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An adaptive generalized finite element method applied to free vibration analysis of straight bars and trusses

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A B S T R A C T

In this paper, a first application of an adaptive Generalized Finite Element Method to free longitudinal vibration analysis of straight bars and trusses is presented. The Generalized Finite Element Method is developed by enriching the standard Finite Element Method space, whose basis performs a partition of unity, with knowledge about the differential equation being solved. The enrichment functions used are dependent on the geometric and mechanical properties of the element. The proposed approach converges very fast and is able to approximate the frequency related to any vibration mode. The variational problem of free vibration is formulated and the main aspects of the adaptive Generalized Finite Element Method are presented and discussed. The efficiency and convergence of the proposed method in vibration analysis of uniform and non-uniform straight bars are checked. The application of this technique in a truss is also presented. The frequencies obtained by the adaptive Generalized Finite Element Method are compared with those obtained by the analytical solution, the Composite Element Method and the *h*-version of Finite Element Method.

1. Introduction

The efficient use of natural resources is a current challenge, leading the most recent projects to focus on the optimum structural solution. As a consequence, engineering structures have become taller, slender, light and cheaper. But these features highlight the significance of the dynamic effects which must be assessed with precision. Accurate and efficient numerical procedures must be developed in order to design safe structures.

Among the structural dynamic methods, analytical solutions of the free vibration problems are known only for simple geometries and specific boundary conditions. The exact solutions may provide adequate insight into the physics of the problem and help in checking the accuracy and the efficiency of numerical methods.

Several studies have been dedicated to the problem of exact solutions for free longitudinal vibration of uniform [1] and non-uniform rods [2–4] and many researchers have developed numerical methods for vibration analysis. The Finite Element Method (FEM) [5] and the Composite Element Method (CEM) [6–8] are examples of these methods.

The free vibration analysis by the standard Finite Element Method gives good results for the lowest frequencies but demands great computational cost to work up the accuracy for the higher frequencies. In the Composite Element Method, the basic shape functions are the same as the Finite Element Method but they are enriched by adding analytical functions related to the solution of a simple similar problem with certain special boundary conditions. Thus the Composite Element

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Method can be understood as an enriched and hierarchical Finite Element Method. It is more accurate than the *h*-version of the Finite Element Method for the same number of degrees of freedom when it is employed in the free vibration analysis of bars and beams [9,10]. It is, however, less accurate than the *p*-version of the Finite Element Method for the lowest frequencies [11]. An improved version of the Composite Element Method for free vibration analysis of beams was recently proposed [12]. This new version differs from that in Ref. [8]. The new analytical shape functions are determined by the boundary conditions of each beam analyzed.

As a further attempt to optimize the enrichment technique, the Generalized Finite Element Method (GFEM), which was conceived on the basis of the Partition of Unity Method [13,14], allows the inclusion of a priori knowledge about the fundamental solution of the governing differential equation. This approach ensures accurate local and global approximations.

The Generalized Finite Element Method was independently proposed by Babuska and colleagues [13–15] and by Duarte and Oden [16,17] under the following names: Special Finite Element Method, Generalized Finite Element Method, Finite Element Partition of Unity Method, *hp* Clouds and Cloud-Based *hp* Finite Element Method. In this sense, several meshless methods recently proposed may be considered special cases of this method. Strouboulis et al. [18] defined otherwise the subclass of methods developed from the Partition of Unity Method including *hp* Cloud Method of Oden and Duarte [16,17], the eXtended Finite Element Method (XFEM) of Belytschko and co-workers [19,20], the Generalized Finite Element Method (GFEM) of Strouboulis et al. [21,22], the Method of Finite Spheres of De and Bathe [23], and the Particle-Partition of Unity Method of Griebel and Schweitzer [24].

Recently several studies have indicated the efficiency of the Generalized Finite Element Method and other methods based on the Partition of Unity Method in problems such as analysis of cracks [25–28], dislocations based on interior discontinuities [29], large deformation of solid mechanics [30] and Helmholtz equation [31,32]. In structural dynamics, the Partition of Unity Finite Element Method, along with the interface element technique, was applied by Hazard and Bouillard [33] to numerical analysis of structures equipped with passive damping layers. Hazard and Bouillard formulated a Mindlin plate element applying the partition of unity technique with polynomial enrichment. They adopted a weak penalty method in order to prescribe the essential boundary conditions.

In this paper, a first application of an adaptive Generalized Finite Element Method to free longitudinal vibration analysis of straight bars and trusses is presented.

2. Variational form of the axial free vibration of bars

Consider a straight bar with axial strains, as illustrated in Fig. 1. The basic hypotheses are [34]: (a) The cross sections which are straight and normal to the axis of the bar before deformation remain straight and normal after deformation and (b) the material is elastic, linear and homogeneous. The vibration of the bar is a time dependent problem. The momentum equation that governs this problem is the partial differential equation

$$\rho A(x)\frac{\partial^2 \overline{u}}{\partial t^2} - \frac{\partial}{\partial x}\left(EA(x)\frac{\partial \overline{u}}{\partial x}\right) = p(x,t), \tag{1}$$

where $A(x)$ is the cross section area, E is the Young modulus, ρ is the specific mass, p is the externally applied axial force per unit length and t is the time. The problem of free vibration consists in finding the axial displacement $\overline{u} = \overline{u}(x,t)$ which satisfies Eq. (1) when $p(x,t) = 0$. The solution $\overline{u} = \overline{u}(x,t)$ must satisfy the boundary and initial conditions defined in the problem.

According to Carey and Oden [35], in order to obtain the variational form of a time dependent problem, one should consider the time t as a real parameter and develop a family of variational problems in t. This consists in selecting test functions $v = v(x)$, independent of t, and applying the weighted-residual method. If the Finite Element Method is used to represent the spatial behavior of the solution, one obtains a system of ordinary differential equations in terms of the time dependent degrees of freedom. This approach is called the semi discrete formulation of the problem.

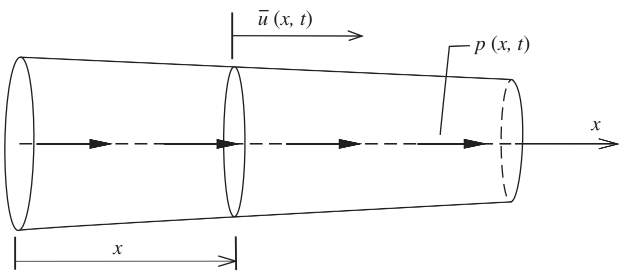

Fig. 1. Straight bar.

