_n+\xi_m+\xi_n)(\zeta_m\mu_m\zeta_n\mu_n+2\kappa_m\xi_m\kappa_n\xi_n)\}$$
$$-2\eta^2\mu\gamma_1^2[\zeta_m\mu_m-\zeta_n\mu_n-(\kappa_m-\kappa_n)(\xi_m+\xi_n)][\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-\eta\mu^2\gamma_1^2[\kappa_m+\kappa_n+2(\xi_m+\xi_n)][\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$+2\eta^3\gamma_1^2[\zeta_m\mu_m(\kappa_m+\xi_m+2\xi_n)+\zeta_n\mu_n(\kappa_n+\xi_n+2\xi_m)-2(\xi_m+\xi_n)(\kappa_m\kappa_n+\xi_m\xi_n)$$
$$-(\kappa_m+\kappa_n)(\xi_m^2+4\xi_m\xi_n+\xi_n^2)][\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]+2\eta\gamma_1^4(\xi_m+\xi_n)|\chi_{mn}|^2|\chi_{nm}|^2 \tag{64f}$$

$$\alpha_6=\frac{\mu^6}{16}+\frac{1}{16}\eta^2\mu^4\left\{3(\kappa_m^2+\kappa_n^2+\xi_m^2+\xi_n^2)+4[\kappa_m\kappa_n+\xi_m\xi_n+(\kappa_m+\kappa_n)(\xi_m+\xi_n)]+2(\zeta_m\mu_m-\zeta_n\mu_n)\right\}$$
$$+\frac{1}{2}\eta^4\mu^2\left\{\zeta_m^2\mu_m^2+\kappa_m^2(\kappa_m+\xi_m+\xi_n)^2-2\zeta_m\mu_m[2\zeta_n\mu_n+\xi_m\xi_n\kappa_m(\xi_m-2\xi_n)]\right.$$
$$+[\xi_m\xi_n-\zeta_n\mu_n+\kappa_n(\xi_m+\xi_n)]^2+2\kappa_m[(\kappa_n\xi_m+\xi_n^2)(\kappa_n+\xi_m)+(\kappa_m^2+\xi_m^2)\xi_n$$
$$\left.-\zeta_n\mu_n(\kappa_n-2\xi_m+\xi_n)-\zeta_m\mu_m(\kappa_n+\xi_m+\xi_n)]\right\}+\eta^6\left\{(\zeta_m\mu_m-\kappa_m\xi_m)[\zeta_m\mu_m(\kappa_n^2+\xi_n^2)\right.$$
$$-2(\zeta_n^2\mu_n^2+\kappa_n^2\xi_n^2)]-[2\zeta_m^2\mu_m^2+2\kappa_m^2\xi_m^2-\zeta_n\mu_n(\kappa_m^2+\xi_m^2)](\zeta_n\mu_n-\kappa_n\xi_n)+\kappa_m^2\kappa_n^2(\xi_m^2+\xi_n^2)$$
$$+\xi_m^2\xi_n^2(\kappa_m^2+\kappa_n^2)+4\zeta_m\mu_m\zeta_n\mu_n(\kappa_m\kappa_n+\kappa_m\xi_m+\kappa_n\xi_m+\kappa_m\xi_n+\kappa_n\xi_n+\xi_m\xi_n)$$
$$\left.-8\kappa_m\kappa_n\xi_m\xi_n(\zeta_m\mu_m-\zeta_n\mu_n)-4(\kappa_m+\xi_m)(\kappa_n+\xi_n)(\zeta_n\mu_n\kappa_m\xi_m-\zeta_m\mu_m\kappa_n\xi_n-\kappa_m\kappa_n\xi_m\xi_n)\right\}$$
$$+\frac{1}{2}\eta\mu^3\gamma_1^2(\kappa_m-\kappa_n)[\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]-\eta^3\mu\gamma_1^2[\zeta_m\mu_m(\kappa_n+\xi_m+4\xi_n)$$
$$-\zeta_n\mu_n(\kappa_m+\xi_n+4\xi_m)-(\kappa_m-\kappa_n)(\xi_m^2+4\xi_m\xi_n+\xi_n^2)][\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-\frac{3}{8}\mu^4\gamma_1^2[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]-\eta^2\mu^2\gamma_1^2(\kappa_m+\xi_m+\xi_n)(\kappa_n+\xi_m+\xi_n)[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})$$
$$+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]-2\eta^4\gamma_1^2[(\zeta_m\mu_m-\kappa_m\xi_n)(\zeta_n\mu_n-\kappa_n\xi_m)-2\zeta_m\mu_m\xi_n\cdot(\kappa_n+\xi_m)$$
$$-2\zeta_n\mu_n\xi_m(\kappa_m+\xi_n)+\kappa_m\kappa_n(\xi_m+\xi_n)^2-\zeta_n\mu_n\xi_m^2-\zeta_m\mu_m\xi_n^2+\xi_m\xi_n(\kappa_m\kappa_n+\xi_m\xi_n)$$
$$+2\xi_m\xi_n(\kappa_m+\kappa_n)(\xi_m+\xi_n)][\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$+\frac{1}{2}\mu^2\gamma_1^4|\chi_{mn}|^2|\chi_{nm}|^2+\eta^2\gamma_1^4(\xi_m^2+4\xi_m\xi_n+\xi_n^2)|\chi_{mn}|^2|\chi_{nM}|^2 \tag{64g}$$

