Journal of Sound and Vibration 329 (2010) 721–732

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

On the forced vibrations of an oscillator with a periodically time-varying mass

W.T. van Horssen, O.V. Pischanskyy *, J.L.A. Dubbeldam

Delft Institute of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

ARTICLE INFO

Article history:
Received 14 January 2009
Received in revised form
22 July 2009
Accepted 2 October 2009
Handling Editor: A.V. Metrikine
Available online 25 October 2009

ABSTRACT

In this paper the forced vibrations of a linear, single degree of freedom oscillator (sdofo) with a time-varying mass will be studied. The forced vibrations are due to small masses which are periodically hitting and leaving the oscillator with different velocities. Since these small masses stay for some time on the oscillator surface the effective mass of the oscillator will periodically vary in time. Additionally, an external harmonic force will be applied to the oscillator with a time-varying mass. Not only solutions of the oscillator equation will be constructed, but also stability properties for the forced vibrations will be presented for various parameter values.

1. Introduction

Systems with time-varying masses frequently occur in practice. Examples of such systems can be found in robotics, rotating crankshafts, conveyor systems, excavators, cranes, biomechanics and in fluid–structure interaction problems [1,2]. The oscillations of electric transmission lines and cables of cable-stayed bridges with water rivulets on the surface are also examples of time-varying dynamic systems [3]. For these mechanical constructions the 1-mode Galerkin approximation of the continuous model will lead to a sdofo-equation. These sdofos are considered to be representative models for testing numerical methods and for studying forces which are acting on the system [4]. In this paper the forced oscillations of a linear sdofo with a (periodically and stepwise changing) time-varying mass will be studied. The free oscillations have been recently studied in [5].

Consider the oscillations of a sdofo with a linear restoring force and a mass which varies in time according to a periodic stepwise dependence. This model is perhaps the simplest model which describes the process of the vibrations of a cable with rainwater located on it. Part of the raindrops hitting the cylinder (i.e. the cable) will remain on the surface of the cylinder for some time, and will subsequently be blown or shaken off after some time. It will be assumed when mass is added to or separated from the oscillator that the position of the center of the (total) mass of the oscillator is not influenced. The following equation of motion for the sdofo can now be derived (see for instance [1, p. 152]):

$$M\ddot{y} = \dot{M}(w - \dot{y}) - ky + F, \tag{1}$$

where $y = y(t)$ is the displacement of the oscillator (see Fig. 1), $M = M(t)$ is the time-varying mass of the oscillator, $w = w(t)$ is the mean velocity at which masses (i.e. raindrops) are hitting or leaving the oscillator, k is the (positive)

* Corresponding author. Tel.: +31 15 2783613; fax: +31 15 2787295.
E-mail addresses: w.t.vanhorssen@tudelft.nl (W.T. van Horssen), a.peschanskyy@ewi.tudelft.nl (O.V. Pischanskyy), j.l.a.dubbeldam@tudelft.nl (J.L.A. Dubbeldam).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2009.10.001

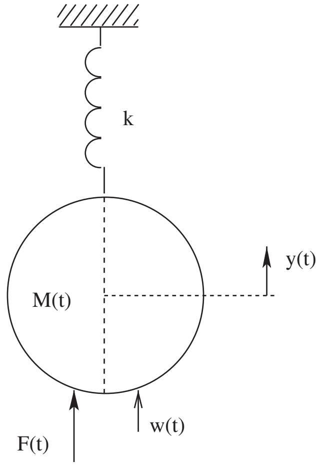

Fig. 1. The single degree of freedom oscillator.

stiffness coefficient in the linear restoring force, $F=F(t)$ or $F=F(t,y,\dot{y})$ is an external force, and the dot denotes differentiation with respect to t. The force F and the velocity w are measured positive in positive y direction (see Fig. 1). In [5] the free vibrations and the stability (for $F\equiv 0$ and $w\equiv 0$ in (1)) of the sdofo have been studied, and in [8] the vibrations and the stability of the sdofo for $F\equiv 0$ and w is different from zero have been considered for the nondegenerate cases. In this paper for the degenerate cases when $F\equiv 0$ and $w\neq 0$, and for some harmonic forcing cases (i.e. F is a harmonic function in t) the oscillations and the stability of the sdofo will be studied in detail. Following [5] it turns out to be convenient to separate the mass $M(t)$ into a time invariant part M_0 and into a time-varying part $m(t)$, that is, $M(t)=M_0-m(t)$, where M_0 is a positive constant, and $M_0-m(t)>0$. Then it follows that (1) can be rewritten in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((M_0-m(t))\frac{\mathrm{d}y}{\mathrm{d}t}\right)+ky=\frac{-\mathrm{d}m}{\mathrm{d}t}w+F. \tag{2}$$

Then, by introducing the time-rescaling $t=\sqrt{M_0/k}\tau$ Eq. (2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left(1-\frac{\tilde{m}(\tau)}{M_0}\right)\frac{\mathrm{d}\tilde{y}(\tau)}{\mathrm{d}\tau}\right)+\tilde{y}(\tau)=\frac{-\tilde{w}(\tau)}{\sqrt{M_0k}}\frac{\mathrm{d}\tilde{m}(\tau)}{\mathrm{d}\tau}+\tilde{F}(\tau), \tag{3}$$

where $\tilde{y}(\tau)=y(\sqrt{M_0/k}\tau)$, $\tilde{m}(\tau)=m(\sqrt{M_0/k}\tau)$, $\tilde{w}(\tau)=w(\sqrt{M_0/k}\tau)$ and $\tilde{F}(\tau)=(1/k)F(\sqrt{M_0/k}\tau)$. In this paper it will be assumed that $h(\tau)=\tilde{m}(\tau)/M_0$ is a periodic step function with $1-h(\tau)>0$, that is,

$$h(\tau)=\begin{cases}\varepsilon & \text{for } 0<\tau<T_0,\\ 0 & \text{for } T_0<\tau<T,\end{cases} \tag{4}$$

and $h(\tau+T)=h(\tau)$, and ε is a constant (in practice usually small) with $0<\varepsilon<1$. It should be observed that ε is defined to be the quotient m/M_0, where m is the added mass at time T_0, and where M_0 is the mass of the oscillator. So, ε can be seen as a measure for the relative mass which is added at time T_0. For convenience the tildes in (3) will be dropped, and a prime will be introduced to denote differentiation with respect to τ, yielding

$$((1-h(\tau))y'(\tau))'+y(\tau)=\frac{-w(\tau)\omega_0}{k}m'(\tau)+F(\tau), \tag{5}$$

where $\omega_0=\sqrt{k/M_0}$ is the natural frequency of the oscillator. The initial displacement and the initial velocity of $y(\tau)$ are given by

$$y(0)=y_0 \quad \text{and} \quad y'(0)=y_0'. \tag{6}$$

The paper is organized as follows. In Section 2 the initial value problem (5)–(6) will be studied with $F(\tau)\equiv 0$. In this case the small masses which are periodically hitting and leaving the oscillator (with nonzero velocities) can be seen as an external force acting on the oscillator. The stability of the solution(s) of the initial value problem will be studied, and the existence of periodic solutions will be investigated. In Section 3 it will be assumed that the force $F(\tau)$ is a harmonic force, that is, $F(\tau)=A\cos(\alpha\tau+\beta)$, where A and β are constants, and where α is the frequency of the external force. Then the following initial value problem for $y(\tau)$ is obtained:

$$((1-h(\tau))y'(\tau))'+y(\tau)=\frac{-w(\tau)\omega_0}{k}m'(\tau)+A\cos(\alpha\tau+\beta), \tag{7}$$

with initial conditions (6). The initial value problem (6)–(7) will be studied in detail in Section 3. The stability of the solutions will be studied as well as the existence of resonance frequencies (depending on α). Finally, in Section 4 of this paper some conclusions will be drawn, and remarks will be made about future research on this subject.

