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Flexural and transversal wave motion in homogeneous isotropic thermoelastic plates by using asymptotic method

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ABSTRACT

The present investigation is concerned with the flexural and transversal wave motion in an infinite, transversely isotropic, thermoelastic plate by asymptotic method. The governing equations for the flexural and transversal motions have been derived from the system of three-dimensional dynamical equations of linear theory of coupled thermoelasticity. The asymptotic operator plate model for free vibrations; both flexural and transversal, in a homogenous thermoelastic plate leads to fifth degree and cubic polynomial secular equations, respectively, that governs frequency and phase velocity of various possible modes of wave propagation at all wavelengths. All the coefficients of differential operator have been expressed as explicit functions of the material parameters. The velocity dispersion equations for the flexural and transversal wave motion have been deduced from the three-dimensional analog of Rayleigh–Lamb frequency equation for thermoelastic plate waves. The approximations for long and short waves and expression for group velocity have also been derived. The thermoelastic Rayleigh–Lamb frequency equations for the considered plate are expanded in power series in order to obtain polynomial frequency and velocity dispersion relations whose equivalence is established with that of asymptotic method. The dispersion curves for phase velocity, group velocity and attenuation coefficient of various flexural and transversal wave modes are shown graphically for aluminum-epoxy material elastic and thermoelastic plates.

1. Introduction

The wave propagation in elastic and thermoelastic plate structures has received much attention [1–5] during the recent years because of its engineering applications. Losin [6,7] studied the asymptotics of flexural and extensional waves in homogeneous isotropic elastic plates. Losin [8] tried to establish the equivalence of dispersion relations obtained from operator plate model and Rayleigh–Lamb frequency equation. He established that terms up to eighth power of nh lead to equivalence in dispersion relations obtained from two methods. The work of Losin [6,7] was further extended to transversely isotropic elastic plate by Sharma and Kumar [9,10]. Zelentsov [11] proposed an asymptotic method for solving transient elastic problem of thin strips by employing combination of Laplace and Fourier transform and obtained asymptotic solutions for large values of Laplace transform parameter. Kirova et al. [12] have studied the asymptotic behavior for linear and nonlinear waves in viscoelastic materials. Ryabenkov and Faizullina [13] proved that asymptotic method is identical with the method of hypothesis and successive approximations for slabs and plates. Agalovyan and

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Gevorkyan [14] solved first boundary-value problem for forced vibrations of an isotropic strip by an asymptotic method. Gales [15] studied asymptotic spatial behavior of solutions in a mixture consisting of two thermoelastic solids.

The coupling between the strain and temperature fields was first studied by Duhamel [16] who derived equations for the distribution of strains in an elastic medium subjected to temperature gradients. Biot [17] gave a satisfactory derivation of the equation of thermal conductivity which included the dilation term based on thermodynamics of irreversible processes. Sharma et al. [18,19] investigated the propagation of thermoelastic waves in homogeneous isotropic plate subjected to stress free/ rigid fixed and thermally insulated/ isothermal conditions in the conventional coupled and generalized theories of thermoelasticity. Sharma and Singh [20] studied the propagation of circularly crested thermoelastic waves in a homogeneous isotropic cylindrical plate subjected to stress free and isothermal conditions. Sharma and Pathania [21] studied the propagation of waves in a homogeneous, transversely isotropic, thermally conducting plate bordered with layers of inviscid liquid or half-space of inviscid liquid on both sides. Gevorgyan [22] investigated the thermoelastic wave propagation in a transversely isotropic heat conducting as well as non-heat conducting elastic materials. The asymptotic expansion of the frequency equation for wave motion in a thermoelastic plate generated by the Rayleigh–Lamb equation does not give an adequate approximation. Senthil and Batra [23] investigated three-dimensional thermomechanical deformations of a simply supported functionally graded rectangular plate. Altukhov [24] obtained the homogenous thermal solutions due to a temperature field for three-dimensional thermoelastic problem for isotropic plates.