$$\alpha_7=\frac{\eta\mu^6}{32}(\kappa_m+\kappa_n+\xi_m+\xi_n)+\frac{\eta^3\mu^4}{8}[(\kappa_m+\kappa_n)(\kappa_m\kappa_n+\xi_m^2+\xi_n^2)+(\xi_m+\xi_n)(\xi_m\xi_n+\kappa_m^2+\kappa_n^2)$$
$$-(\zeta_m\mu_m-2\zeta_n\mu_n)(\kappa_m+\xi_m)-(\zeta_n\mu_n-2\zeta_m\mu_m)(\kappa_n+\xi_n)]+\frac{\eta^5\mu^2}{2}\left\{[(\zeta_m\mu_m-\kappa_m\xi_m)^2\right.$$
$$+2\zeta_m\mu_m\kappa_n\xi_n-\zeta_n\mu_n(\kappa_m^2+\xi_m^2)](\kappa_n+\xi_n)+[(\zeta_n\mu_n-\kappa_n\xi_n)^2+2\zeta_n\mu_n\kappa_m\xi_m-\zeta_m\mu_m(\kappa_n^2+\xi_n^2)]$$
$$\left.\cdot(\kappa_m+\xi_m)-2\zeta_m\mu_m\zeta_n\mu_n(\kappa_m+\kappa_n+\xi_m+\xi_n)\right\}-2\eta^7(\zeta_m\mu_m-\kappa_m\xi_m)(\zeta_n\mu_n-\kappa_n\xi_n)\{-\kappa_m\kappa_n\xi_m$$
$$+\zeta_n\mu_n(\kappa_m+\xi_m)+\zeta_m\mu_m(\kappa_n+\xi_n)-\xi_n[\kappa_m\kappa_n+\xi_m(\kappa_m+\kappa_n)]\}$$
$$-\frac{1}{2}\eta^2\mu^3\gamma_1^2[\zeta_m\mu_m-\zeta_n\mu_n-(\kappa_m-\kappa_n)(\xi_m+\xi_n)][\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mm})\mathrm{Im}(\chi_{nm})]$$
$$+2\eta^4\mu\gamma_1^2[\zeta_n\mu_n\xi_m(\kappa_m+\xi_m+\xi_n)-\zeta_m\mu_m\xi_n(\kappa_n+\xi_n+\xi_m)$$
$$+\xi_m\xi_n(\kappa_m-\kappa_n)(\xi_m+\xi_n)][\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$

$$-\frac{1}{8}\eta\mu^4\gamma_1^2[\kappa_m+\kappa_n+2(\xi_m+\xi_n)][\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-\frac{1}{2}\eta^3\mu^2\gamma_1^2[2(\zeta_n\mu_n\xi_m+\zeta_m\mu_m\xi_n)-\zeta_n\mu_n(\kappa_m+\xi_n)-\zeta_m\mu_m(\kappa_n+\xi_m)$$
$$+(\kappa_m+\kappa_n)(\xi_m^2+\xi_n^2)+2(\xi_m+\xi_n)(\kappa_n\kappa_m+\xi_n\xi_m)][\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-2\eta^5\gamma_1^2[\zeta_m\mu_m\zeta_n\mu_n(\xi_m+\xi_n)-2\xi_m\xi_n(\zeta_m\mu_m\kappa_n+\zeta_n\mu_n\kappa_m)-\zeta_m\mu_m(\kappa_n+\xi_m)\xi_n^2-\zeta_n\mu_n\xi_m^2$$
$$\cdot(\kappa_m+\xi_n)+2\kappa_n\kappa_m\xi_n\xi_m(\xi_m+\xi_n)+(\kappa_m+\kappa_n)(\xi_m^2+\xi_n^2)][\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$\cdot\frac{1}{2}\eta\mu^2\gamma_1^4(\xi_m+\xi_n)|\chi_{mn}|^2|\chi_{nm}|^2+2\eta^3\gamma_1^4\xi_n\xi_m(\xi_m+\xi_n)|\chi_{mn}|^2|\chi_{nm}|^2 \tag{64h}$$

$$\alpha_8=\frac{\mu^8}{256}+\frac{1}{64}\eta^2\mu^6(\kappa_m^2+\kappa_n^2+2\zeta_m\mu_m+2\zeta_n\mu_n+\xi_m^2+\xi_n^2)+\frac{1}{16}\eta^4\mu^4[(\zeta_n\mu_n-\kappa_n\xi_n)^2$$
$$+(\zeta_m\mu_m-\kappa_m\xi_m)^2+(\kappa_m^2+2\zeta_m\mu_m+\xi_m^2)(\kappa_n^2+2\zeta_n\mu_n+\xi_n^2)]$$
$$+\frac{1}{4}\eta^6\mu^2[(\kappa_n^2+2\zeta_n\mu_n+\xi_n^2)(\zeta_m\mu_m-\kappa_m\xi_m)^2+(\kappa_m^2+2\zeta_m\mu_m+\xi_m^2)(\zeta_n\mu_n-\kappa_n\xi_n)^2]$$
$$+\eta^8(\zeta_m\mu_m-\kappa_m\xi_m)^2(\zeta_n\mu_n-\kappa_n\xi_n)^2+\frac{1}{16}\eta\sigma^5\gamma_1^2(\kappa_m-\kappa_n)$$
$$\cdot[\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]-\frac{1}{4}\eta^3\mu^3\gamma_1^2[\zeta_m\mu_m(\kappa_n+\xi_m)-\zeta_n\mu_n(\kappa_m+\xi_n)$$
$$-(\kappa_m-\kappa_n)(\xi_m^2+\xi_n^2)][\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-\eta^5\mu\gamma_1^2\left\{\zeta_m\mu_m[\zeta_n\mu_n(\xi_m-\xi_n)+\xi_n^2(\kappa_n+\xi_m)]-\xi_m^2[\xi_n^2(\kappa_m-\kappa_n)+\zeta_n\mu_n(\kappa_n+\xi_m)]\right\}$$
$$\cdot[\mathrm{Im}(\chi_{mn})\mathrm{Re}(\chi_{nm})-\mathrm{Re}(\chi_{mn})\mathrm{Im}(\chi_{nm})]-\frac{1}{32}\mu^6\gamma_1^2[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-\frac{1}{8}\eta^2\mu^4\gamma_1^2(\kappa_m\kappa_n+\zeta_m\mu_m+\zeta_n\mu_n+\xi_m^2+\xi_n^2)[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-\frac{1}{8}\eta^4\mu^2\gamma_1^2\left\{\xi_m^2(\zeta_n\mu_n+\xi_n^2)+\zeta_m\mu_m(\zeta_n\mu_n-\kappa_n\xi_m+\xi_n^2)\right.$$
$$\left.+\kappa_m[\kappa_n(\xi_m^2+\xi_n^2)-\zeta_n\mu_n\xi_n]\right\}[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$-2\eta^6\gamma_1^2\xi_m\xi_n(\zeta_m\mu_m-\kappa_m\xi_m)(\zeta_n\mu_n-\kappa_n\xi_n)[\mathrm{Re}(\chi_{mn})\mathrm{Re}(\chi_{nm})+\mathrm{Im}(\chi_{mn})\mathrm{Im}(\chi_{nm})]$$
$$+\frac{1}{16}\mu^4\gamma_1^4|\chi_{mn}|^2|\chi_{nm}|^2+\frac{1}{4}\eta^2\mu^2\gamma_1^4(\xi_m^2+\xi_n^2)|\chi_{mn}|^2|\chi_{nm}|^2+\eta^4\gamma_1^4\xi_m^2\xi_n^2|\chi_{mn}|^2|\chi_{nm}|^2 \tag{64i}$$