2. The case $F \equiv 0$

In this section the initial value problem (5)–(6) with $F \equiv 0$ will be studied, or equivalently

$$((1-h(\tau))y'(\tau))' + y(\tau) = \frac{-w(\tau)}{\omega_0} h'(\tau), \quad \tau > 0, \tag{8}$$

with $y(0) = y_0$, $y'(0) = y'_0$, $\omega_0 = \sqrt{k/M_0}$, and where $h(\tau)$ is given by (4). This section is organized as follows. In Section 2.1 a representation for the solution $y(\tau)$ of the initial value problem will be given. The stability properties of the solution(s) will be discussed in Section 2.2, and in Section 2.3 the existence of periodic solutions will be investigated.

2.1. A representation of the solution

It is obvious that the derivative of $h(\tau)$ with respect to τ for $0<\tau<T_0$ and $T_0<\tau<T$ is equal to 0. Thus, for $0<\tau<T_0$ Eq. (8) becomes

$$(1-\varepsilon)y'' + y = 0. \tag{9}$$

The initial value problem for (9) can easily be solved, yielding

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_1}(\tau) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{10}$$

where matrix $\mathbf{M_1}(\tau)$ is given by

$$\mathbf{M_1}(\tau) = \begin{pmatrix} \cos(\tau(1-\varepsilon)^{-1/2}) & (1-\varepsilon)^{1/2}\sin(\tau(1-\varepsilon)^{-1/2}) \\ -(1-\varepsilon)^{-1/2}\sin(\tau(1-\varepsilon)^{-1/2}) & \cos(\tau(1-\varepsilon)^{-1/2}) \end{pmatrix}.$$

At $\tau = T_0$ the function $h(\tau)$ has a jump discontinuity. Consider the infinitesimal small time-interval $T_0^- \le \tau \le T_0^+$, where $T_0^- = T_0 - 0$, $T_0^+ = T_0 + 0$. For this interval the following conditions can be formulated: the displacement of the oscillator is continuous, and the impulse of the system at $\tau = T_0^+$ is equal to the impulse of the system at $\tau = T_0^-$ plus the impulse of the raindrop (which hits the oscillator). The continuity of the displacement at $\tau = T_0$ implies that $y(T_0^-) = y(T_0^+)$, and the impulse condition can be obtained by integrating (8) with respect to τ from $\tau = T_0^-$ to $\tau = T_0^+$, yielding $y'(T_0^+) - (1-\varepsilon)y'(T_0^-) = \varepsilon w(T_0)/\omega_0$. And so,

$$\begin{pmatrix} y(T_0^+) \\ y'(T_0^+) \end{pmatrix} = \mathbf{M_2}(\mathrm{T}_0) \begin{pmatrix} y(T_0^-) \\ y'(T_0^-) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon w(T_0)/\omega_0 \end{pmatrix} = \mathbf{M_2}(\mathrm{T}_0)\mathbf{M_1}(\mathrm{T}_0) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon w(T_0)/\omega_0 \end{pmatrix}, \tag{11}$$

where $\mathbf{M_2}(\mathrm{T}_0)$ is given by $\mathbf{M_2}(\mathrm{T}_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}$. For $T_0<\tau<T$ Eq. (8) has the following form:

$$y'' + y = 0 \tag{12}$$

and the solution of Eq. (12) is given by

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_3}(\tau)\mathbf{M_2}(\mathrm{T}_0)\mathbf{M_1}(\mathrm{T}_0) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{M_3}(\tau) \begin{pmatrix} 0 \\ \varepsilon w(T_0)/\omega_0 \end{pmatrix}, \tag{13}$$

where $\mathbf{M_3}(\tau)$ is given by $\mathbf{M_3}(\tau) = \begin{pmatrix} \cos(\tau-T_0) & \sin(\tau-T_0) \\ -\sin(\tau-T_0) & \cos(\tau-T_0) \end{pmatrix}$.

At $\tau = T$ the function $h(\tau)$ has again a jump discontinuity. Consider the infinitesimal small time-interval $T^- \le \tau \le T^+$, where $T^- = T - 0$, $T^+ = T + 0$. For this interval the following conditions can be formulated: the displacement of the oscillator is continuous, and the impulse of the system at $\tau = T^+$ is equal to the impulse of the system at $\tau = T^-$ plus the impulse of the raindrop (which leaves the oscillator). The continuity of the displacement at $\tau = T$ simply implies that $y(T^-) = y(T^+)$, and the impulse condition can be obtained by integrating (8) with respect to τ from $\tau = T^-$ to $\tau = T^+$, yielding $(1-\varepsilon)y'(T^+) - y'(T^-) = -\varepsilon w(T)/\omega_0$. And so,

$$\begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} = \mathbf{M_4}(\mathrm{T}) \begin{pmatrix} y(T^-) \\ y'(T^-) \end{pmatrix} + \begin{pmatrix} 0 \\ -\dfrac{\varepsilon w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix}, \tag{14}$$

where $\mathbf{M_4}(\mathrm{T})$ is given by $\mathbf{M_4}(\mathrm{T}) = \begin{pmatrix} 1 & 0 \\ 0 & (1-\varepsilon)^{-1} \end{pmatrix}$. So, the solution of Eq. (8) on the interval $0<\tau\le T^+$ has been constructed, and at $\tau = T^+$ the solution is given by

$$\begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{W}, \tag{15}$$

where

$$\mathbf{A} = \mathbf{M_4}(\mathrm{T}^+)\ \mathbf{M_3}(\mathrm{T}^+)\mathbf{M_2}(\mathrm{T}_0)\ \mathbf{M_1}(\mathrm{T}_0) = \begin{pmatrix} ab - cd(1-\varepsilon)^{1/2} & bc(1-\varepsilon)^{1/2} + ad(1-\varepsilon) \\ -ad(1-\varepsilon)^{-1} - bc(1-\varepsilon)^{-1/2} & -cd(1-\varepsilon)^{-1/2} + ab \end{pmatrix}, \tag{16}$$

where

$$a = \cos(T_0(1-\varepsilon)^{-1/2}), \quad b = \cos(T - T_0),$$

$$c = \sin(T_0(1-\varepsilon)^{-1/2}), \quad d = \sin(T - T_0), \tag{17}$$

and where

$$\mathbf{W} = \mathbf{M_4}(\mathrm{T}^+)\ \mathbf{M_3}(\mathrm{T}^+)\begin{pmatrix} 0 \\ \dfrac{\varepsilon w(T_0)}{\omega_0} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{-\varepsilon w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix} = \begin{pmatrix} \dfrac{\varepsilon w(T_0)}{\omega_0}\sin(T-T_0) \\ \dfrac{\varepsilon(w(T_0)\cos(T-T_0) - w(T))}{\omega_0(1-\varepsilon)} \end{pmatrix}. \tag{18}$$