Assuming periodic solutions $\overline{u}(x,t) = e^{i\omega t}u(x)$, where ω is the natural frequency, the free vibration of a bar becomes an eigenvalue problem with variational statement: find a pair (λ, u), with $u \in H^1(0,L)$ and $\lambda \in \mathbf{R}$, so that

$$B(u,v) = \lambda F(u,v) \tag{2}$$

for all admissible test functions $v \in H^1(0,L)$, where $\lambda = \omega^2$ and, $B: H^1 \times H^1 \mapsto \mathbf{R}$ and $F: H^1 \times H^1 \mapsto \mathbf{R}$ are bilinear forms obtained from

$$B(u,v) = \int_0^L EA \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x, \tag{3}$$

$$F(u,v) = \int_0^L \rho A u v\,\mathrm{d}x. \tag{4}$$

In numerical methods, finite dimensional subspaces of approximation $H^h \subset H^1(0,L)$ are chosen and the variational statement becomes: find $\lambda_h \in \mathbf{R}$ and $u_h \in H^h(0,L)$ so that

$$B(u_h, v) = \lambda_h F(u_h, v), \forall v \in H^h. \tag{5}$$

The approximated solution $u_h(x)$ can be written, for a discrete system with N degrees of freedom, in the following form:

$$u_h(x) = \sum_{j=1}^{N} u_j \phi_j(x), \tag{6}$$

where ϕ_j are the basis functions of the subspace of approximation H^h and u_j are the corresponding degrees of freedom. It will be shown that different subspaces of approximation are proposed for the Finite Element Method, the Composite Element Method and the Generalized Finite Element Method.

3. Finite element method

The standard Finite Element Method uses polynomials shape functions in the approximated solution which can be expressed generally in matrix form as

$$u_h^e(\xi) = \mathbf{N}^{\mathrm{T}}\mathbf{q}, \tag{7}$$

where $\mathbf{N}$ is the matrix of shape functions and $\mathbf{q}$ is the displacement vector. The polynomial functions may be of any order, the most simple being the linear ones. Taking the uniform bar element (Fig. 2) with one degree of freedom per node, the terms of the approximated solution (Eq. (7)) using linear Lagrangian polynomials as local shape functions are defined in the master element domain as

$$\mathbf{N}^{\mathrm{T}} = [1-\xi \quad \xi], \tag{8}$$

$$\mathbf{q}^{\mathbf{T}} = [u_1 \quad u_2], \tag{9}$$

$$\xi = \frac{x}{L_e}, \tag{10}$$

where L_e is the element length, and, u_1 and u_2 are the nodal displacements.

4. Composite element method

The standard Finite Element Method shape functions can be enriched by the addition of non-polynomial functions related to the closed form solutions from the classical theory of free vibration analysis.

Weaver Junior and Loh [36] used analytical solutions as shape functions in order to represent lateral displacements in the local vibration analysis of trusses. The same approach was applied by Ganesan and Engels [37] in order to obtain a hierarchical model of finite elements of Euler–Bernoulli beams. Zeng [7,8] developed elements of trusses, Euler–Bernoulli beams and frames using this approach to vibration analysis. In the studies of Zeng [6–8] this technique was called "Composite Element Method" and its formulation is briefly explained here.

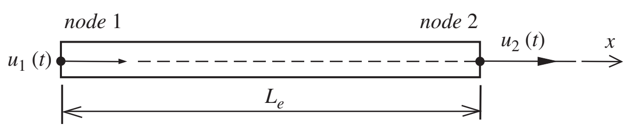

Fig. 2. Uniform bar element.

According to the Composite Element Method, the approximated solution in the master element domain is the sum of two components:

$$u_h^e = u_{\mathrm{FEM}} + u_{\mathrm{CT}}, \tag{11}$$

or in matrix form

$$u_h^e = \mathbf{N}^{\mathrm{T}}\mathbf{q} + \text{Ø}^{\mathrm{T}}\mathbf{c}, \tag{12}$$

where u_{FEM} is the Finite Element Method displacement field function based on nodal degrees of freedom, u_{CT} is the classical theory displacement field function based on field degrees of freedom, **N** is the shape function vector of the standard Finite Element Method, **q** is the nodal displacement vector, Ø is the vector of analytical functions from the classical theory and **c** is the field displacement vector.

The u_{FEM} displacement field component for the bar element (Fig. 2) is usually the linear Finite Element Method solution defined in Eqs. (7)–(10). The classical theory displacement field component u_{CT} of the bar is obtained by the following expressions [7]:

$$\text{Ø}^{\mathrm{T}}(\xi) = [F_1 \;\; F_2 \;\; \ldots \;\; F_n], \tag{13}$$

$$\mathbf{c}^{\mathrm{T}} = [c_1 \;\; c_2 \;\; \ldots \;\; c_n], \tag{14}$$

$$F_r = \sin(r\pi\xi), \quad r = 1, 2, \ldots, n, \tag{15}$$

where c_i are the field degrees of freedom and F_r are the analytical functions obtained from the solution of the free vibration problem of a rod with constrained end displacements.

The new degrees of freedom related to the enrichment shape functions F_r do not have particular physical meaning and they were called c degrees of freedom by Zeng [7]. The enrichment proposed by the Composite Element Method produces hierarchical models and better results than those obtained by h-version of the Finite Element Method [9,10]. The hierarchical version produced by increasing the number of analytical functions F_r is known as c-version.

5. Generalized finite element method

The Generalized Finite Element Method is a Galerkin method whose main goal is the construction of a finite dimensional subspace of approximating functions using local knowledge about the solution that ensures accurate local and global results. The Generalized Finite Element Method was initially named Partition of Unity Finite Element Method by Melenk and Babuska [13], and the local enrichment in the approximation subspace is incorporated by the partition of unity approach. The standard Finite Element Method may be considered as a special case of the Generalized Finite Element Method.

The approximated solution proposed by the Generalized Finite Element Method in the master element domain may be written as the sum of two components:

$$u_h^e = u_{\mathrm{FEM}} + u_{\mathrm{ENRICHED}}, \tag{16}$$

where u_{FEM} is the Finite Element Method component based on nodal degrees of freedom and u_{ENRICHED} is the enriched component by the partition of unity approach based on field degrees of freedom. In this sense, the bar approximated solution on a master element is

$$u_h^e(\xi) = \sum_{i=1}^{2} \eta_i(\xi) u_i + \sum_{i=1}^{2} \eta_i(\xi) \left[\sum_{j=1}^{n_l} (\gamma_{ij}(\xi) a_{ij} + \varphi_{ij}(\xi) b_{ij}) \right], \tag{17}$$

where η_i are the partition of unity functions, γ_{ij} and φ_{ij} are the enrichment functions, n_l is the number of enrichment levels, u_i are the nodal displacements (nodal degrees of freedom) and, a_{ij} and b_{ij} are the field degrees of freedom related to the enrichment functions γ_{ij} and φ_{ij}, respectively. For the linear bar element (Fig. 2), the partition of unity functions are those given by Eq. (8).