The Routh–Hurvitz criterion gives the sufficient and necessary condition of the stability of non-zero solutions to (61) as

$$\Delta_1=\alpha_1>0;\Delta_2=\begin{vmatrix}\alpha_1&\alpha_0\\\alpha_3&\alpha_2\end{vmatrix}>0,\Delta_3=\begin{vmatrix}\alpha_1&\alpha_0&0\\\alpha_3&\alpha_2&\alpha_1\\\alpha_5&\alpha_4&\alpha_3\end{vmatrix}>0,\Delta_4=\begin{vmatrix}\alpha_1&\alpha_0&0&0\\\alpha_3&\alpha_2&\alpha_1&\alpha_0\\\alpha_5&\alpha_4&\alpha_3&\alpha_2\\\alpha_7&\alpha_6&\alpha_5&\alpha_4\end{vmatrix}>0,$$

$$\Delta_5=\begin{vmatrix}\alpha_1&\alpha_0&0&0&0\\\alpha_3&\alpha_2&\alpha_1&\alpha_0&0\\\alpha_5&\alpha_4&\alpha_3&\alpha_2&\alpha_1\\\alpha_7&\alpha_6&\alpha_5&\alpha_4&\alpha_3\\0&\alpha_8&\alpha_7&\alpha_6&\alpha_5\end{vmatrix}>0,\Delta_6=\begin{vmatrix}\alpha_1&\alpha_0&0&0&0&0\\\alpha_3&\alpha_2&\alpha_1&\alpha_0&0&0\\\alpha_5&\alpha_4&\alpha_3&\alpha_2&\alpha_1&\alpha_0\\\alpha_7&\alpha_6&\alpha_5&\alpha_4&\alpha_3&\alpha_2\\0&\alpha_8&\alpha_7&\alpha_6&\alpha_5&\alpha_4\\0&0&0&\alpha_8&\alpha_7&\alpha_6\end{vmatrix}>0,$$

$$\Delta_7=\begin{vmatrix}\alpha_1&\alpha_0&0&0&0&0&0\\\alpha_3&\alpha_2&\alpha_1&\alpha_0&0&0&0\\\alpha_5&\alpha_4&\alpha_3&\alpha_2&\alpha_1&\alpha_0&0\\\alpha_7&\alpha_6&\alpha_5&\alpha_4&\alpha_3&\alpha_2&\alpha_1\\0&\alpha_8&\alpha_7&\alpha_6&\alpha_5&\alpha_4&\alpha_3\\0&0&0&\alpha_8&\alpha_7&\alpha_6&\alpha_5\\0&0&0&0&0&\alpha_8&\alpha_7\end{vmatrix}>0,\Delta_8=\alpha_8>0 \tag{65}$$

Inequality (65) means the stability conditions are

$$|\gamma_1| < \gamma_{\min} = \min\{\gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}, \gamma^{(5)}, \gamma^{(6)}, \gamma^{(7)}, \gamma^{(8)}\} \tag{66}$$

4.2. Principal parametric resonance

The transformation

$$A_n(T_1) = a_n(T_1)\mathrm{e}^{\mathrm{i}\mu T_1/2}; B_n(T_1) = b_n(T_1)\mathrm{e}^{\mathrm{i}\mu T_1/2} \tag{67}$$

changes Eqs. (54b) and (54d) into an autonomous system

$$\dot{a}_n + \frac{\mathrm{i}\mu}{2}a_n + \eta\mu_n b_n + \eta\kappa_n a_n + \gamma_1\chi_n\overline{a}_n = 0$$

$$\dot{b}_n + \frac{\mathrm{i}\mu}{2}b_n + \eta\xi_n b_n + \eta\zeta_n a_n = 0 \tag{68}$$

To investigate the stability of the non-zero solutions of Eq. (68), separate of those solutions into real and imaginary parts as

$$a_n(T_1) = p_1(T_1) + \mathrm{i}q_1(T_1); b_n(T_1) = p_2(T_1) + \mathrm{i}q_2(T_1) \tag{69}$$

where $p_1(T_1)$, $q_1(T_1)$, $p_2(T_1)$, and $q_2(T_1)$ are real functions with respect to T_1. Substituting Eq. (69) into (68) and separating the resulting equations into real and imaginary parts lead to

$$p_1 = -[\eta\kappa_n + \gamma_1 \operatorname{Re}(\chi_n)]p_1 + \left[\frac{\mu}{2} - \gamma_1 \operatorname{Im}(\chi_n)\right]q_1 - \eta\mu_n p_2$$

$$\dot{q}_1 = -\left[\frac{\mu}{2} + \gamma_1 \operatorname{Im}(\chi_n)\right]p_1 - [\eta\kappa_n - \gamma_1 \operatorname{Re}(\chi_n)]q_1 - \eta\mu_n q_2$$

$$\dot{p}_2 = -\eta\zeta_n p_1 - \eta\xi_n p_2 + \frac{\mu}{2}q_2$$

$$\dot{q}_2 = -\eta\zeta_n q_1 - \frac{\mu}{2}p_2 - \eta\xi_n q_2 \tag{70}$$

The characteristic equation of Eq. (70) is

$$\begin{vmatrix} -\eta\kappa_n - \gamma_1\operatorname{Re}(\chi_n) - \lambda & \frac{\mu}{2} - \gamma_1\operatorname{Im}(\chi_n) & -\eta\mu_n & 0 \\ -\frac{\mu}{2} - \gamma_1\operatorname{Im}(\chi_n) & \gamma_1\operatorname{Re}(\chi_n) - \eta\kappa_n - \lambda & 0 & -\eta\mu_n \\ -\eta\zeta_n & 0 & -\eta\xi_n - \lambda & \frac{\mu}{2} \\ 0 & -\eta\zeta_n & -\frac{\mu}{2} & -\eta\xi_n - \lambda \end{vmatrix} = 0 \tag{71}$$