Moreover, it is pertinent to mention here that the dispersion relations reported in the works of Losin [6,7] were of sixth degree polynomial equations in frequency/phase velocity instead of tenth degree as reported in [9,10]. However, the corresponding equivalence relations obtained by Losin [8] in case of symmetric (extensional) and skew symmetric (flexural) motions of elastic plate are also tenth degree polynomial equations in phase velocity (see terms under the braces of Eqs. (7) and (14) in [8]). Equivalence of these relations has been established by considering terms upto eighth power of $\eta = nh$. The present work is an attempt to find a frequency and velocity dispersion relation from three-dimensional analog of the thermoelastic Rayleigh–Lamb frequency equation that would be sufficient to govern the flexural, and transversal and wave motions. The asymptotic method applied by Protsenko [25] for thin n-shelled elastic structures and by the authors [6,7,9] in case of elastic plates is employed in this investigation. The dispersion relation obtained here is also tenth degree polynomial in frequency/phase velocity which is in agreement with [9] and its equivalence relation also agrees with Eq. (14) of Losin [8] in non-dimensional form.

2. Formulation of the problem

We consider free wave motion in a homogenous isotropic coupled thermoelastic plate of thickness $2h$ initially at uniform temperature T_0 in the undisturbed state. The origin of Cartesian coordinate system $oxyz$ is taken at any point o in the middle plane of the plate and z-axis is pointed along the thickness of the plate. We assume that the plate is infinite in x and y directions which thus occupies the region

$$\Omega = \{-\infty < x, y < \infty, -h \le z \le h\}$$

In the region Ω, the corresponding basic governing equations of linear thermoelasticity in the absence of body forces and heat sources, are given by

$$\left\{(\lambda+2\mu)\frac{\partial^2}{\partial x^2}+\mu\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\right\}u+(\lambda+\mu)\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}\right)-\beta\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\left(\mu\frac{\partial^2}{\partial x^2}+(\lambda+2\mu)\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial z^2}\right)v+(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial y\,\partial z}\right)-\beta\frac{\partial T}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2} \tag{2}$$

$$\left\{\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+(\lambda+2\mu)\frac{\partial^2}{\partial z^2}\right\}w+(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x\,\partial z}+\frac{\partial^2 v}{\partial y\,\partial z}\right)-\beta\frac{\partial T}{\partial z}=\rho\frac{\partial^2 w}{\partial t^2} \tag{3}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)T-\frac{\rho c_e}{K}\frac{\partial T}{\partial t}-\frac{\beta T_0}{K}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \tag{4}$$

where $\beta=(3\lambda+2\mu)\alpha_t$, α_t is coefficient of linear thermal expansion; λ, μ are Lamé parameters; $T=T(x,y,z,t)$ is temperature change; u, v and w are displacement components; K is thermal conductivity; ρ is mass density and c_e is the specific heat at constant strain.

The surfaces of plate are assumed to be stress free and thermally insulated. Thus the boundary conditions on the surfaces $z=\pm h$ of the plate to be satisfied are

$$\sigma_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0$$

$$\sigma_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\sigma_{zz} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + (\lambda + 2\mu)\frac{\partial w}{\partial z} - \beta T = 0$$

$$\frac{\partial T}{\partial z} = 0 \tag{5}$$

We define the quantities

$$(x', y', z') = \frac{\omega^*}{c_1}(x, y, z), (u', v', w') = \frac{\rho\omega^* c_1}{\beta T_0}(u, v, w)$$

$$T' = \frac{T}{T_0}, t' = \omega^* t, h' = \frac{\omega^* h}{c_1}, c_1^2 = \frac{\lambda + 2\mu}{\rho}, c_2^2 = \frac{\mu}{\rho}$$

$$\delta^2 = \frac{c_2^2}{c_1^2} = \frac{\mu}{\lambda + 2\mu}, \varepsilon = \frac{\beta^2 T_0}{\rho c_e(\lambda + 2\mu)}, \omega^* = \frac{(\lambda + 2\mu)c_e}{K}, \sigma'_{ij} = \frac{\sigma_{ij}}{\beta T_0} \tag{6}$$

where ω^* is characteristics frequency, ε is thermoelastic-coupling constant, c_1 and c_2 are the velocities of longitudinal and transverse waves, respectively.