In this study, the enrichment functions will be written as

$$\gamma_{1j} = \sin(\beta_j L_e \xi), \tag{18}$$

$$\gamma_{2j} = \sin(\beta_j L_e(\xi - 1)), \tag{19}$$

$$\varphi_{1j} = \cos(\beta_j L_e \xi) - 1, \tag{20}$$

$$\varphi_{2j} = \cos(\beta_j L_e(\xi - 1)) - 1, \tag{21}$$

$$\beta_j = \sqrt{\frac{\rho}{E}}\mu_j, \quad j = 1, 2, \ldots, n_l, \tag{22}$$

where L_e is the element length, E is the Young modulus, ρ is the specific mass and μ_j is a frequency related to the enrichment level j. The coefficients β_j are coupling terms between the spatial and time dependent parts of the governing differential equation. It must be noted that when $\gamma_{ij} = 0$ and $\varphi_{ij} = 0$, or $n_l = 0$, the proposed method corresponds to the standard Finite Element Method. As can be seen, the enriched component of the solution, u_{ENRICHED}, is different from that of the Composite Element Method, u_{CT}.

The enrichment functions (Eqs. (18)–(21)) are those obtained from the space of the fundamental solutions of the differential equation governing the uniform bar free vibration in order to include some knowledge about the differential equation being solved. They incorporate geometric and mechanical properties of the elements because these functions are dependent of the parameter β_j (Eq. (22)) which is function of length, mass density and Young modulus of the elements. These functions were mainly chosen in order to generate shape functions that have compact support in the element domain leading to global solution continuity. In this sense, the introduction of boundary conditions follows the standard finite element procedure.

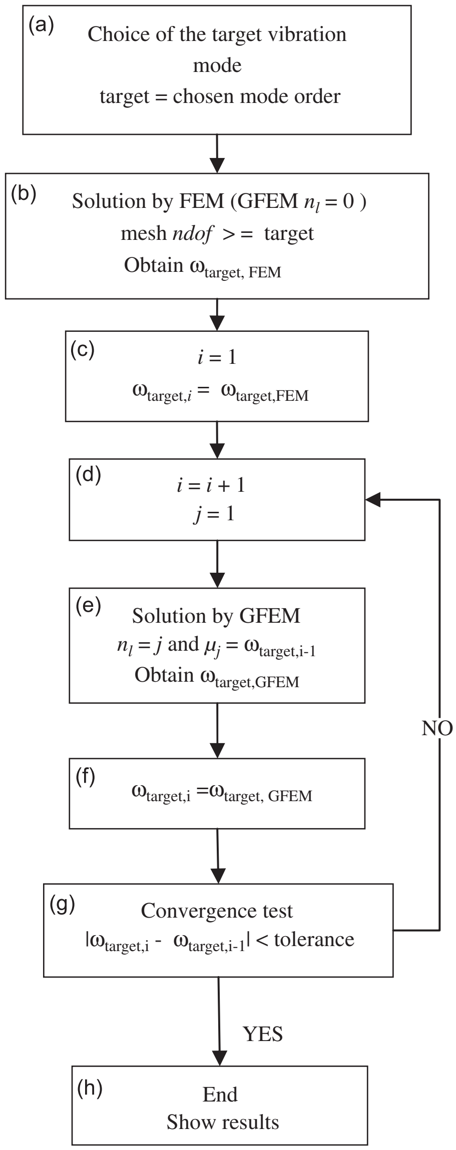

Fig. 3. Flowchart of the Adaptive Generalized Finite Element Method (GFEM).

The enrichment functions, γ_{ij} and φ_{ij} (Eqs. (18)–(21)), and their degrees of freedom, a_{ij} and b_{ij} (Eq. (17)), are associated to the elements. The nodal degrees of freedom u_i are associated to the nodes as in the standard Finite Element Method. In this approach, the degrees of freedom a_{ij} and b_{ij} (Eq. (17)) do not have particular physical meaning.

The idea behind the adaptive Generalized Finite Element Method is similar to the Rayleigh's quotient to evaluate a specific eigenvalue. The Rayleigh's quotient is a scalar λ_r given by [38,39]

$$\lambda_r = \frac{\mathbf{u}_r^{\mathrm{T}}\mathbf{K}\mathbf{u}_r}{\mathbf{u}_r^{\mathrm{T}}\mathbf{M}\mathbf{u}_r}, \tag{23}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices of the analyzed system, respectively. It is proven that if the arbitrary vector $\mathbf{u_r}$ coincides with one of the system eigenvectors, then the quotient λ_r reduces to the associated eigenvalue. This quotient has stationary values in the neighborhood of the system eigenvectors. In this sense, the Rayleigh's quotient can be applied iteratively to improve an eigenvalue from an initial trial eigenvector.

The adaptive Generalized Finite Element Method is an iterative approach whose main goal is to increase the accuracy of the frequency (eigenvalue) related to a chosen vibration mode with order denoted by "target order". The flowchart with blocks A to H presented in Fig. 3 represents the adaptive process. In this flowchart, ω_{target} corresponds to the frequency related to the target mode. The first step of the adaptive Generalized Finite Element process (blocks A to C) consists in obtaining an approximation of the target frequency by the standard Finite Element Method (Generalized Finite Element Method with n_l=0) with a coarse mesh. The finite element mesh used in the analysis has to be as coarse as is necessary to capture a first approximation of the target frequency. The subsequent steps (blocks D to G) consist in applying the Generalized Finite Element Method with just one enrichment level (n_l=1) to the same finite element mesh assuming the frequency μ_j (j=1, blocks D and E) of the enrichment functions (Eqs. (18)–(21)) as the target frequency obtained in the last step. Thus, no mesh refinement is necessary along the iterative process.

Both the standard Finite Element Method and the adaptive Generalized Finite Element Method allow as many frequencies as the total number of degrees of freedom to be obtained. However, in this approach just the precision of the target frequency is effectively improved by the iterative process. The other frequencies present errors similar to those obtained by the standard Finite Element Method with the same mesh. In order to improve the precision of another frequency, it is necessary to perform a new analysis by the adaptive Generalized Finite Element Method, taking this new one as the target frequency. Few steps are necessary for the method to converge, for each target frequency, and the number of degrees of freedom is smaller than those of the standard Finite Element Method to achieve a similar precision, resulting in an appreciative global performance.

An alternative p adaptive scheme arises by increasing the number of enrichment levels (n_l) in each step, but it will be left for future study.

6. Applications

The present adaptive method is applied to problems with known analytical solutions such as uniform and non-uniform bars. A simple practical application in truss vibration analysis is also performed. The applications presented below are very simple for two main reasons: the incipient state of the research, and because the revised literature offers an extensive basis of comparison only on simple applications. The examples are useful to compare the Finite Element Method, the Composite Element Method and the Generalized Finite Element Method performances. Once the proposed method is verified, it can be applied in practical situations.

These examples are solved by the h-version of the Finite Element Method with a regular mesh, the c-version of the Composite Element Method and the adaptive Generalized Finite Element Method, in order to compare the accuracy of their results. The number of degrees of freedom (ndof) considered in each analysis is the total number of effective degrees of freedom after introduction of boundary conditions. The analyzed cases are hypothetical and the dimensions are generic, thus the units are omitted.

The adaptive Generalized Finite Element Method was coded in the software Maple. The integrations and eigenvalue problems were solved using intrinsic Maple functions. The eigenvalues were computed by the QR method.