Direct calculation of the determinant leads to the characteristic equation as

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0 \tag{72}$$

with

$$\alpha_1 = 2\eta(\kappa_n + \xi_n)$$

$$\alpha_2 = \frac{\mu^2}{2} + \eta^2(\kappa_n^2 - 2\zeta_n\mu_n + 4\kappa_n\xi_n + \xi_n^2) - \gamma_1^2|\chi_n|^2$$

$$\alpha_3 = \frac{\eta\mu^2}{2}(\kappa_n + \xi_n) - 2\eta^3(\zeta_n\mu_n - \kappa_n\xi_n)(\kappa_n + \xi_n) - 2\eta\xi_n\gamma_1^2|\chi_n|^2$$

$$\alpha_4 = \frac{\mu^4}{16} + \frac{\eta^2\mu^2}{4}(\kappa_n^2 + 2\zeta_n\mu_n + \xi_n^2) + \eta^4(\zeta_n\kappa_n - \mu_n\xi_n)^2 - |\chi_n|^2\gamma_1^2\left(\frac{\mu^2}{4} + \eta^2\xi_n^2\right) \tag{73}$$

The Routh–Hurvitz criterion gives the sufficient and necessary condition of the stability of non-zero solutions to (70) as

$$\Delta_1 = \alpha_1 > 0; \Delta_2 = \begin{vmatrix} \alpha_1 & 1 \\ \alpha_3 & \alpha_2 \end{vmatrix} > 0, \Delta_3 = \begin{vmatrix} \alpha_1 & 1 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ 0 & \alpha_4 & \alpha_3 \end{vmatrix} > 0, \Delta_4 = \alpha_4 > 0 \tag{74}$$

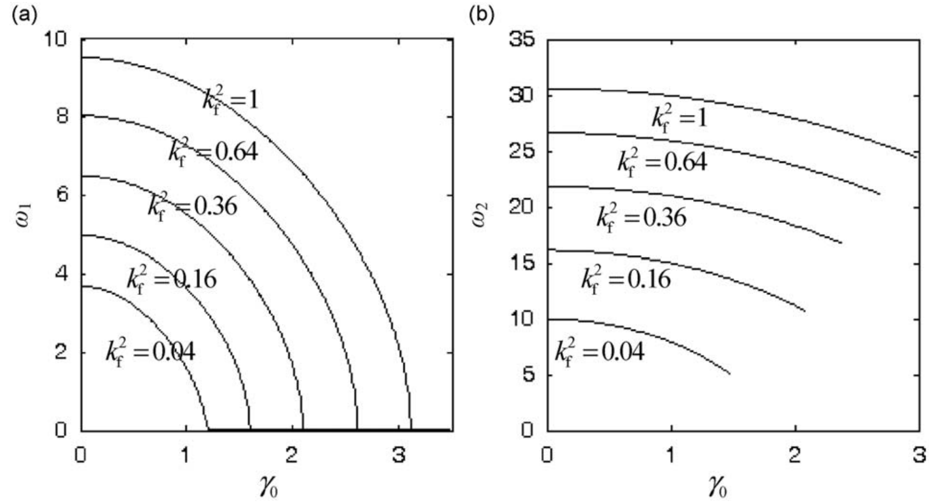

Fig. 2. Natural frequencies changing with the mean axial speed for different bending stiffness: (a) the first natural frequencies and (b) the second natural frequencies.

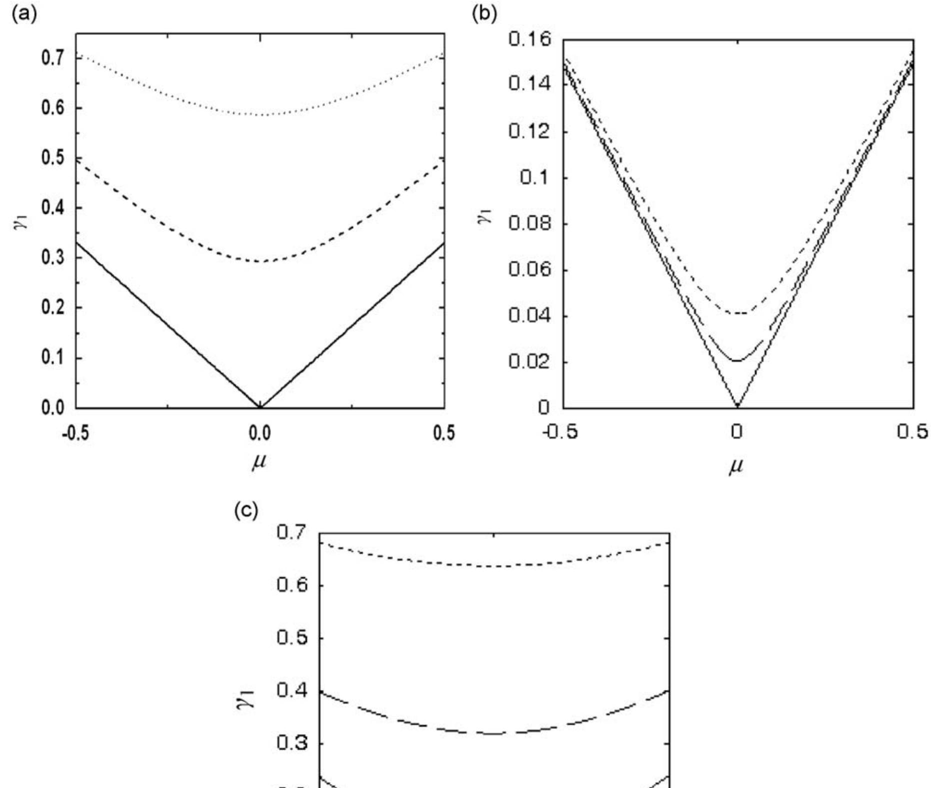

Fig. 3. The effect of viscosity coefficient on the stability boundaries: (a) the summation resonance, (b) the first principal resonance and (c) the second principal resonance.