Upon introducing quantities (6) in governing Eqs. (1)–(4) and boundary conditions (5), we obtain (on suppressing dashes for convenience)

$$\left\{\frac{\partial^2}{\partial x^2} + \delta^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right\}u + (1 - \delta^2)\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial z}\right) - \frac{\partial T}{\partial x} = \frac{\partial^2 u}{\partial t^2} \tag{7}$$

$$\left(\delta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \delta^2\frac{\partial^2}{\partial z^2}\right)v + (1 - \delta^2)\left(\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial y\,\partial z}\right) - \frac{\partial T}{\partial y} = \frac{\partial^2 v}{\partial t^2} \tag{8}$$

$$\left\{\delta^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial z^2}\right\}w + (1 - \delta^2)\left(\frac{\partial^2 u}{\partial x\,\partial z} + \frac{\partial^2 v}{\partial y\,\partial z}\right) - \frac{\partial T}{\partial z} = \frac{\partial^2 w}{\partial t^2} \tag{9}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)T - \frac{\partial T}{\partial t} - \varepsilon\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{10}$$

$$\sigma_{xz} = \delta^2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

$$\sigma_{yz} = \delta^2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\sigma_{zz} = (1 - 2\delta^2)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial z} - T = 0$$

$$\frac{\partial T}{\partial z} = 0 \tag{11}$$

3. Solution of the problem

We assume harmonic wave solution of the form

$$(u, T, v, w)(x, y, z, t) = \vec{u}(z)\exp\{-\mathrm{i}(\vec{r}\cdot\vec{n} - \omega t)\} \tag{12}$$

where $\vec{u}(z) = (U(z), \theta(z), V(z), W(z))$ is amplitude vector, $\omega = \vec{\omega}(n)$ is the circular frequency depending on the wavenumber $\vec{n} = (n_1, n_2)$ and $\vec{r} = (x, y)$.

On applying solution (12) to governing Eqs. (7)–(11) and observing that the coefficient of operator D^2 $(= \mathrm{d}^2/\mathrm{d}z^2)$ is a non-singular matrix of order four, the resulting system of equations can be written in matrix form as

$$(D^2 - Q_1 D - R_1)\vec{u}(z) = 0 \text{in the domain } \Omega \tag{13}$$

$$\sigma_c(z, n) = (D - S_1)\vec{u}(z) = \vec{0} \text{on } z = \pm h \tag{14}$$

where

$$Q_1 = \mathrm{i}n\begin{bmatrix} 0 & 0 & 0 & \overline{n}_1(\delta^{-2}-1) \\ 0 & 0 & 0 & \varepsilon c \\ 0 & 0 & 0 & \overline{n}_2(\delta^{-2}-1) \\ \overline{n}_1(1-\delta^2) & -\mathrm{i}\omega^{-1}c & \overline{n}_2(1-\delta^2) & 0 \end{bmatrix},$$

$$R_1 = n^2\begin{bmatrix} \delta^{-2}\overline{n}_1^2+\overline{n}_2^2-\delta^{-2}c^2 & -\mathrm{i}\omega^{-1}\overline{n}_1 c\delta^{-2} & \overline{n}_1\overline{n}_2(\delta^{-2}-1) & 0 \\ \overline{n}_1\varepsilon c & \overline{n}_1^2+\overline{n}_2^2+\mathrm{i}\omega^{-1}c^2 & \varepsilon\overline{n}_2 c & 0 \\ (\delta^{-2}-1)\overline{n}_1\overline{n}_2 & -\mathrm{i}\omega^{-1}\overline{n}_2 c\delta^{-2} & \overline{n}_1^2+\delta^{-2}\overline{n}_2^2-\delta^{-2}c^2 & 0 \\ 0 & 0 & 0 & \delta^2(\overline{n}_1^2+\overline{n}_2^2)-c^2 \end{bmatrix},$$

$$S_1 = \mathrm{i}n\begin{bmatrix} 0 & 0 & 0 & \overline{n}_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{n}_2 \\ (1-2\delta^2)\overline{n}_1 & -\mathrm{i}\omega^{-1}c & (1-2\delta^2)\overline{n}_2 & 0 \end{bmatrix} \tag{15}$$

$\sigma_c(z,n) = [\delta^{-2}\sigma_{xz}\ \ T_{,z}\ \ \delta^{-2}\sigma_{yz}\ \ \sigma_{zz}]^t$ is modified stress vector.

Here $\overline{n}_1 = n_1/n$, $\overline{n}_2 = n_2/n$, $n = |\vec{n}| = \sqrt{n_1^2+n_2^2}$, $\vec{n} = n\hat{n}$, $\hat{n} = (\overline{n}_1, \overline{n}_2)$, $D^m\vec{u}(z) = \mathrm{d}^m\vec{u}(z)/\mathrm{d}z^m$, $\vec{c}(\vec{n}) = \vec{\omega}(n)/n$ is the phase velocity, $c(n) = |\vec{c}(\vec{n})|$ is the phase speed of a traveling wave and $\hat{n}$ is the unit direction vector.