To obtain the solution on the interval $0<\tau\leq(n+1)T^+$, the procedure should be repeated n more times, yielding for $\tau=(n+1)T^+$:

$$\begin{pmatrix} y((n+1)T^+) \\ y'((n+1)T^+) \end{pmatrix} = \mathbf{A}^{\mathrm{n}+1}\begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \sum_{r=0}^{n}\mathbf{A}^{\mathrm{r}}\mathbf{W}. \tag{19}$$

The properties of matrix $\mathbf{A}$ are known from [5]. For $\mathbf{W}=(0\ \ 0)^{\mathrm{T}}$ the oscillator is unstable when at least one of the eigenvalues λ_1 or λ_2 is such that $|\lambda_j|>1$, or when $\lambda_1=\lambda_2$ with $|\lambda_j|=1$ and the dimension of the corresponding eigenspace is equal to one. In all other cases the oscillator is stable for $\mathbf{W}=(0\ \ 0)^{\mathrm{T}}$. These results are summarized in Table 1, where $\lambda_{1,2}=\frac{1}{2}\mathrm{tr}(\mathbf{A})\pm\frac{1}{2}\sqrt{D}$ with $D=(\mathrm{tr}(\mathbf{A}))^2-4$, and $\mathrm{tr}(\mathbf{A})$ is the trace of matrix $\mathbf{A}$ (see also [5]). The stability of the oscillator when $\mathbf{W}\neq\mathbf{0}$ will be determined in the next subsection.

2.2. On the stability of the oscillator

From the previous subsection (see (15)–(19)) it follows that the solution of Eq. (8) at $\tau=(n+1)T^+$ and at $\tau=nT^+$ can be linked by

$$\begin{pmatrix} y_{n+1} \\ y'_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} y_n \\ y'_n \end{pmatrix} + \mathbf{W}, \tag{20}$$

where $y_{n+1}=y((n+1)T^+)$, $y'_{n+1}=y'((n+1)T^+)$ and where $\mathbf{A}$ and $\mathbf{W}$ are given by (16) and (18), respectively. The solution of the system of difference equation (20) is given by (19). However, the representation (19) is not very convenient to determine the stability of the oscillator (due to an external force, that is, due to $\mathbf{W}\neq 0$). Also the use of a fundamental matrix for system (20) will lead to a representation (see for instance [6, p. 124]) from which it is not very convenient to determine the stability. Now a diagonalization method will be used to obtain a representation of the solution from which the stability of the oscillator can be determined immediately. From [7, p. 6] it follows that if the eigenvalues λ_1,λ_2 of a 2×2 matrix $\mathbf{A}$ are distinct or if the two eigenvalues are coinciding and the dimension of the corresponding eigenspace is 2, then from any set of linearly independent corresponding eigenvectors v_1, v_2 a matrix $\mathbf{P}$ can be formed, which is invertible and

Table 1
Stability properties of the oscillator when $\mathbf{W}=\mathbf{0}$.

Stability properties for $\mathrm{tr}(\mathbf{A})$	The oscillator for $\mathbf{W}=\mathbf{0}$ is
$-2<\mathrm{tr}(\mathbf{A})<2$ $(\lvert\lambda_{1,2}\rvert=1)$	Stable
$\mathrm{tr}(\mathbf{A})<-2$ or $\mathrm{tr}(\mathbf{A})>2$ $(\lvert\lambda_j\rvert>1$ for $j=1$ or $j=2)$	Unstable
$\mathrm{tr}(\mathbf{A})=2$ $(\lambda_1=\lambda_2=1)$	Only stable when $c=d=0$ and $ab=1$ in matrix $\mathbf{A}$, else unstable
$\mathrm{tr}(\mathbf{A})=-2$ $(\lambda_1=\lambda_2=-1)$	Only stable when $c=d=0$ and $ab=-1$ in matrix $\mathbf{A}$, else unstable

$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} = \text{diag}[\lambda_1, \lambda_2]$. Let

$$\begin{pmatrix} y_n \\ y'_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} x_n \\ x'_n \end{pmatrix}, \tag{21}$$

and substitute the transformation (21) into (20). Then, multiply the left- and the right-hand sides of the so-obtained equation by the inverse matrix of $\mathbf{P}$. So, we can rewrite (20) in the following form:

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_n \\ x'_n \end{pmatrix} + \mathbf{G}, \tag{22}$$

where

$$\mathbf{G} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \mathbf{P}^{-1}\mathbf{W}. \tag{23}$$

Then x_n and x'_n can be obtained, yielding

$$\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} + \sum_{r=0}^{n-1} \begin{pmatrix} \lambda_1^r & 0 \\ 0 & \lambda_2^r \end{pmatrix} \mathbf{G}. \tag{24}$$

Then use (21) and (23) in (24) to obtain for $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$:

$$\begin{pmatrix} y_n \\ y'_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \mathbf{P}^{-1} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{P} \begin{pmatrix} \dfrac{1-\lambda_1^n}{1-\lambda_1} & 0 \\ 0 & \dfrac{1-\lambda_2^n}{1-\lambda_2} \end{pmatrix} \mathbf{P}^{-1}\mathbf{W}. \tag{25}$$

For the eigenvalues $\lambda_{1,2} = 1$ and the dimension of the corresponding eigenspace is two, it is obvious from (24) that the solution (20) is unbounded, and that the oscillator is unstable for $\mathbf{W} \neq \mathbf{0}$. In [5] it has been shown that for $\mathbf{W} = \mathbf{0}$ the solution of (8) is bounded in this case. From [5], Eqs. (20)–(22) can be seen that the eigenvalues $\lambda_{1,2}$ of matrix $\mathbf{A}$ can be only coinciding for $\lambda_1 = \lambda_2 = 1$, or $\lambda_1 = \lambda_2 = -1$, and if one of the eigenvalues is equal to 1 (or -1) then the other eigenvalue is also equal to 1 (or -1). The case $\lambda_{1,2} = 1$ (and the dimension of the corresponding eigenspace is two) has just been considered, and for the case $\lambda_{1,2} = -1$ (and the dimension of the corresponding eigenspace is two) it follows from (25) that the solution is bounded, and so for $\lambda_1 = \lambda_2 = -1$ (and the dimension of the corresponding eigenspace is two) the oscillator is stable. For all other noncoinciding values of $\lambda_{1,2}$ the stability properties of the oscillator easily follow from (25).

Now the following case still has to be considered: matrix $\mathbf{A}$ has two coinciding eigenvalues and the dimension of the corresponding eigenspace is one (implying that matrix $\mathbf{A}$ cannot be diagonalized). For this case the Jordan-form matrix method can be used as for instance described in [6,7]. It can be shown that again an invertible matrix $\mathbf{P}$ exists such that

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{J} = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}. \tag{26}$$

Instead of (22) the following system will be obtained:

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} x_n \\ x'_n \end{pmatrix} + \mathbf{G}. \tag{27}$$

For $\lambda_{1,2} = 1$ x_n and x'_n can be determined from (27), yielding

$$\begin{cases} x_n = x_0 + n x'_0 + nG_1 + \dfrac{n(n-1)}{2} G_2, \\ x'_n = x'_0 + nG_2. \end{cases} \tag{28}$$

In (28) it can be seen that several terms are multiplied by n, so the vibrations of the oscillator will grow in time. For $\lambda_{1,2} = -1$, one obtains

$$\begin{cases} x_n = (-1)^n x_0 - (-1)^{n+1} n \left(x'_0 - \dfrac{G_2}{2} \right) + \left(G_1 + \dfrac{G_2}{2} \right) \cos^2 \left(\dfrac{\pi(n+1)}{2} \right), \\ x'_n = (-1)^n x'_0 + G_2 \cos^2 \left(\dfrac{\pi(n+1)}{2} \right). \end{cases} \tag{29}$$

Again there are several unbounded terms in (29), so the vibrations of the oscillator will also grow in time. All of the stability properties of the oscillator (for $\mathbf{W} \neq \mathbf{0}$) are summarized in Table 2.