As an intrinsic imposition of the adaptive method, each target frequency is obtained by different iterative analyses. The mesh used in each analysis is the coarser one, that is, just as coarse as is necessary to capture a first approximation of the

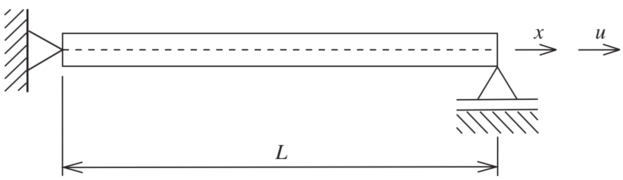

Fig. 4. Uniform fixed-free bar.

target frequency. The reader should be advised that the frequencies presented, other than the target one, are just to emphasize the ability of the adaptive method to achieve one specific (target) frequency each time it is run.

6.1. Uniform fixed-free bar

The free axial vibration of a fixed-free bar (Fig. 4) with length L, elasticity modulus E, mass density ρ and uniform cross section area A, has exact natural frequencies (ω_r) given by [34]

$$\omega_r = \frac{(2r-1)\pi}{2L}\sqrt{\frac{E}{\rho}}, r = 1, 2, \ldots \,. \tag{24}$$

In order to compare the exact solution with the approximated ones, in this example a non-dimensional eigenvalue χ_r given by

$$\chi_r = \frac{\rho L^2 \omega_r^2}{E}, \tag{25}$$

will be used.

Four different adaptive Generalized Finite Element analyses are performed in order to obtain the first four frequencies. The behavior of the relative error in each analysis is presented in Fig. 5. In order to capture a first approximation of the target vibration frequency, for the first frequency (Fig. 5a), the finite element mesh must have at least one element (one effective degree of freedom), for the second frequency (Fig. 5b), it must have at least two elements (two effective degrees of freedom), and so on.

Table 1 presents the relative errors obtained by the numerical methods. The Finite Element Method (FEM) solution is obtained with 100 elements, that is, 100 effective degrees of freedom. The Composite Element Method (CEM) solution is obtained with just one element and 15 enrichment functions that correspond to one nodal degree of freedom and 15 field degrees of freedom resulting in 16 effective degrees of freedom. The analyses by the adaptive Generalized Finite Element Method (GFEM) have no more than 13 degrees of freedom in each iteration. For example, the fourth frequency is obtained taking 4 degrees of freedom in the first iteration and 13 degrees of freedom in the two subsequent ones.

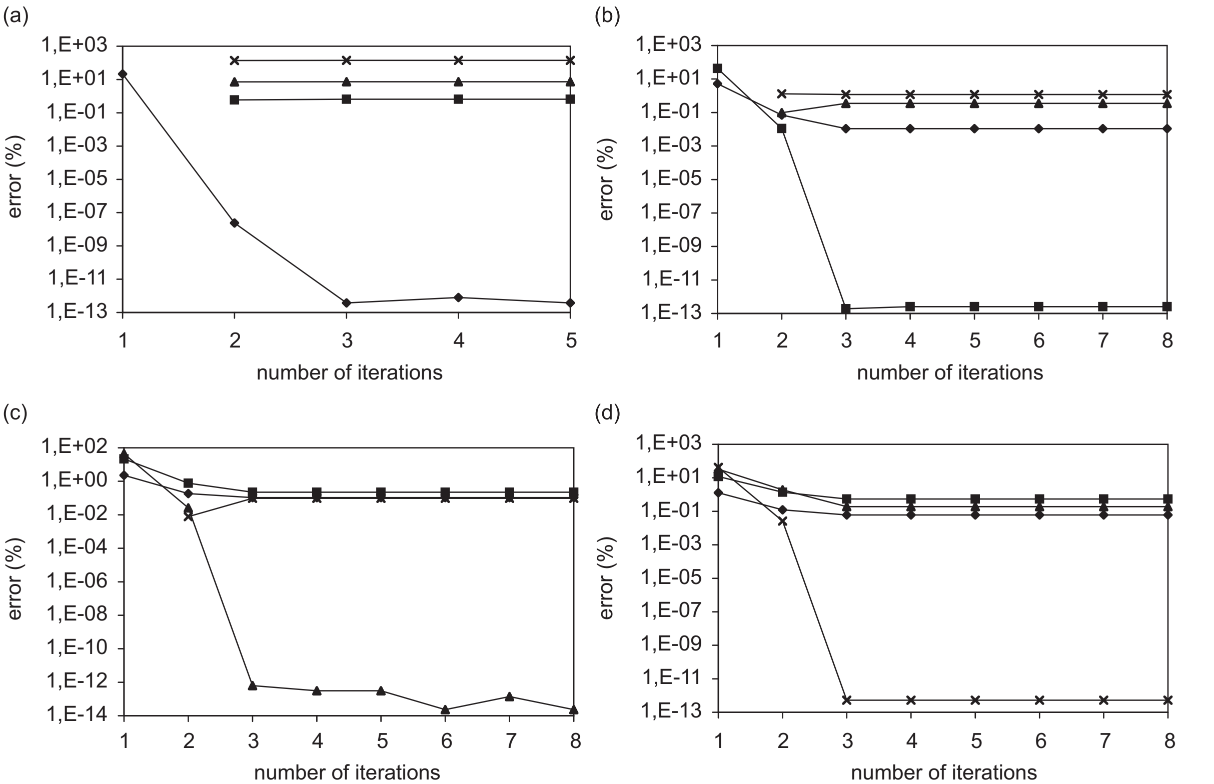

Fig. 5. Error in the Adaptive Generalized Finite Element analyses of fixed-free uniform bar. (a) Analysis 1: 1st target frequency—1 element mesh, (b) analysis 2: 2nd target frequency—2 element mesh, (c) analysis 3: 3rd target frequency—3 element mesh and (d) analysis 4: 4th target frequency—4 element mesh. —◆—: 1st frequency; —■—: 2nd frequency; —▲—: 3rd frequency; and —×—: 4th frequency.

Table 1
Results to free vibration of uniform fixed-free bar.

Frequency	Exact solution	FEM (100e) ndof[a]=100	CEM (1e 15c) ndof[a]=16	Adaptive GFEM (after 3 iterations)	
r	Eigenvalue χ_r	Error (%)	Error (%)	Error (%)	ndof in iterations[b]
1	2.46740	2.056e−3	8.936e−4	3.780e−13	1 × 1 dof+2 × 4 dof
2	22.20661	1.851e−2	8.188e−3	1.920e−13	1 × 2 dof+2 × 7 dof
3	61.68503	5.141e−2	2.299e−2	6.335e−13	1 × 3 dof+2 × 10 dof
4	120.90265	1.008e−1	4.579e−2	5.289e−13	1 × 4 dof+2 × 13 dof

[a] ndof=effective number of degrees of freedom after introduction of boundary conditions.
[b] $1 \times n$ dof+$2 \times m$ dof indicates first iteration (FEM) with n degrees of freedom and the other two iterations (GFEM) with m degrees of freedom in each analysis.