If we consider the waves propagating along x-axis, so that $(\overline{n}_1, \overline{n}_2) = (1,0)$, then from Eqs. (13) and (14), we obtain

$$(D^2 - QD - R)\vec{u}(z) = 0 \text{in the domain } \Omega$$

$$\sigma_c(z,n) = (D-S)\vec{u}(z) = \vec{0}\text{on } \ z = \pm h$$

where

$$Q = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & \delta^{-2}-1 \\ 0 & 0 & 0 & \varepsilon c \\ 0 & 0 & 0 & 0 \\ 1-\delta^2 & -\mathrm{i}\omega^{-1}c & 0 & 0 \end{bmatrix};$$

$$R = \begin{bmatrix} \delta^{-2}(1-c^2) & -\mathrm{i}\omega^{-1}c\delta^{-2} & 0 & 0 \\ \varepsilon c & 1+\mathrm{i}\omega^{-1}c^2 & 0 & 0 \\ 0 & 0 & 1-\delta^{-2}c^2 & 0 \\ 0 & 0 & 0 & \delta^2-c^2 \end{bmatrix};$$

$$S = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-2\delta^2 & -\mathrm{i}\omega^{-1}c & 0 & 0 \end{bmatrix} \tag{16}$$

4. Boundary value problem and its solution

Assuming that $\sigma_c(z,n)$ has a finite asymptotic expansion of the form

$$\sigma_c(z,n) = \sum_{m=0}^{N} \sigma_c^{(m)}(\zeta)\varepsilon_T^m + o(\varepsilon_T^m), \zeta = \frac{z}{h}, \varepsilon_T = nh$$

and approximating it by the partial sum of Taylor series expansions in $z\,(-h<z<h)$, about $z=0$, we obtain

$$\sigma_c(z) = \sum_{m=0}^{N} \sigma_c^{(m)}(0)\frac{z^m}{m!} + o(z^N) \tag{17}$$

where $\sigma_c(z) = (D-S)\vec{u}(z)$, $\sigma_c^{(m)} = \mathrm{d}^m\sigma_c/\mathrm{d}z^m$ and the second argument is omitted for convenience.

In case the surfaces $z = \pm h$ of the plate are subjected to stress free and thermally insulated/boundary conditions, one can write

$$\sigma_c(h) \pm \sigma_c(-h) = 0 \tag{18}$$

Using the asymptotic expansion (17) for $\sigma_c(z)$ in Eq. (18), one arrives at the following asymptotic boundary value problem

$$D^2\vec{u}(z) = (QD+R)\vec{u}(z) \text{ in the domain } \Omega \tag{19}$$

$$(\vec{u}^{(1)}(0) - S(n)\vec{u}(0)) + \frac{h^2}{2}(\vec{u}^{(3)}(0) - S(n)\vec{u}^{(2)}(0)) + \frac{h^4}{24} + (\vec{u}^{(5)}(0) - S(n)\vec{u}^{(4)}(0)) \approx 0 \tag{20}$$

$$(\vec{u}^{(2)}(0) - S(n)\vec{u}^{(1)}(0)) + \frac{h^2}{6}(\vec{u}^{(4)}(0) - S(n)\vec{u}^{(3)}(0)) + \frac{h^4}{24} + (\vec{u}^{(6)}(0) - S(n)\vec{u}^{(5)}(0)) \approx 0 \tag{21}$$

where $u^{(m)} = \mathrm{d}^m u/\mathrm{d}z^m$.

Eqs. (20) and (21) are valid at the thermally insulated and stress free surfaces of the plate.