The calculations yield

$$\Delta_2 = \tfrac{1}{2}\eta(\kappa_n + \xi_n)\Big\{\mu^2 + 4\eta^2[(\kappa_n + \xi_n)^2 + \kappa_n\xi_n - \zeta_n\mu_n]\Big\} - 2\eta\kappa_n|\chi_n|^2\gamma_1^2$$

$$\Delta_3 = 4\eta^2\{\eta^2(\kappa_n + \xi_n)^2(\kappa_n\xi_n - \zeta_n\mu_n)[\mu^2 + \eta^2(\kappa_n + \xi_n)^2] + \kappa_n\xi_n|\chi_n|^4\gamma_1^4 + \eta^2(\kappa_n + \xi_n)^2(\zeta_n\mu_n - 2\kappa_n\xi_n)|\chi_n|^2\gamma_1^2\}$$

$$\Delta_4 = \frac{\mu^4}{16} + \frac{\eta^2\mu^2}{4}(\kappa_n^2 + 2\zeta_n\mu_n + \xi_n^2) + \eta^4(\kappa_n\xi_n - \zeta_n\mu_n)^2 - |\chi_n|^2\gamma_1^2\left(\frac{\mu^2}{4} + \eta^2\xi_n^2\right) \tag{75}$$

Inequality (74) means the stability conditions are

$$|\gamma_1| < \gamma_{\min} = \min\{\gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\} \tag{76}$$

where

$$\gamma^{(2)} = \frac{1}{2|\chi_{nm}|\sqrt{\kappa_n}}\sqrt{\eta(\kappa_n + \xi_n)\{\mu^2 + 4\eta^2[(\kappa_n + \xi_n)^2 + \kappa_n\xi_n - \zeta_n\mu_n]\}}$$

$$\gamma^{(3)} = \frac{1}{\sqrt{2\kappa_n\xi_n}|\chi_{nm}|}[\eta(\kappa_n + \xi_n)\sqrt{\eta^2(\kappa_n + \xi_n)^2\zeta_n^2\mu_n^2 - 4\mu^2\kappa_n\xi_n(\kappa_n\xi_n - \zeta_n\mu_n)} - \eta^2(\kappa_n + \xi_n)^2(\zeta_n\mu_n - 2\kappa_n\xi_n)]^{1/2}$$

$$\gamma^{(4)} = \frac{1}{2|\chi_{nm}|}\sqrt{\frac{\mu^4 + 4\eta^2\mu^2(\kappa_n^2 + 2\zeta_n\mu_n + \xi_n^2) + 16\eta^4(\zeta_n\kappa_n - \mu_n\xi_n)^2}{\mu^2 + 4\eta^2\xi_n^2}} \tag{77}$$

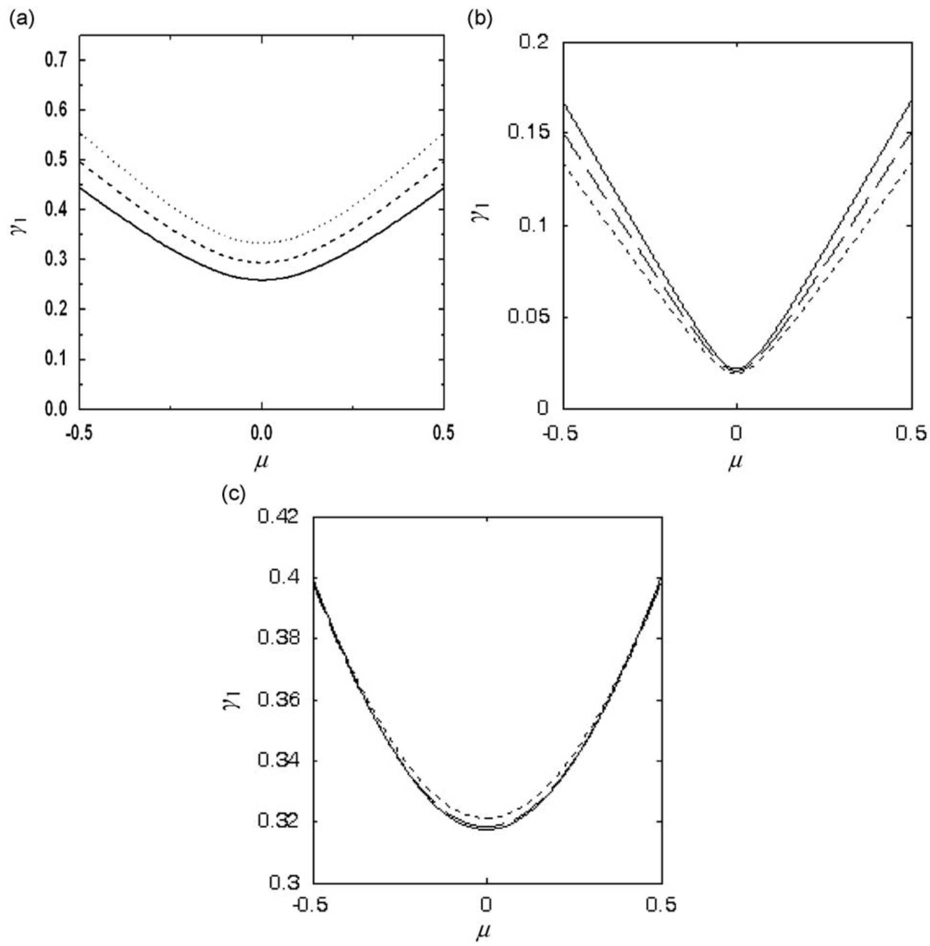

Fig. 4. The effect of the mean axial speed on the stability boundaries: (a) the summation resonance, (b) the first principal resonance and (c) the second principal resonance.

5. Numerical examples

Consider an axially moving viscoelastic Timoshenko beam with $A=9\times10^{-3}\,\mathrm{m}^2$, $P=10^7\,\mathrm{N}$, $L=0.3\,\mathrm{m}$, $E=169\times10^9\,\mathrm{Pa}$, $G=66\times10^9\,\mathrm{Pa}$, and $\kappa=5/6$ for different value of J. Eq. (13) $k_1=71.28$ and $k_2=0.006575k_f^2$. For given values of the dimensionless mean axial speed γ_0 and the dimensionless bending stiffness k_f, based on Eqs. (37) and (38), the nth values β_{jn} (j=1, 2, 3, 4) and the corresponding natural frequency ω_n can be calculated numerically for given the set of parameters. Fig. 2 presents the first and second natural frequencies for different mean axial speed and bending stiffness. The numerical results indicate that the natural frequencies decrease with the increasing mean axial speed and the decreasing bending stiffness.