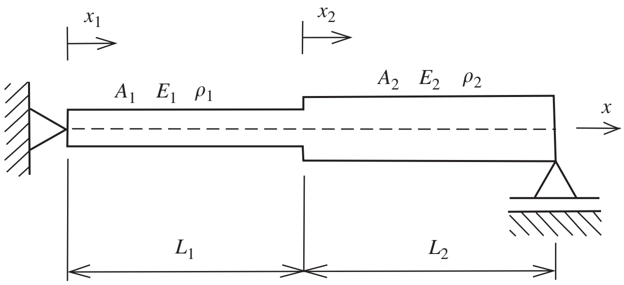

Fig. 6. Stepped fixed-free bar composed of two materials.

For the uniform fixed-free bar, one notes that the adaptive Generalized Finite Element Method reaches greater precision than the h-version of Finite Element Method and the c-version of Composite Element Method. The adaptive process converges rapidly requiring three iterations in order to achieve each target frequency with precision of the 10^{-13} order. The first four frequencies with similar precision are reached by the standard Finite Element Method software Ansys with 410 truss elements (LINK8) that corresponds to 410 effective degrees of freedom after introduction of boundary conditions. Results have shown that for each adaptive cycle run (blocks D to G in Fig. 3) the accuracy of the target frequency considerably improves, reaching rapidly a narrow interval of convergence; however, they have also shown no effect on the accuracy of the other frequencies. For example, in analysis 4 (Fig. 5d) the target frequency (4th frequency) achieves a precision of 10^{-13} order after three iterations while the first three frequencies achieve a precision of 10^{-1} order.

6.2. Fixed-free stepped bar with variable materials and transversal sections

A stepped fixed-free bar composed of two different materials (Fig. 6), with lengths L_1=L_2, elasticity moduli E_2=$2E_1$, cross section areas A_2=$2A_1$, and mass densities ρ_2=$8\rho_1$, is analyzed as follows:

The exact natural frequencies (ω_r) are obtained solving the characteristic equation

$$\frac{E_1A_1\kappa_1\cos(\kappa_1L_1)\cos(\alpha\kappa_1L_2)-\alpha E_2A_2\kappa_1\sin(\kappa_1L_1)\sin(\alpha\kappa_1L_2)}{\cos(\alpha\kappa_1L_2)}=0, \tag{26}$$

where

$$\alpha=\sqrt{\frac{\rho_2E_1}{\rho_1E_2}}, \tag{27}$$

$$\kappa_1=\omega_r\sqrt{\frac{\rho_1}{E_1}}, \tag{28}$$

$$\kappa_2=\omega_r\sqrt{\frac{\rho_2}{E_2}}. \tag{29}$$

This problem is analyzed by considering the h-version of Finite Element Method with a regular mesh and the c-version of Composite Element Method with a two element mesh. Six adaptive Generalized Finite Element analyses are performed in order to obtain each of the first six frequencies. The behavior of the relative error in each one of the first four analyses is

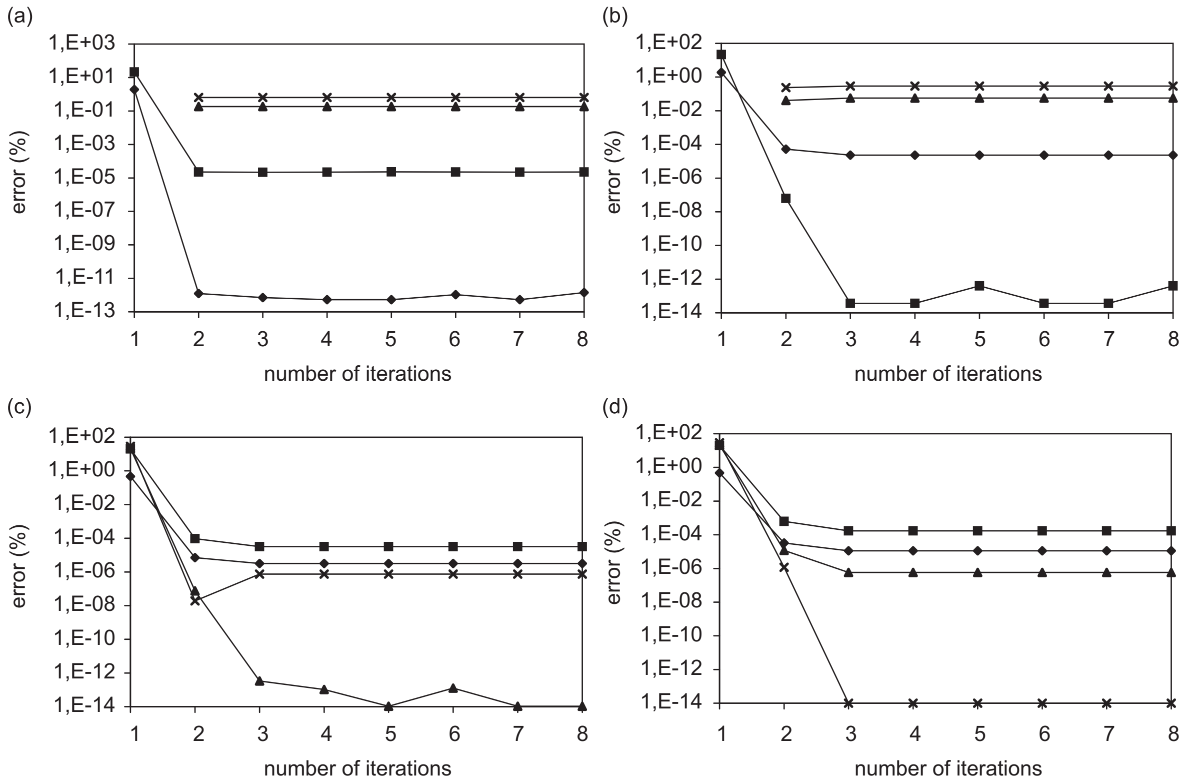

Fig. 7. Error in the Adaptive Generalized Finite Element analyses of two materials fixed-free uniform bar. (a) analysis 1: 1st target frequency—2 element mesh, (b) analysis 2: 2nd target frequency—2 element mesh, (c) analysis 3: 3rd target frequency—4 element mesh and (d) analysis 4: 4th target frequency—4 element mesh. —◆—: 1st frequency; —■—: 2nd frequency; —▲—: 3rd frequency; and —×—: 4th frequency.

Table 2
Results to free vibration of two materials stepped fixed-free bar.