The successive differentiation of Eq. (19) four times with the help of Eqs. (20) and (21) leads to

$$\left[I + \frac{h^2n^2}{2}A + \frac{h^4n^4}{24}B\right]\vec{u}^{(1)}(0) - n\left[S - \frac{h^2n^2}{2}C - \frac{h^4n^4}{24}N\right]\vec{u}(0) = \vec{0} \tag{22}$$

$$\left[Q - S + \frac{h^2n^2}{6}E + \frac{h^4n^4}{120}F\right]\vec{u}^{(1)}(0) - n\left[R - \frac{h^2n^2}{6}G - \frac{h^4n^4}{120}H\right]\vec{u}(0) = \vec{0} \tag{23}$$

where

$$A = R + (Q-S)Q, B = G + EQ, C = (Q-S)R$$

$$E = C + AQ, F = BQ + N, G = AR, H = BR, N = ER$$

It is noticed that the matrices A and B have block diagonal structures given by

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} = (a_{ij})_{4\times4}, B = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix} = (b_{ij})_{4\times4}$$

where $a_{11} = 1 + 2\gamma - v_s^2$, $a_{12} = -2\mathrm{i}\omega^{-1}\delta v_s$, $a_{21} = \delta^3\varepsilon v_s$, $a_{22} = 1 + \xi v_s^2$, $a_{33} = 1 - v_s^2$, $a_{44} = \delta^2(2 - v_s^2) - 1$, $b_{11} = 1 + 4\gamma - a_1 v_s^2 + v_s^4$, $b_{12} = -4\mathrm{i}\omega^{-1}\delta v_s - a_2 v_s^3$, $b_{21} = a_3 v_s - a_4 v_s^3$, $b_{22} = 1 + a_5 v_s^2 - a_6 v_s^4$, $b_{33} = (1 - v_s^2)^2$, $b_{44} = -(4\gamma + 3) - a_7 v_s^2 + a_8 v_s^4$, $\gamma = 1 - \delta^2$, $\xi = \mathrm{i}\omega^{-1}\delta^2(1-\varepsilon)$, $\xi' = \mathrm{i}\omega^{-1}\delta^2\varepsilon$; $v_s = c/\delta$, $a_1 = 2[2(1+\xi') - \delta^2(\delta^2 + \xi')]$, $a_2 = 2\mathrm{i}\omega^{-1}\delta^2(\xi - \delta^2)$, $a_3 = 2\varepsilon\delta(1 + 2\gamma)$, $a_4 = \delta\varepsilon[2(1-\xi) - \delta^2(\delta^2 - \xi)]$, $a_5 = 2(\xi - \xi')$, $a_6 = \xi^2 - \delta^2\xi'$, $a_7 = 4\delta^2 - 7$, $a_8 = 2(-\gamma + a_9)$, $a_9 = \delta^2(\xi' + \delta^2)$.

The coefficient of $\vec{u}^{(1)}$ in Eq. (22) being a non-singular square matrix of order four and therefore one can obtain the resolving operator from Eqs. (22) and (23) as

$$P\vec{u}(0) = \left(P_0 + P_2\frac{h^2n^2}{6} + P_4\frac{h^4n^4}{120}\right)\vec{u}(0) = \vec{0} \tag{24}$$

where

$$P_0 = R + (Q-S)M^{-1}S$$

$$P_2 = G + EM^{-1}S - 3(Q-S)M^{-1}C$$

$$P_4 = H + FM^{-1}S - 10EM^{-1}C - 5(Q-S)M^{-1}N$$

Here the matrix $M = (m_{ij})_{4\times4}$; $m_{ij} = \delta_{ij} + (h^2n^2/2)a_{ij} + (h^4n^4/24)b_{ij}$; $i,j = 1,2,3,4$ and M^{-1} is its inverse. The matrix of the operator P has in general a block diagonal structure of the form

$$P = diag(P_L, P_{S_1}, P_{S_2})\vec{u} = 0 \tag{25}$$

and thus we have

$$P_L\begin{bmatrix} U \\ \theta \end{bmatrix} = 0, P_{S_1}[V] = 0, P_{S_2}[W] = 0$$

where $P_L = (p_{ij})_{2\times2}$, $P_{S_1} = (p_{33})_{1\times1}$ and $P_{S_2} = (p_{44})_{1\times1}$, respectively, govern the extensional, transversal and flexural in plane motion of the plate.

Eq. (25) has a non-trivial solution if and only if

$$|P| \equiv \det P_L \cdot \det P_{S_1} \cdot \det P_{S_2} = 0 \tag{26}$$

This leads to the secular equations

$$p_{11}p_{22} - p_{12}p_{21} = 0, p_{33} = 0, p_{44} = 0 \tag{27}$$

Eq. (26) is the three-dimensional analog of the Rayleigh–Lamb frequency for a thermoelastic plate. The first, second and third equations in system (27) are the corresponding frequency equations for extensional, transversal and flexural waves motion, respectively. We study only transversal and flexural wave motion one by one in the following sections.