Consider an axially moving viscoelastic Timoshenko beam with k_1=71.28, k_2=0.0042, k_f=0.8, and γ_0=2. The stability boundaries for the summation resonance of first two modes and the first and second principal resonances in plane μ–γ_1 for different viscosity coefficient are shown in Fig. 3. The solid lines denote η=0.0, the dashed lines denote η=0.0005, and the dotted lines denote η=0.001. The larger viscosity coefficient leads to the larger instability threshold of γ_1 for given μ, and the smaller instability range of μ for given γ_1. The stability boundaries in the resonance involving higher order modes, such as the summation resonance and the second principal resonance, are more sensitive to the viscosity coefficient.

Consider an axially moving viscoelastic Timoshenko beam with k_1=71.28, k_2=0.0042, k_f=0.8, and η=0.0005. The stability boundaries for the summation resonance of first two modes and the first and second principal resonance in plane μ–γ_1 for different mean axial speed are shown in Fig. 4. The solid lines denote γ_0=1.9, the dashed lines denote γ_0=2.0, and the dotted lines denote γ_0=2.1. The decreasing mean axial speed makes the stability boundaries in the summation resonance move towards the decreasing direction of γ_1 in plane μ–γ_1 and the instability regions become narrow. However, the tendencies in the principal resonances are opposite. Physically, a possible explanation is the mean axial speed makes the system unstable, while, with the use of the material time derivative in the constitutive relation, reflected by Eqs. (5) and (6), the

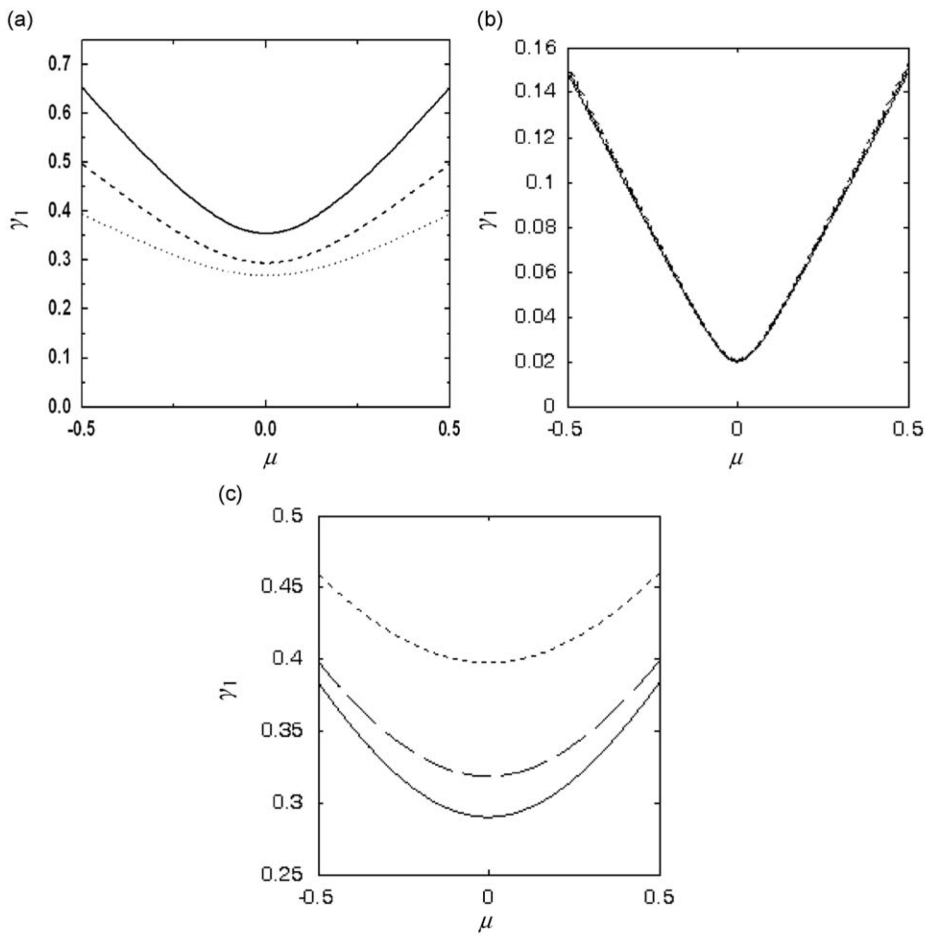

Fig. 5. The effect of shear deformation on the stability boundaries: (a) the summation resonance, (b) the first principal resonance and (c) the second principal resonance.

axial speed also has the effect to increase the damping that makes system stable. As the stability boundaries in the principal resonances are more sensitive to the viscosity coefficient, the opposite changing tendencies occur.

Consider an axially moving viscoelastic Timoshenko beam with γ_0=2, k_2=0.0042, k_f=0.8, and η=0.0005. The stability boundaries for the summation resonance of first two modes and the first and second principal resonances in plane μ–γ_1 for different shear deformation are shown in Fig. 5. The solid lines denote k_1=54, the dashed lines denote k_1=71.28, and the dotted lines denote k_1=108. Once again, the changing effect of shear deformation leads to the opposite tendencies in the summation resonance and the principal resonances. In the summation (principal) resonance, the larger shear deformation effects lead to the smaller (larger) instability threshold of γ_1 for given μ, and the larger (smaller) instability range of μ for given γ_1.

Consider an axially moving viscoelastic Timoshenko beam with γ_0=2, k_1=71.28, k_f=0.8, and η=0.0005. The stability boundaries for the summation resonance of first two modes and the first and second principal resonances in plane μ–γ_1 for different effects of rotary inertia are shown in Fig. 6. The solid lines denote k_2=0.0017, the dashed lines denote k_2=0.003, and the dotted lines denote the coefficient is k_2=0.0042. In the summation resonance and the second principal resonance, the larger rotary inertia leads the larger instability threshold, while in the first principal resonance, the changing tendency is opposite.