Frequency	Exact solution	FEM (16e) ndof[a]=16	Ansys ndof[a]=1000	CEM (2e 8c) ndof[a]=18	Adaptive GFEM (after 3 iterations)	
r	χ_r	Error (%)	Error (%)	Error (%)	Error (%)	ndof in iterations[b]
1	0.056616	2.911e−2	2.608e−3	5.596e−4	7.109e−13	1×2 dof+2×10 dof
2	2.467401	1.233	3.780e−13	3.465e−2	3.648e−14	1×2 dof+2×10 dof
3	8.431192	2.636	2.137e−4	8.587e−2	3.371e−13	1×4 dof+2×20 dof
4	11.421249	4.146	1.836e−4	1.185e−1	1.000e−14	1×4 dof+2×20 dof
5	22.206610	11.36	2.667e−3	3.328e−1	1.000e−14	1×6 dof+2×30 dof
6	36.544977	9.828	4.055e−3	4.106e−1	8.166e−13	1×6 dof+2×30 dof

[a] ndof=effective number of degrees of freedom after introduction of boundary conditions.
[b] $1\times n$ dof+$2\times m$ dof indicates first iteration (FEM) with n degrees of freedom and the other two iterations (GFEM) with m degrees of freedom.

presented in Fig. 7. In order to determine the first and the second frequencies (Figs. 7a and b), the finite element mesh must have at least two elements, for the third and fourth frequencies (Figs. 7c and d), it must have at least four elements, and so on.

Table 2 presents the relative error for the first six non-dimensional eigenvalues $\chi_r=(\kappa_1 L_1)^2$ obtained by the numerical methods. The Finite Element Method (FEM) solution is obtained with 16 elements, that is, 16 effective degrees of freedom. The problem is also solved by the Finite Element Method software Ansys with a 1000 truss element (LINK8) that corresponds to 1000 effective degrees of freedom. The Composite Element Method (CEM) solution is obtained with two elements and 8 enrichment functions that correspond to two nodal degrees of freedom and 16 field degrees of freedom. Each analysis by the adaptive Generalized Finite Element Method (GFEM) has no more than 30 degrees of freedom in each iteration.

One notes that the adaptive Generalized Finite Element Method presents greater precision for the first six natural frequencies than the h-version of the Finite Element Method and the c-version of the Composite Element Method. Like in the first application, the target frequency convergence rate in all adaptive analyses is very high and the results have shown that the process allows the accuracy of the target frequency to be improved without significant effect on the precision of the other frequencies.

6.3. Fixed-fixed bar with sinusoidal variation of cross section area

In this topic, the longitudinal free vibration of a fixed-fixed non-uniform bar with sinusoidal variation of cross section area, length L, elasticity modulus E and mass density ρ is analyzed. The boundary conditions are $\overline{u}(0,t)=0$ and $\overline{u}(L,t)=0$, and the cross section area varies as

$$A(x) = A_0 \sin^2\left(\frac{x}{L}+1\right), \tag{30}$$

where A_0 is a reference cross section area.

Kumar and Sujith [3] have presented exact analytical solutions for longitudinal free vibration of bars with sinusoidal and polynomial area variations. The equation of motion of axial vibration is reduced to analytically solvable differential equations using appropriate transformations.

This problem is analyzed by the h-version of Finite Element Method and the adaptive Generalized Finite Element Method. Four adaptive analyses are performed in order to obtain each of the first four frequencies. The behavior of the relative error in each analysis is presented in Fig. 8.

Table 3 presents the first four non-dimensional eigenvalues ($\beta_r = \omega_r L\sqrt{\rho/E}$) and their relative errors obtained by these methods. The Finite Element Method (FEM) solution is obtained with 100 elements, that is, 99 effective degrees of freedom after introduction of boundary conditions. The analyses by the adaptive Generalized Finite Element Method (GFEM) have maximum number of degrees of freedom in each iteration ranging from 9 to 24.

One notes that the adaptive Generalized Finite Element Method reaches more precise values than the Finite Element Method with even less degrees of freedom. The errors are greater than those from the uniform bars because the exact

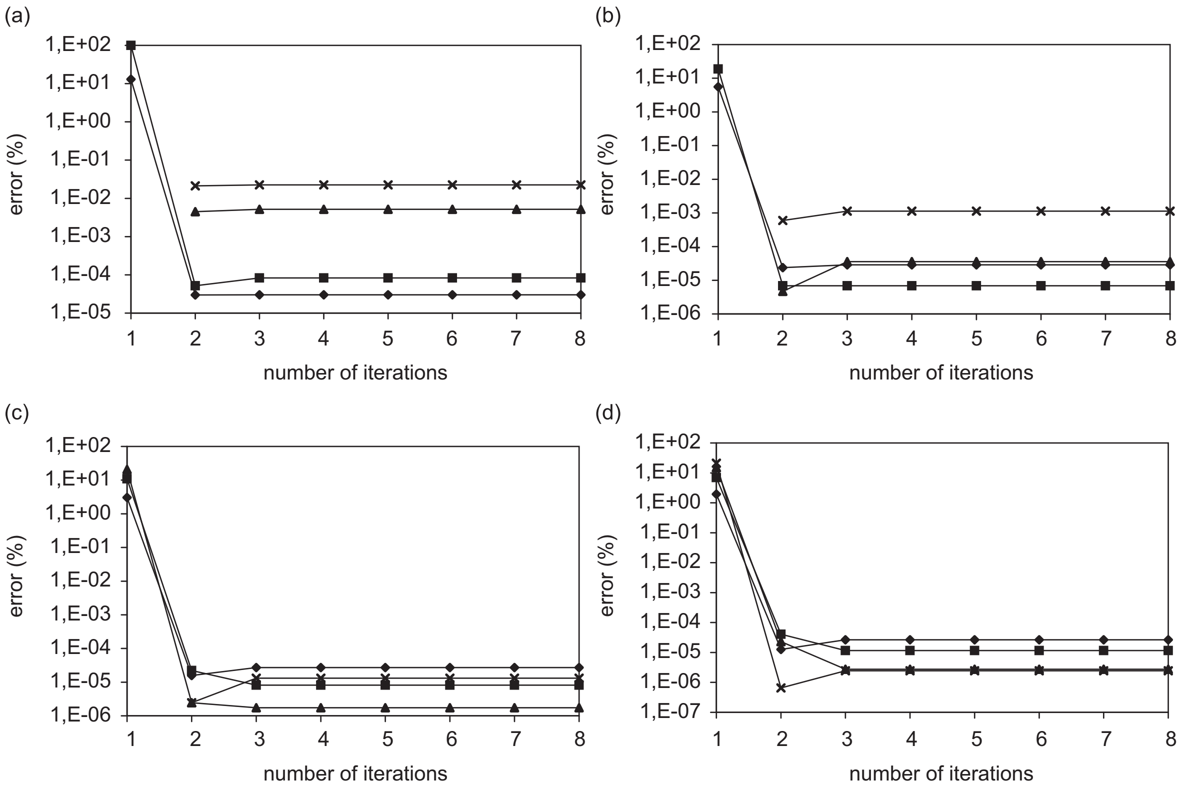

Fig. 8. Error in the Adaptive Generalized Finite Element analyses of fixed-fixed bar with sinusoidal area variation. (a) Analysis 1: 1st target frequency—2 element mesh, (b) analysis 2: 2nd target frequency—3 element mesh, (c) analysis 3: 3rd target frequency—4 element mesh and (d) analysis 4: 4th target frequency—5 element mesh. —◆—: 1st frequency; —■—: 2nd frequency; —▲—: 3rd frequency; and —×—: 4th frequency.

Table 3
Results to free vibration of fixed-fixed bar with sinusoidal variation of area.