Table 2
Stability properties of the oscillator when $\mathbf{W}\neq\mathbf{0}$.

Stability properties for tr(A)	The oscillator for $\mathbf{W}\neq\mathbf{0}$ is
$-2<\mathrm{tr}(\mathbf{A})<2$ ($\lvert\lambda_{1,2}\rvert=1$)	Stable
$\mathrm{tr}(\mathbf{A})<-2$ or $\mathrm{tr}(\mathbf{A})>2$ ($\lvert\lambda_j\rvert>1$ for $j=1$ or $j=2$)	Unstable
$\mathrm{tr}(\mathbf{A})=2$ ($\lambda_1=\lambda_2=1$)	Unstable
$\mathrm{tr}(\mathbf{A})=-2$ ($\lambda_1=\lambda_2=-1$)	Only stable when $c=d=0$ and $ab=-1$ in matrix $\mathbf{A}$, else unstable

2.3. On the existence of periodic solutions

In this subsection the existence of qT- periodic solutions (with $q\in\mathbf{Z}^+$) for Eq. (8) will be investigated. Since a small mass hits and leaves the oscillator with period T, it is natural to study the question whether qT-periodic solutions exist or not. In [8] a uniqueness result about the existence of T-periodic solutions for (8) has recently been presented. In this section the existence or nonexistence, and the uniqueness or nonuniqueness of qT-periodic solutions for Eq. (8) will be discussed in detail. To study these properties the map (20) will be used, that is,

$$y_{n+1}=\mathbf{A}y_n+\mathbf{W}, \tag{30}$$

where $y_n=(y(nT^+),y'(nT^+))^{\mathrm{T}}$, and where $\mathbf{A}$ and $\mathbf{W}$ are given by (16) and (18), respectively. For a T-periodic solution of (8) it follows from (30) that $y_{n+1}=y_n=y_{n-1}=\cdots=y$, and so y follows from (30):

$$y=\mathbf{A}y+\mathbf{W}\iff(\mathbf{I}-\mathbf{A})y=\mathbf{W}. \tag{31}$$

So, a unique, T-periodic solution of Eq. (8) exists when matrix $\mathbf{I}-\mathbf{A}$ is invertible, or equivalently $det(\mathbf{I}-\mathbf{A})\neq0$, or equivalently 1 is not an eigenvalue of matrix $\mathbf{A}$, or equivalently $\mathrm{tr}(\mathbf{A})\neq2$. When $\mathrm{tr}(\mathbf{A})=2$ or equivalently $\lambda=1$ is an eigenvalue of matrix $\mathbf{A}$ then there are two possibilities: there are no T-periodic solutions of Eq. (8), or there are infinitely many T-periodic solutions. From (24) and (28) it is obvious that for $\mathbf{W}\neq(0,0)^{\mathrm{T}}$ there are no T-periodic solutions, and that for $\mathbf{W}\equiv(0,0)^{\mathrm{T}}$ there are infinitely many T-periodic solutions.

For a qT-periodic solution of (8) with $q\in\mathbf{Z}^+$ and $q\geq2$ it follows from (30) that $y_{n+q}=y_n=y_{n-q}=\cdots=y$, and so it follows from (30) that

$$\begin{aligned}y_{n+q}&=\mathbf{A}y_{n+q-1}+\mathbf{W}=\mathbf{A}(\mathbf{A}y_{n+q-2}+\mathbf{W})+\mathbf{W}=\cdots\implies y=\mathbf{A}^qy+(\mathbf{A}^{q-1}+\cdots+\mathbf{A}+\mathbf{I})\mathbf{W}\iff(\mathbf{I}-\mathbf{A}^q)y\\&=(\mathbf{A}^{q-1}+\cdots+\mathbf{A}+\mathbf{I})\mathbf{W}\iff\end{aligned} \tag{32}$$

$$(\mathbf{A}^{q-1}+\cdots+\mathbf{A}+\mathbf{I})(\mathbf{I}-\mathbf{A})y=(\mathbf{A}^{q-1}+\cdots+\mathbf{A}+\mathbf{I})\mathbf{W}\iff(\mathbf{A}^{q-1}+\cdots+\mathbf{A}+\mathbf{I})((\mathbf{I}-\mathbf{A})y-\mathbf{W})=\mathbf{O}. \tag{33}$$

So, a unique, qT-periodic solution of Eq. (8) exists (see (32)) when matrix $\mathbf{I}-\mathbf{A}^q$ is invertible, or equivalently $det(\mathbf{I}-\mathbf{A}^q)\neq0$, or equivalently 1 is not an eigenvalue of matrix $\mathbf{A}^q$, or equivalently those λ's with $\lambda^q=1$ are not eigenvalues of matrix $\mathbf{A}$. When λ is an eigenvalue of matrix $\mathbf{A}$, and $\lambda^q=1$, and $\lambda\neq1$ (the case of T-periodic solutions has already been studied) then $\mathbf{A}^{q-1}+\cdots+\mathbf{A}+\mathbf{I}$ is not invertible, and Eq. (33) has at least one solution $y=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{W}$. And so, Eq. (33) has infinitely many solutions, that is, there are infinitely many qT-periodic solutions (with $q\geq2$) of Eq. (8) for all vectors $\mathbf{W}$. It can be shown in an elementary way that $\lambda^q=1$ and λ is an eigenvalue of matrix $\mathbf{A}$ is equivalent with $\mathrm{tr}(\mathbf{A})=2\cos(2n\pi/q)$ for at least one n in the set $0,1,2,\ldots,q-1$. The results obtained so far about the existence (and uniqueness) of qT-periodic solutions of Eq. (8), can be summarized as follows. Let λ be an eigenvalue of matrix $\mathbf{A}$, and let q be an element in $\mathbf{Z}^+$. Then,

(i) If $\lambda=1$ ($\Leftrightarrow\mathrm{tr}(\mathbf{A})=2$) then there are only T-periodic solutions when $\mathbf{W}\equiv(0,0)^{\mathrm{T}}$.
(ii) If $\lambda^q=1$ and $\lambda\neq1$ for a certain $q\geq2$ ($\Leftrightarrow\mathrm{tr}(\mathbf{A})=2\cos(2n\pi/q)$ for at least one n in the set $0,1,2,\ldots,q-1$) then there are infinitely many qT-periodic solutions of Eq. (8) for all vectors $\mathbf{W}$.
(iii) If $\lambda^q\neq1$ then there is a unique qT- periodic solution of Eq. (8) for all vectors $\mathbf{W}$.