Consider an axially moving viscoelastic Timoshenko beam with k_1=71.28, k_2=0.006575 k_f^2, γ_0=2, and η=0.0005. The stability boundaries for the summation resonance of first two modes and the first and second principal resonance in plane μ–γ_1 for different bending stiffness are shown in Fig. 7. The solid lines denote k_f=0.6, the dashed lines denote k_f=0.8, and the dotted lines denote k_f=1. The increasing bending stiffness makes the stability boundaries move towards the increasing direction of γ_1 in plane μ–γ_1 and the instability regions become narrow. The stability boundaries in the principal parametric resonances are more sensitive to the stiffness.

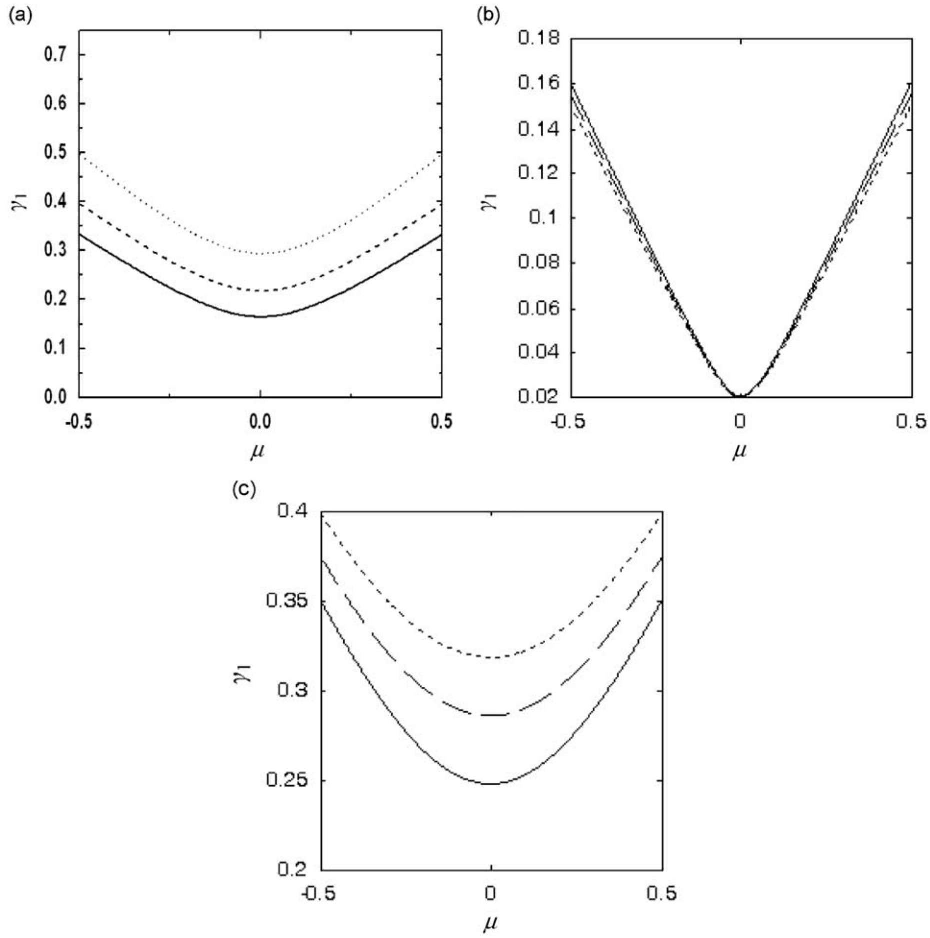

Fig. 6. The effect of rotary inertia on the stability boundaries: (a) the summation resonance, (b) the first principal resonance and (c) the second principal resonance.

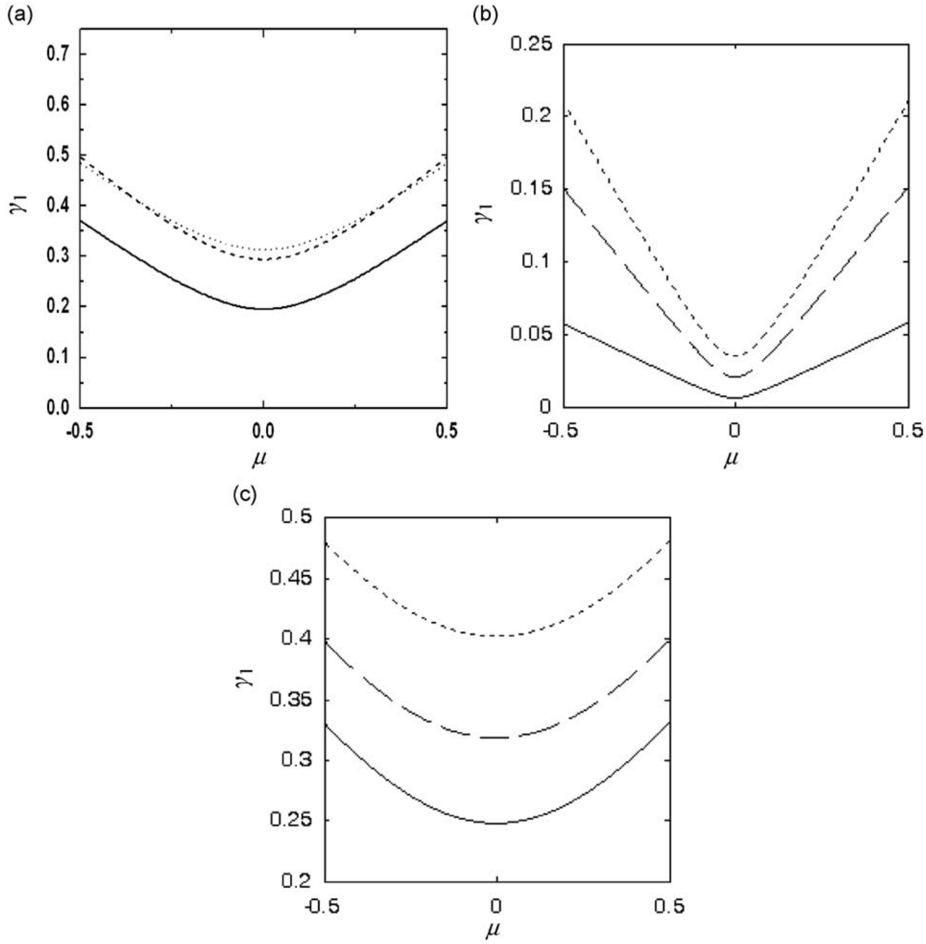

Fig. 7. The effect of k_f on the instability region: (a) the summation resonance, (b) the first principal resonance and (c) the second principal resonance.