Frequency	Exact solution[a]	FEM (100e) ndof[b]=99		Adaptive GFEM (after 3 iterations)		
r	β_r	β_r	Error (%)	β_r	Error (%)	ndof in iterations[c]
1	2.978189	2.978330	4.737e−3	2.978188	2.997e−5	1×1 dof+2×9 dof
2	6.203097	6.204151	1.699e−2	6.203097	6.871e−6	1×2 dof+2×14 dof
3	9.371576	9.375094	3.753e−2	9.371576	1.731e−6	1×3 dof+2×19 dof
4	12.526519	12.534827	6.632e−2	12.526519	2.441e−6	1×4 dof+2×24 dof

[a] Results from Ref. [3].
[b] ndof=effective number of degrees of freedom after introduction of boundary conditions;
[c] $1\times n$ dof+$2\times m$ dof indicates first iteration (FEM) with n degrees of freedom and the other two iterations (GFEM) with m degrees of freedom.

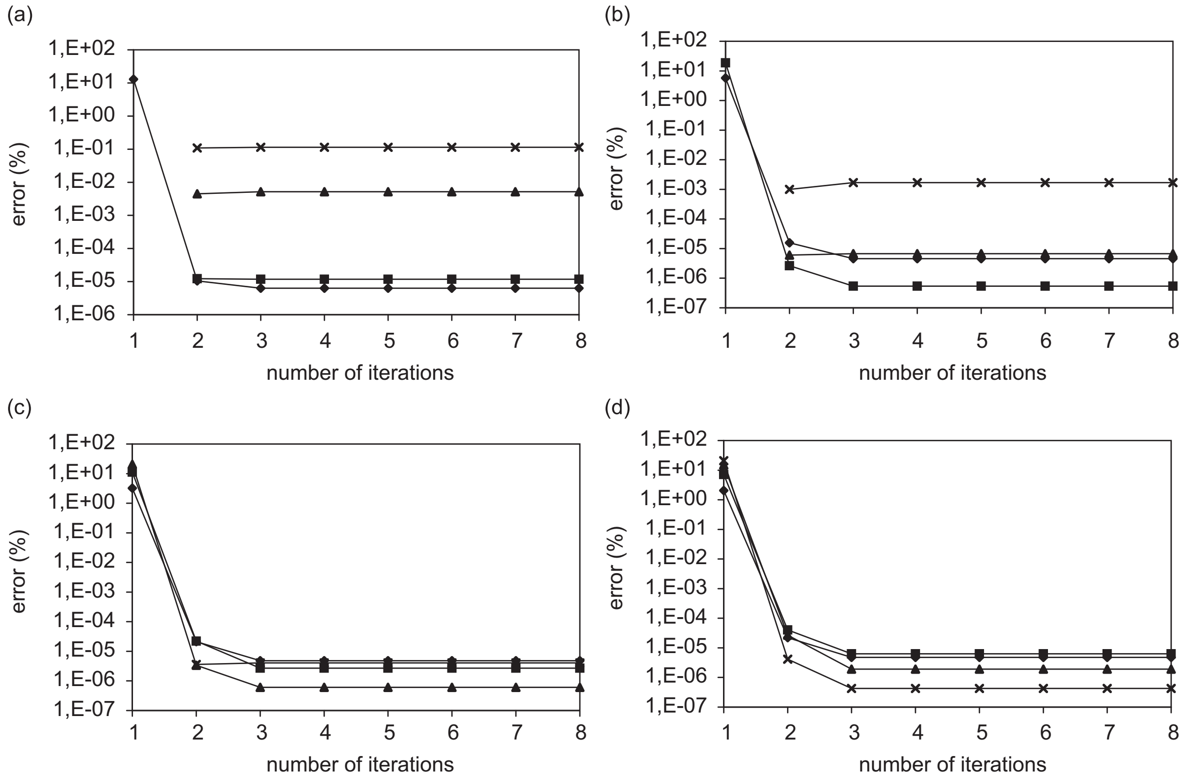

Fig. 9. Error in the Adaptive Generalized Finite Element analyses of fixed-fixed bar with polynomial area variation. (a) Analysis 1: 1st target frequency—2 element mesh, (b) analysis 2: 2nd target frequency—3 element mesh, (c) analysis 3: 3rd target frequency—4 element mesh and (d) analysis 4: 4th target frequency—5 element mesh. —◆—: 1st frequency; —■—: 2nd frequency; —▲—: 3rd frequency; and —×—: 4th frequency.

vibration modes of non-uniform bars cannot be exactly represented by the trigonometric functions used as enrichment functions; however, the precision is acceptable for engineering applications. Each analysis by the adaptive Generalized Finite Element Method is able to refine the target frequency until the exhaustion of the approximation capacity of the enriched subspace. Thus the precision can be improved by using a more refined mesh in the adaptive process.

6.4. Fixed-fixed bar with polynomial variation of cross section area

In this topic, the longitudinal free vibration of a fixed-fixed non-uniform bar with polynomial variation of cross section area, length L, elasticity modulus E and mass density ρ is analyzed. The boundary conditions are $\overline{u}(0,t)=0$ and $\overline{u}(L,t)=0$, and the cross section area varies as

$$A(x)=A_0\left(\frac{x}{L}+1\right)^4, \tag{31}$$

where A_0 is the cross section area at x=0.

Kumar and Sujith [3] have presented the general analytical solutions for longitudinal vibration of bars with polynomial area variation. The characteristic equation for free vibration of a fixed-fixed bar with the 4th order polynomial area variation is

$$J_{-3/2}(\beta_r)J_{3/2}(2\beta_r) - J_{3/2}(\beta_r)J_{-3/2}(2\beta_r) = 0, \tag{32}$$

where

$$\beta_r = \omega_r L\sqrt{\frac{\rho}{E}}, \tag{33}$$

$J_{3/2}$ and $J_{-3/2}$ are Bessel functions of the first kind of order $\frac{3}{2}$ and $-\frac{3}{2}$.

The free vibration of a bar with this area variation is analyzed by the h-version of Finite Element Method and the adaptive Generalized Finite Element Method. Again four adaptive analyses are performed in order to obtain each of the first four frequencies. The behavior of the relative error in each analysis is presented in Fig. 9.

Table 4 presents a comparison of the relative error of the first four non-dimensional eigenvalues (β_r) between both numerical approaches. The Finite Element Method (FEM) solution is obtained with 100 elements, that is, 99 effective degrees of freedom. The analyses by the adaptive Generalized Finite Element Method (GFEM) have no more than 24 degrees of freedom in each iteration.

Here, the same comments on the sinusoidal example are due. However, even for problems where the exact solution is not represented by trigonometric functions, the results from the adaptive method are accurate.

6.5. Fifteen bar truss

The free axial vibration of a truss formed by 15 straight bars is analyzed to illustrate the application of the adaptive Generalized Finite Element Method in structures formed by bars. This problem was proposed by Zeng [7] in order to check the Composite Element Method. The geometry of the truss is presented in Fig. 10. All bars in the truss have cross section area A=0.001 m^2, mass density ρ=8000 kg m^{-3} and elasticity modulus E=2.1 $\times$ 10^{11} N m^{-2}.