3. The case with an external, harmonic force and $w(t)\equiv0$

In this section the initial value problem (6)–(7) with $w(t)\equiv0$ will be studied, that is,

$$((1-h(\tau))y'(\tau))'+y(\tau)=A\cos(\alpha\tau+\beta),\quad\tau>0, \tag{34}$$

with $y(0) = y_0$, $y'(0) = y_0'$, $\omega_0 = \sqrt{k/M_0}$, where $h(\tau)$ is given by (4), and where α, A and β are constants. This section is organized as follows. In Section 3.1 a representation for the solution $y(\tau)$ of the initial value problem will be given. The amplitude increase after one period T will be discussed in Section 3.2, and in Section 3.3 the stability properties of the solution and the resonance cases will be investigated.

3.1. A representation of the solution

As in the previous section a map will be constructed which relates the solution at $\tau = (n+1)T + 0^+$ to the solution at $\tau = nT + 0^+$. For simplicity the following notation will be introduced: $y_n(0^+) = y(nT + 0^+)$, $y_{n+1}(0^+) = y((n+1)T + 0^+)$, $y_n(\tau^*) = y(nT + \tau^*)$ with $0 < \tau^* \le T + 0^+$. Starting at $\tau = nT + 0^+$ the solution will now be constructed (leading to the solution at $\tau = (n+1)T + 0^+$). For $nT < \tau < nT + T_0$ or equivalently for $0 < \tau^* < T_0$ Eq. (34) becomes

$$(1-\varepsilon)y'' + y = A\cos(\alpha\tau + \beta). \tag{35}$$

For $\alpha^2 \neq 1/(1-\varepsilon)$ a particular solution of (35) is given by

$$y_p(\tau) = y_{1p}\cos(\alpha\tau + \beta), \tag{36}$$

where

$$y_{1p} = \frac{A\phi^2}{\phi^2 - \alpha^2}, \quad \phi = (1-\varepsilon)^{-1/2}. \tag{37}$$

The initial value problem (with $\alpha^2 \neq \phi^2$) can easily be solved for $0 < \tau^* < T_0$, yielding

$$\begin{pmatrix} y_n(\tau) \\ y_n'(\tau^*) \end{pmatrix} = \mathbf{M_1}(\tau^*)\begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{N_1}(\tau^*)\begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{38}$$

where

$$\mathbf{N_1}(\tau^*) = \begin{pmatrix} y_{1p}(c^*j - \alpha\phi^{-1}a^*l + f^*) & y_{1p}(a^*l + \alpha\phi^{-1}c^*j - g^*) \\ y_{1p}(\phi c^*j + \alpha a^*l - \alpha g^*) & y_{1p}(\alpha a^*j - \phi c^*l - \alpha f^*) \end{pmatrix}, \quad \mathbf{M_1}(\tau^*) = \begin{pmatrix} a^* & c^*\phi^{-1} \\ -\phi c^* & a^* \end{pmatrix},$$

and where a^*, c^*, j, l, f^*, g^* are given by

$$a^* = \cos(\phi\tau^*), \quad c^* = \sin(\phi\tau^*), \quad j = \cos(\beta), \quad l = \sin(\beta),$$

$$f^* = \cos(\alpha\tau^* + \beta), \quad g^* = \sin(\alpha\tau^* + \beta). \tag{39}$$

For $\alpha^2 = \phi^2$ a particular solution of (35) on the time-interval $nT < \tau < nT + T_0$ is given by $y_p(\tau) = (A/2)\phi\tau\sin(\phi\tau + \beta)$, and an expression almost similar to (38) can be given. At $\tau^* = T_0$ the function $h(\tau)$ in (34) has a jump discontinuity. As in Section 2 of this paper it follows for $\tau^* = T_0^+$ that

$$\begin{pmatrix} y_n(\tau^*) \\ y_n'(\tau^*) \end{pmatrix} = \mathbf{M_2}\mathbf{M_1}(T_0)\begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{M_2}\mathbf{N_1}(T_0)\begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{40}$$

where $\mathbf{M_2} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \phi^{-2} \end{smallmatrix}\right)$. For $T_0 < \tau^* < T$ Eq. (34) is given by

$$y'' + y = A\cos(\alpha\tau + \beta), \tag{41}$$

and for $\alpha^2 \neq 1$ a particular solution of (41) can be written as

$$y_p(\tau) = y_{2p}\cos(\alpha\tau + \beta), \tag{42}$$

where $y_{2p} = A/(1-\alpha^2)$. The initial value problem (with $\alpha^2 \neq 1$) can easily be solved for $T_0 < \tau^* < T$, yielding

$$\begin{pmatrix} y_n(\tau^*) \\ y_n'(\tau^*) \end{pmatrix} = \mathbf{M_3}(\tau^*)\mathbf{M_2}\mathbf{M_1}(T_0)\begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + (\mathbf{M_3}(\tau^*)\mathbf{M_2}\mathbf{N_1}(T_0) + \mathbf{N_3}(\tau^*))\begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{43}$$

where

$$\mathbf{N_3}(\tau^*) = \begin{pmatrix} y_{2p}(\alpha d^*g - b^*f + p^*) & y_{2p}(\alpha d^*f + b^*g - q^*) \\ y_{2p}(\alpha b^*g + d^*f - \alpha q^*) & y_{2p}(\alpha b^*f - d^*g - \alpha p^*) \end{pmatrix} \quad \text{and} \quad \mathbf{M_3}(\tau^*) = \begin{pmatrix} b^* & d^* \\ -d^* & b^* \end{pmatrix},$$

and where b^*, d^*, p^*, q^*, f, g are given by

$$b^* = \cos(\tau^* - T_0), \quad d^* = \sin(\tau^* - T_0), \quad p^* = \cos(\alpha\tau^* + \beta),$$

$$q^* = \sin(\alpha\tau^* + \beta), \quad f = \cos(\alpha T_0 + \beta), \quad g = \sin(\alpha T_0 + \beta). \tag{44}$$

For $\alpha^2 = 1$ a particular solution of (41) on the time-interval $nT + T_0 < \tau < (n+1)T$ is given by $y_p(\tau) = (A/2)\tau\sin(\tau + \beta)$, and an expression almost similar to (43) can be given. At $\tau^* = T$ the function $h(\tau)$ in (34) has again a jump discontinuity. As in

Section 2 of this paper it follows for $\tau^* = T^+$ that (observe that $y_{n+1}(0^+) = y_n(T^+)$ and $y'_{n+1}(0^+) = y'_n(T^+)$):

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{M_4 M_3}(T)\mathbf{M_2 M_1}(T_0)\begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} + (\mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0) + \mathbf{M_4 N_3}(T))\begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{45}$$

where $\mathbf{M_4} = \begin{pmatrix} 1 & 0 \\ 0 & \phi^2 \end{pmatrix}$. From (45) the following map can be obtained:

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{A}\begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} + \mathbf{W}_n, \tag{46}$$

where $\mathbf{A}$ is given by (16), and where $\mathbf{W}_n$ is given by

$$\mathbf{W}_n = (\mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0) + \mathbf{M_4 N_3}(T))\begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}. \tag{47}$$

Comparing (47) to (20) it should be observed that the nonhomogeneous term now explicitly depends on n. The solution of the system of difference equation (46) is given by

$$\begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} = \mathbf{A}^n\begin{pmatrix} y_0(0^+) \\ y'_0(0^+) \end{pmatrix} + \sum_{r=0}^{n-1}\mathbf{A}^r\mathbf{W}_r. \tag{48}$$