In cases above discussed, the stability boundary in the first principal parametric resonance is much lower than that in the second principal parametric resonance and slightly lower than that in the summation resonance of first two modes. Therefore, it can be inferred that the instability occurs more possibly in low-order, especially the first order, principal parametric resonance.

Finally, the results based on the Timoshenko model here are contrasted with those based on the Euler–Bernoulli model [21]. Consider an axially moving viscoelastic Timoshenko beam with k_1=71.28, k_2=0.0042, k_f=0.8, γ_0=2.0, and η=0.0005 and an axially moving viscoelastic Euler beam with k_f=0.8, γ_0=2.0, and η=0.0005. The stability boundaries for the summation resonance of first two modes and the first and second principal resonances in plane μ–γ_1 are shown in Fig. 8. The solid lines denote the Timoshenko model and the dashed lines denote the Euler–Bernoulli model. In the principal (summation) resonances, the Timoshenko model leads to the larger (smaller) instability threshold.

6. Conclusions

This paper is devoted to parametric vibration of an axially accelerating beam. The beam, modeled by the Timoshenko thick beam theory and constituted by the Kelvin model using the material time derivative, moves at an axial speed fluctuating harmonically about a constant mean speed. The governing equations are derived from the physical laws, the constitutive relation, and linear geometrical equations. The method of multiple scales is applied to analyze the governing equation in summation and principal parametric resonances. The Routh–Hurvitz criterion is employed to establish the sufficient and necessary condition of the stability. Based on the approximate analytical results, numerical evaluations demonstrate the following conclusions.

(1) If the axial speed variation frequency approaches the sum of any two natural frequencies, instability may occur for lager enough the axial speed variation amplitude in the summation parametric resonance. The smaller viscosity coefficient, the smaller mean axial speed, the larger shear deformation effect, the smaller rotary inertia effect, and the smaller bending

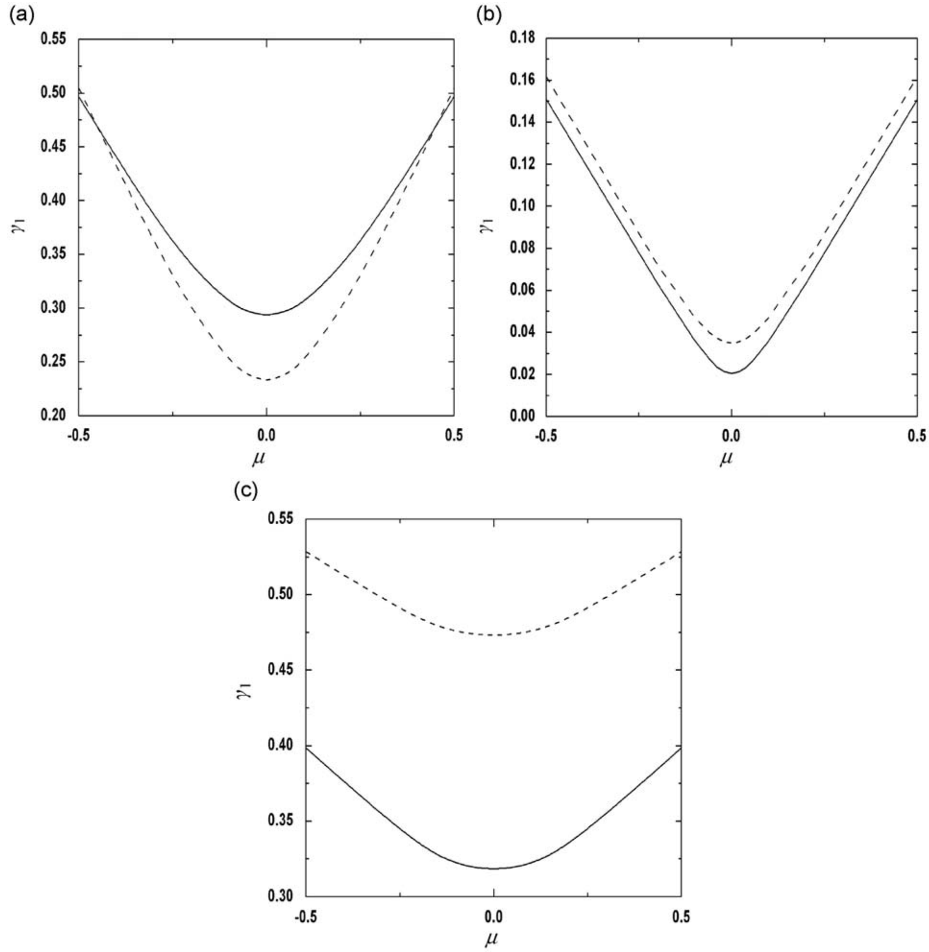

Fig. 8. The comparison of the stability boundaries for different models: (a) the summation resonance, (b) the first principal resonance and (c) the second principal resonance.

stiffness lead to the smaller necessary instability threshold of the speed variation amplitude and the larger required frequency closeness in the summation parametric resonance. The Euler–Bernoulli model leads to the larger instability threshold in the summation resonances than the Timoshenko model.

(2) If the axial speed variation frequency approaches the 2 times of any natural frequency, instability may occur for lager enough the axial speed variation amplitude in the principal parametric resonance. The smaller viscosity coefficient, the larger mean axial speed, the smaller shear deformation effect, and the smaller bending stiffness lead to the smaller necessary instability threshold of the speed variation amplitude and the larger required frequency closeness in the principal parametric resonance. The Euler–Bernoulli model leads to the smaller instability threshold in the principal resonances than the Timoshenko model.

(3) The necessary instability threshold of the speed variation amplitude in the first principal parametric resonance is much smaller than that in the second principal parametric resonance and slightly smaller than that in the summation resonance of first two modes.

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