All analyses used 15 element mesh, the minimum number of C^0 type elements necessary to represent the truss geometry. The Finite Element Method and the c-version of Composite Element Method are applied. Fourteen analyses by the adaptive Generalized Finite Element Method are performed in order to improve the accuracy of each of the first 14 natural frequencies. The frequencies obtained by each analysis are presented in Table 5. The Finite Element Method (FEM) solution is obtained with 15 elements, that is, 14 effective degrees of freedom after introduction of boundary conditions. The c-version of the Composite Element Method (CEM) solution is obtained with 15 elements and 1, 2, 4 and 6 enrichment functions that correspond to 14 nodal degrees of freedom and 15, 30, 60 and 90 field degrees of freedom, respectively. All analyses by the adaptive Generalized Finite Element Method (GFEM) have 14 degrees of freedom in the first iteration and 74 degrees of freedom in the following two.

Table 4
Results to free vibration of fixed-fixed bar with polynomial variation of area.

Frequency	Exact solution	FEM (100e) ndof[a]=99		Adaptive GFEM (after 3 iterations)		
r	β_r	β_r	Error (%)	β_r	Error (%)	ndof in iterations[b]
1	3.286007	3.286175	5.130e−3	3.286007	6.330e−6	1 × 1 dof+2 × 9 dof
2	6.360678	6.361800	1.763e−2	6.360678	5.409e−7	1 × 2 dof+2 × 14 dof
3	9.477196	9.480820	3.823e−2	9.477196	6.061e−7	1 × 3 dof+2 × 19 dof
4	12.605890	12.614341	6.704e−2	12.605890	4.269e−7	1 × 4 dof+2 × 24 dof

[a] ndof=effective number of degrees of freedom after introduction of boundary conditions.
[b] 1 × n dof+2 × m dof indicates first iteration (FEM) with n degrees of freedom and the other two iterations (GFEM) with m degrees of freedom.

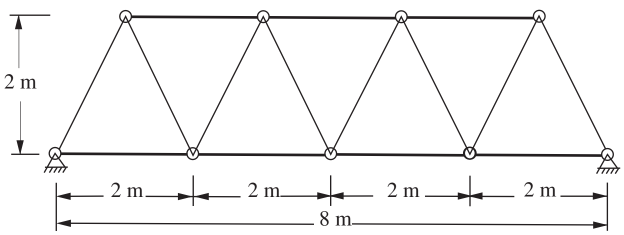

Fig. 10. Fifteen bar truss.

Table 5
Natural frequencies of the 15 bar truss.

r	FEM (15e) ndof[a]=14 ω_r (rad s^{-1})	CEM[b] (15e 1c) ndof[a]=29 ω_r (rad s^{-1})	CEM[b] (15e 2c) ndof[a]=44 ω_r (rad s^{-1})	CEM (15e 4c) ndof[a]=74 ω_r (rad s^{-1})	CEM (15e 6c) ndof[a]=104 ω_r (rad s^{-1})	Adap. GFEM (15e 3i) ndof[a]=74[c] ω_r (rad s^{-1})
1	682.272384	679.824639	679.821793	679.791992	679.788127	679.786383
2	1149.296348	1139.376074	1139.341796	1139.223267	1139.207727	1139.200586
3	1612.350232	1582.388830	1582.181618	1581.837705	1581.792424	1581.771367
4	2519.866118	2411.837021	2410.250884	2409.132928	2408.982871	2408.911573
5	2715.759047	2604.117985	2601.848419	2600.647590	2600.484076	2600.405466
6	2968.220775	2818.301138	2815.435334	2813.921841	2813.716162	2813.617278
7	3573.361130	3300.489329	3293.258728	3290.816121	3290.475697	3290.308232
8	4207.781031	3824.741153	3811.365219	3808.112160	3807.646059	3807.411395
9	5134.736048	4507.646123	4480.524910	4476.024121	4475.351429	4475.001704
10	5399.565563	4746.383128	4707.913787	4702.826748	4702.039754	4701.621014
11	7163.278724	6189.385029	6069.269438	6060.720374	6059.228964	6058.376768
12	7471.073109	6493.482868	6341.616890	6331.997689	6330.275690	6329.278684
13	7586.074215	6623.729777	6455.516182	6445.563179	6443.724911	6442.644839
14	8462.586195	7386.073478	7381.063755	7380.481418	7380.395797	7380.351301

[a] ndof=effective number of degrees of freedom after introduction of boundary conditions.
[b] Results from Ref. [7].
[c] First iteration (FEM) with 14 degrees of freedom and the other two iterations (GFEM) with 74 degrees of freedom.

This special case is not well suited to the h-version of Finite Element Method since it demands the adoption of restraints at each internal bar node in order to avoid modeling instability. The Finite Element Method analysis of this truss can be improved by applying bar elements of higher order (p-version) or beam elements. The results show that both the c-version of Composite Element Method and the adaptive Generalized Finite Element Method converge to the same frequencies.

7. Conclusion

The main contribution of the present study consisted in proposing an adaptive Generalized Finite Element Method for vibration analysis. This study performed a preliminary formulation of free vibration analysis of straight bars and trusses by the proposed method. The Generalized Finite Element Method results were compared with those obtained by the h-version of Finite Element Method and the c-version of the Composite Element Method.

In this adaptive Generalized Finite Element Method, trigonometric enrichment functions depending on geometric and mechanical properties of the elements were added to the linear Finite Element Method shape functions by the partition of unity approach. This technique allows an accurate adaptive process that converges very fast and is able to refine the frequency related to a specific vibration mode. In addition the enrichment functions are easily obtained and the introduction of boundary conditions follows the standard finite element procedure.

The results have shown that the adaptive Generalized Finite Element Method achieves narrower precision than the c-version of Composite Element Method and the h-version of Finite Element Method in free longitudinal vibration analysis of uniform and non-uniform straight bars for the same number of degrees of freedom. It has been observed that even for problems where the exact solutions are not represented by trigonometric functions, like non-uniform bars, the results from the adaptive method are accurate with relatively few degrees of freedom. This method has been applied in free vibration analysis of trusses showing results very close to those of the Composite Element Method.

It is worth remarking that the adaptive Generalized Finite Element Method is an iterative process that requires less computational effort than it appears. Instead of dealing with a matrix of $(n+2m)\times(n+2m)$ dimension, the problem is divided in one of $n\times n$, and two of $2m\times 2m$ matrices, significantly reducing the amount of arithmetic. Since the adaptive approach requires much less degrees of freedom than the standard Finite Element Method, the adaptive process spends less computational effort in order to obtain similar accuracy.

The adaptive Generalized Finite Element Method has shown to be efficient in the analysis of longitudinal vibration of bars and has indicated that it can be applied even for a coarse discretization scheme in complex practical problems. Future research will extend this adaptive method to other structural elements like beams, plates and shells.

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