For $\alpha^2 \neq \phi^2$ and $\alpha^2 \neq 1$ the vector $\mathbf{W}_r$ is given by

$$\mathbf{W}_r = \begin{pmatrix} w_{11}\cos(\alpha rT) + w_{12}\sin(\alpha rT) \\ w_{21}\cos(\alpha rT) + w_{22}\sin(\alpha rT) \end{pmatrix}, \tag{49}$$

where

$$w_{11} = y_{1p}b\left(\frac{\alpha}{\phi}cl - aj + f\right) + y_{1p}\frac{d}{\phi^2}(\phi cj + \alpha al - \alpha g) + y_{2p}(\alpha dg - bf + p),$$

$$w_{12} = y_{1p}b\left(\frac{\alpha}{\phi}cj + al - g\right) + y_{1p}\frac{d}{\phi^2}(\alpha aj - \phi cl - \alpha f) + y_{2p}(\alpha df + bg - q),$$

$$w_{21} = -y_{1p}d\phi^2\left(\frac{\alpha}{\phi}cl - aj + f\right) + y_{1p}b(\phi cj + \alpha al - \alpha g) + y_{2p}\phi^2(\alpha bg + df - \alpha q),$$

$$w_{22} = -y_{1p}d\phi^2\left(\frac{\alpha}{\phi}cj + al - g\right) + y_{1p}b(\alpha aj - \phi cl - \alpha f) + y_{2p}\phi^2(\alpha bf - dg - \alpha p).$$

For $\alpha^2 = \phi^2$ the vector $\mathbf{W}_r$ is given by (49) with

$$w_{11} = y_{1p}2b(\phi T_0 g - cl) + y_{1p}\frac{2d}{\phi}(\phi T_0 f + cj) + y_{2p}(\phi dg - bf + p),$$

$$w_{12} = y_{1p}2b(\phi T_0 f - cj) - y_{1p}\frac{2d}{\phi}(\phi T_0 g + cl) + y_{2p}(\phi df + bg - q),$$

$$w_{21} = -y_{1p}2\phi^2 d(\phi T_0 g - cl) + y_{1p}2\phi b(\phi T_0 f + cj) + y_{2p}\phi^2(\phi bg + df - \phi q),$$

$$w_{22} = -y_{1p}2\phi^2 d(\phi T_0 f - cj) - y_{1p}2\phi b(\phi T_0 g + cl) + y_{2p}\phi^2(\phi bf - dg - \phi p).$$

And for $\alpha^2 = 1$ the vector $\mathbf{W}_r$ is given by (49) with

$$w_{11} = y_{1p}b\left(\frac{cl}{\phi} - aj + f\right) + y_{1p}\frac{d}{\phi^2}(\phi cj + al - g) + y_{2p}(2(T - T_0)q + p - p_1),$$

$$w_{12} = y_{1p}b\left(\frac{cj}{\phi} + al - g\right) + y_{1p}\frac{d}{\phi^2}(aj - \phi cl - f) + y_{2p}(2(T - T_0)p - q - q_1),$$

$$w_{21} = -y_{1p}\phi^2 d\left(\frac{cl}{\phi} - aj + f\right) + y_{1p}b(\phi cj + al - g) + y_{2p}\phi^2(2(T - T_0)p + q + q_1),$$

$$w_{22} = -y_{1p}\phi^2 d\left(\frac{cj}{\phi} + al - g\right) + y_{1p}b(aj - \phi cl - f) - y_{2p}\phi^2(2(T - T_0)q + p + p_1).$$

The coefficients a, b, c, d are given by (17), j, l are given by (39), and $p = \cos(\alpha T + \beta)$, $q = \sin(\alpha T + \beta)$, $p_1 = \cos(T - 2T_0 - \beta)$, $q_1 = \sin(T - 2T_0 - \beta)$.

3.2. The amplitude increase after one period T due to harmonic forcing

In this section the possible amplitude increase of the displacement function $y(\tau)$ (after one period T) due to the external, harmonic force will be studied. From (48) and (49) it can easily be seen that this increase is completely determined by

$$w_{11}\cos(\alpha nT) + w_{12}\sin(\alpha nT) = \gamma\sin(\alpha nT + \delta), \tag{50}$$

where $\gamma = \sqrt{w_{11}^2 + w_{12}^2}$, and where δ is given by $\sin(\delta) = w_{11}/\gamma$ and $\cos(\delta) = w_{12}/\gamma$. The maximum amplitude response (in absolute value) is γ. Obviously, γ depends on $\alpha, A, T_0, T, \delta$, and ε. In Fig. 2 γ as a function of α is plotted for $A = 1$, $T_0 = 100$, $T = 200$, $\delta = \pi/7$ and $\varepsilon = 0.3$. In Fig. 2 it can be seen that there are two peaks. These two peaks are a consequence of the change of mass of the oscillator, and so the oscillator actually has two resonance frequencies (1 and $(1-\varepsilon)^{-1/2}$). Since only one period T for the amplitude response is considered these maximum amplitude responses are of course bounded. In Fig. 3 an optimization program has been used to show the maximum amplitude responses when $A = 1$, T_0 and T are varied such that $0 < T_0 < T < 20$, $\delta = \pi/7$, and $\varepsilon = 0.3$. Similar results can be obtained for other values of A, T_0, T, δ, and ε. For instance, in Fig. 4 the results have been shown for $A = 1$, $0 < T_0 < T < 100$, $\delta = \pi/7$, and $\varepsilon = 0.3$.

3.3. Stability properties of the solution, and resonance

In this subsection the stability properties and boundness of the solution of (46) will be studied. In fact the solution has to satisfy (45), where $\mathbf{M_4}$, $\mathbf{M_3}(T)$, $\mathbf{M_2}$, $\mathbf{M_1}(T_0)$, $\mathbf{N_1}(T_0)$, and $\mathbf{N_3}$ are defined in Section 3.1. It should be observed that in (45) the matrices $\mathbf{M_4} \cdot \mathbf{M_3}(T) \cdot \mathbf{M_2} \cdot \mathbf{M_1}(T_0)$ denoted by $\mathbf{A}$ and $\mathbf{M_4} \cdot \mathbf{M_3}(T) \cdot \mathbf{M_2} \cdot \mathbf{N_1}(T_0) + \mathbf{M_4} \cdot \mathbf{N_3}(T)$ denoted by $\mathbf{B}$ are both n independent matrices. Then the system of two first-order ordinary difference equation (45) will be reduced to a single

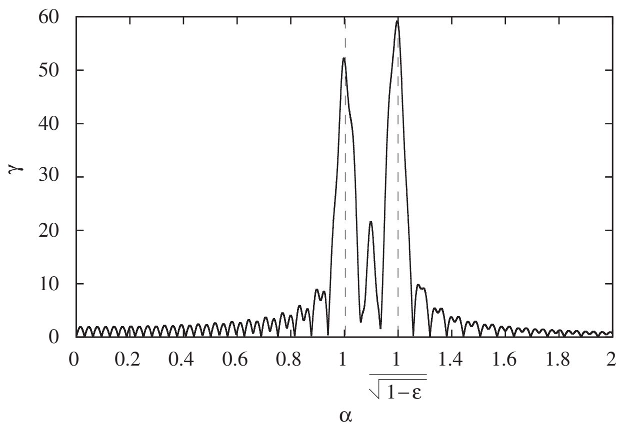

Fig. 2. The maximum amplitude response γ as a function of α for $A = 1$, $T_0 = 100$, $T = 200$, $\delta = \pi/7$, and $\varepsilon = 0.3$.

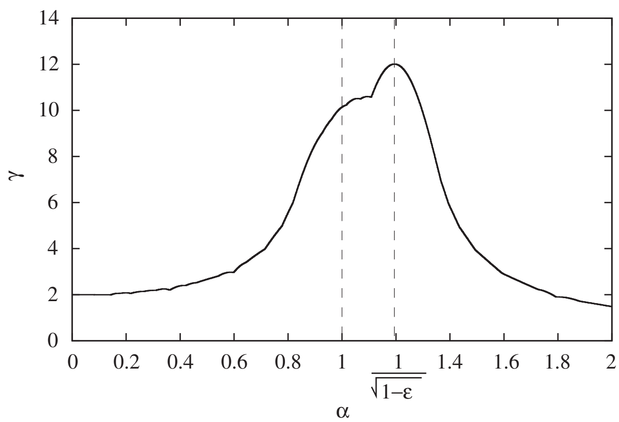

Fig. 3. The maximum amplitude response γ as a function of α for $A = 1$, $0 < T_0 < T < 20$, $\delta = \pi/7$, and $\varepsilon = 0.3$.

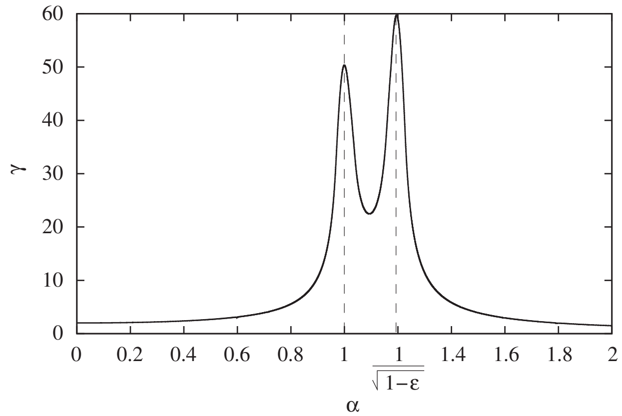

Fig. 4. The maximum amplitude response γ as a function of α for $A=1$, $0<T_0<T<100$, $\delta=\pi/7$, and $\varepsilon=0.3$.

second-order difference equation for $y_n(0^+)=y_n$, yielding

$$y_{n+2}-(a_{11}+a_{22})y_{n+1}+(a_{11}a_{22}-a_{12}a_{21})y_n=c_0\cos(\alpha nT)+s_0\sin(\alpha nT)+c_1\cos(\alpha(n+1)T)+s_1\sin(\alpha(n+1)T), \quad (51)$$

where a_{ij} $(i,j=1,2)$ are the components of matrix $\mathbf{A}$, and $c_0=b_{21}a_{12}-b_{11}a_{22}$, $s_0=b_{22}a_{12}-b_{12}a_{22}$, $c_1=b_{11}$, $s_1=b_{12}$, and where b_{ij} $(i,j=1,2)$ are the components of matrix $\mathbf{B}=\mathbf{M_4}\cdot\mathbf{M_3}(T)\cdot\mathbf{M_2}\cdot\mathbf{N_1}(T_0)+\mathbf{M_4}\cdot\mathbf{N_3}(T)$ which are explicitly given by the components $w_{i,j}$ $(i,j=1,2)$ of matrix $\mathbf{W}_\mathrm{r}$ in Eq. (49). In Eq. (51) $a_{11}+a_{22}=\mathrm{tr}(\mathbf{A})$ is the trace of matrix $\mathbf{A}$, and $a_{11}a_{22}-a_{12}a_{21}=det(\mathbf{A})$ is the determinant of matrix $\mathbf{A}$ which is equal to 1 (see [5]). The solution y_n of (51) can be written as

$$y_n=y_{h,n}+y_{p0,n}+y_{p1,n}, \quad (52)$$

where $y_{h,n}$ is the solution of the homogeneous equation (related to (51)):

$$y_{h,n+2}-\mathrm{tr}(\mathbf{A})y_{h,n+1}+y_{h,n}=0, \quad (53)$$

and where $y_{pm,n}$ (with $m=0,1$) are the particular solutions of (51) satisfying

$$y_{pm,n+2}-\mathrm{tr}(\mathbf{A})y_{pm,n+1}+y_{pm,n}=c_m\cos(\alpha(n+m)T)+s_m\sin(\alpha(n+m)T). \quad (54)$$

The roots of the characteristic equation belonging to the homogeneous equation (53) are given by $\lambda_{1,2}=\frac{1}{2}\mathrm{tr}(\mathbf{A})\pm\frac{1}{2}\sqrt{(\mathrm{tr}(\mathbf{A}))^2-4}$, and are, of course, coinciding with the eigenvalues of matrix $\mathbf{A}$. The corresponding stability properties of the homogeneous solution $y_{h,n}$ can be found in Table 1 or in [5]. The particular solutions $y_{pm,n}$ of (54) can be found in the following way. First one looks for a particular solution $y_{pm,n}$ in the form

$$y_{pm,n}=C_{1m}\cos(\alpha(n+m)T)+C_{2m}\sin(\alpha(n+m)T), \quad (55)$$

where C_{1m} and C_{2m} are constants to be determined. By substituting (55) into (54), and then by collecting the coefficients of $\cos(\alpha(n+m)T)$ and of $\sin(\alpha(n+m)T)$ it follows that C_{1m} and C_{2m} have to satisfy

$$\begin{pmatrix}\cos(2\alpha T)-\mathrm{tr}(\mathbf{A})\cos(\alpha T)+1 & \sin(2\alpha T)-\mathrm{tr}(\mathbf{A})\sin(\alpha T)\\ -\sin(2\alpha T)+\mathrm{tr}(\mathbf{A})\sin(\alpha T) & \cos(2\alpha T)-\mathrm{tr}(\mathbf{A})\cos(\alpha T)+1\end{pmatrix}\begin{pmatrix}C_{1m}\\ C_{2m}\end{pmatrix}=\begin{pmatrix}c_m\\ s_m\end{pmatrix}. \quad (56)$$

The difference equation (51) has a unique solution when two initial conditions are given. And so, the particular solutions $y_{pm,n}$ can be determined uniquely. To have a unique particular solution $y_{pm,n}$ it follows from (56) that the determinant of the coefficient matrix in (56) should be nonzero. When the determinant is equal to zero then there are infinitely many solutions or there is no solution. This will occur when

$$\begin{cases}\cos(2\alpha T)-\mathrm{tr}(\mathbf{A})\cos(\alpha T)+1=0 \quad \text{and}\\ \sin(2\alpha T)-\mathrm{tr}(\mathbf{A})\sin(\alpha T)=0,\end{cases} \quad (57)$$

or equivalently when

$$\mathrm{tr}(\mathbf{A})=2\cos(\alpha T). \quad (58)$$

So, the particular solutions $y_{pm,n}$ can be determined uniquely when $\mathrm{tr}(\mathbf{A})\neq 2\cos(\alpha T)$. When $\mathrm{tr}(\mathbf{A})=2\cos(\alpha T)$ the particular solutions $y_{pm,n}$ will have the following form:

$$y_{pm,n}=n(\tilde{C}_{1m}\cos(\alpha(n+m)T)+\tilde{C}_{2m}\sin(\alpha(n+m)T)), \quad (59)$$

Table 3
Stability properties of the oscillator with a harmonic external force when $\mathbf{W}_n \neq \mathbf{0}$.

Stability properties for $\mathrm{tr}(\mathbf{A})$	The oscillator for $\mathbf{W}_n \neq \mathbf{0}$ is
$-2 < \mathrm{tr}(\mathbf{A}) < 2$ $(\lvert\lambda_{1,2}\rvert = 1)$	Only unstable when $\mathrm{tr}(\mathbf{A}) = 2\cos(\alpha T)$, else stable
$\mathrm{tr}(\mathbf{A}) < -2$ or $\mathrm{tr}(\mathbf{A}) > 2$ $(\lvert\lambda_j\rvert > 1$ for $j = 1$ or $j = 2)$	Always unstable
$\mathrm{tr}(\mathbf{A}) = 2$ $(\lambda_1 = \lambda_2 = 1)$	Only stable when $c = d = 0$ and $ab = 1$ in matrix $\mathbf{A}$, and αT is not an even multiple of π, else unstable
$\mathrm{tr}(\mathbf{A}) = -2$ $(\lambda_1 = \lambda_2 = -1)$	Only stable when $c = d = 0$ and $ab = -1$ in matrix $\mathbf{A}$, and αT is not an odd multiple of π, else unstable

where $\tilde{C}_{1m}$ and $\tilde{C}_{2m}$ are constants to be determined. By substituting (59) into (54), and then by collecting the coefficients of $\cos(\alpha(n+m)T)$ and of $\sin(\alpha(n+m)T)$ it follows that $\tilde{C}_{1m}$ and $\tilde{C}_{2m}$ have to satisfy

$$\begin{pmatrix} 2\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) & 2\sin(2\alpha T) - \mathrm{tr}(\mathbf{A})\sin(\alpha T) \\ -2\sin(2\alpha T) + \mathrm{tr}(\mathbf{A})\sin(\alpha T) & 2\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) \end{pmatrix} \begin{pmatrix} \tilde{C}_{1m} \\ \tilde{C}_{2m} \end{pmatrix} = \begin{pmatrix} c_m \\ s_m \end{pmatrix}. \tag{60}$$

Again to have a unique particular solution $y_{pm,n}$ (in the form (59)) it follows from (60) that the determinant of the coefficient matrix in (60) should be nonzero. When the determinant is equal to zero there are infinitely many solutions or there is no solution. This will occur when

$$\begin{cases} 2\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) = 0 \quad \text{and} \\ 2\sin(2\alpha T) - \mathrm{tr}(\mathbf{A})\sin(\alpha T) = 0, \end{cases} \tag{61}$$

or equivalently when

$$\mathrm{tr}(\mathbf{A}) = \pm 2 \quad \text{and} \quad \sin(\alpha T) = 0. \tag{62}$$

So, when $\mathrm{tr}(\mathbf{A}) = 2\cos(\alpha T)$ and αT is not a multiple of π then the particular solution $y_{pm,n}$ will grow linearly in n (see (59)). The condition (58), that is, $\mathrm{tr}(\mathbf{A}) = 2\cos(\alpha T)$ is called a resonance condition. The case $\mathrm{tr}(\mathbf{A}) = 2\cos(\alpha T)$ and αT is a multiple of π still has to be studied. When αT is an even multiple of π the system of difference equation (45) becomes

$$\begin{pmatrix} y_{n+1}(0^+) \\ y_{n+1}'(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{B} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{63}$$

and $\mathrm{tr}(\mathbf{A}) = 2\cos(\alpha T) = 2$. System (63) with $\mathrm{tr}(\mathbf{A}) = 2$ already has been studied in Section 2.2 of this paper. From Table 2 it follows that the solution of (63) is unstable. Similarly, when αT is an odd multiple of π the system of difference equation (45) becomes

$$\begin{pmatrix} y_{n+1}(0^+) \\ y_{n+1}'(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{B} \begin{pmatrix} (-1)^n \\ 0 \end{pmatrix}, \tag{64}$$

and $\mathrm{tr}(\mathbf{A}) = 2\cos(\alpha T) = -2$. Since the eigenvalues of matrix $\mathbf{A}$ are both equal to -1 it is not difficult to see that the particular solution of (64) will contain unbounded terms in n. So, also in this case the solution of (64) is unstable. All of the stability properties of the solution of the oscillator equation (34) with an external harmonic force are summarized in Table 3.

4. Conclusions and remarks

In this paper the stability properties of the forced vibrations of a linear, single degree of freedom oscillator with a periodically and stepwise changing time-varying mass have been studied. Two types of forcing have been studied. First, a forcing has been investigated, due to a mass which hits the oscillator, stays for some time at the oscillator, and then leaves the oscillator. The stability properties of the oscillator, and the existence and (non) uniqueness of periodic vibrations have been studied in detail in Section 2 of this paper. Secondly, an external, harmonic forcing has been studied for an oscillator to which a mass (with zero velocity) is added for some time, and then is taken away (with zero velocity). For this case an interesting resonance condition has been found, and the stability properties of the oscillator problem have been presented in Section 3 of this paper. When both forcing types are applied to the oscillator the results as obtained in Section 2 and in

Section 3 of this paper can be combined, because the differential equation describing the problem is linear. It is also interesting to see in Section 3 that due to the changing mass and due to the external harmonic forcing the instability region shows two peaks. For a similar oscillator equation with a constant mass and an external, harmonic forcing one usually has one peak in the instability region. This larger instability region might perhaps explain in part the instability mechanism for rain-wind induced oscillations of cables in windfields. Usually cables in windfields are stable, but due to rain these cables can become unstable. Water addition to the cables, water drop off, and water rivulets on these cables (and so, changing aerodynamic forcing acting on the cable), and changing eigenfrequencies of the cable system certainly enlarge the instability regions of these cables.

To obtain more realistic mathematical models for these rain-wind induced oscillations of cables in windfields one might consider periodically and multi-stepwise changing time-varying masses. Other external forces (such as nonlinear drag-and-lift forces, damping forces, and so on) can also be included in the model equation. The aforementioned extensions to the model equation can be interesting subjects for future research.

References

[1] H. Irschik, H.J. Holl, Mechanics of variable-mass systems—part 1: balance of mass and linear momentum, *Applied Mechanical Review* 5 (2) (2004) 145–160.
[2] L. Cveticanin, Self-excited vibrations of the variable mass rotor/fluid system, *Journal of Sound and Vibration* 212 (4) (1998) 685–702.
[3] A.H.P. van der Burgh, Hartono, A.K. Abramian, A new model for the study of rain-wind-induced vibrations of a simple oscillator, *International Journal of Nonlinear Mechanics* 41 (2006) 345–358.
[4] H.J. Holl, A.K. Belyaev, H. Irschik, Simulation of the Duffing-oscillator with time-varying mass by a BEM in time, *Computers and Structures* 73 (1999) 177–186.
[5] W.T. van Horssen, A.K. Abramian, Hartono, On the free vibrations of an oscillator with a periodically time-varying mass, *Journal of Sound and Vibration* 298 (2006) 1166–1172.
[6] S.N. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag New York Inc., New York, 1996.
[7] L. Perko, *Differential Equations and Dynamical Systems*, third ed., Springer-Verlag New York, Inc., New York, 2001.
[8] D. Núñez, P.J. Torres, On the motion of an oscillator with a periodically time-varying mass, *Nonlinear Analysis RWA* 10 (2009) 1976–1983.