5. Transversal motion of a plate

The second equation in the system of Eqs. (27) governs the transversal vibration, since the operator $P_{S_1} = p_{33}$ affects the displacement v only. According to structure (24) of the operator P, the second equation of (27) has the form

$$v_s^6 - \left(3 + \frac{20}{n^2h^2}\right)v_s^4 + \left(3 + \frac{40}{n^2h^2} + \frac{120}{n^4h^4}\right)v_s^2 - \left(1 + \frac{20}{n^2h^2} + \frac{120}{n^4h^4}\right) = 0 \tag{28}$$

Eq. (28) leads to values of phase velocity as

$$c = \pm\delta,\ \pm\delta\left(1 + \frac{2(5 \pm i\sqrt{5})}{n^2h^2}\right)^{1/2} \tag{29}$$

The substitution $v_s = \omega_s/n$ transforms the velocity Eq. (28) into the frequency equation which is obtained as

$$\omega_s^6 - \left(3n^2 + \frac{20}{h^2}\right)\omega_s^4 + \left(3n^4 + \frac{40n^2}{h^2} + \frac{120}{h^4}\right)\omega_s^2 - \left(3n^6 + \frac{20n^4}{h^2} + \frac{120n^2}{h^4}\right) = 0 \tag{30}$$

where $\omega_s = \omega_s(n)$.

Differentiating Eq. (30) with respect to n, the expression for group velocity $c_g(n) = \mathrm{d}\omega(n)/\mathrm{d}n$ of the transversal motion is $c_g = 1/v_s = n/\omega_s$ in terms of phase velocity v_s and frequency ω_s, respectively, by using the relation $v_s(n)=\omega_s(n)/n$. Eq. (30) is the dispersion equation for the transversal motion of a plate in terms of frequency $\omega_s = \omega/\delta$. Clearly the wave motion under consideration is dispersive in character and attenuating in space in case of complex phase velocity in addition to its dependence on parameters nh and δ.

The substitution $-n^2 \to \nabla^2$, $-\mathrm{i}n_1 \to \partial/\partial x$, $-\mathrm{i}n_2 \to \partial/\partial y$, $\mathrm{i}\omega \to \partial/\partial t$, in Eq. (30) and consequent rearrangements of the terms lead to the following asymptotic differential equation:

$$\left\{120\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) + 20h^2\left(\frac{\partial^4}{\partial t^4} - 2\nabla^2\frac{\partial^2}{\partial t^2} + \nabla^4\right) + h^4\left(\frac{\partial^6}{\partial t^6} - 3\nabla^2\frac{\partial^4}{\partial t^4} + 3\nabla^4\frac{\partial^2}{\partial t^2} - 3\nabla^6\right)\right\}V = 0 \tag{31}$$

Eq. (31) governs the transversal motion of the plate.

5.1. Long and short wavelength waves

The limiting forms of secular Eq. (28) in case of long wave length $(nh \to 0)$ and short wave length $(nh \to \infty)$ approximations are, respectively, obtained as

$$\omega_s^2\left(\omega_s^4 - \frac{20}{h^2}\omega_s^2 + \frac{120}{h^4}\right) = 0$$

$$(v_s^2 - 1)^3 = 0 \tag{32}$$

where $v_s = \omega_s/n$ and $\omega_s = \omega/\delta$. The first equation of Eqs. (32) provides us one trivial roots $(\omega_s = 0)$ and corresponding phase velocity $v_s = 1$. Moreover the group velocity is same as phase velocity and hence the waves are dispersion less in character. The biquadratic under the braces provides us a pair of complex conjugate roots and the corresponding modes of wave propagation are dissipative with time and travels with phase velocity given by $v_s = \mathrm{Re}(\omega_s)/n$ and dissipative constant $D_s = I_m(\omega_s)$. The second equation of (32) leads to root $v_s = 1$ of multiplicity three and the corresponding wave modes are non-dissipative.

6. Flexural motion of a plate

The third equation of system (27) governs the flexural vibrations; because the operator $P_{S_2} = p_{44}$ affects the displacement component w only. According to the structure (24) of the operator P, the third equation in the system of Eqs. (27) has the form

$$f_0 v_s^{10} + g_1 v_s^8 + g_3 v_s^6 + g_6 v_s^4 + g_9 v_s^2 + g_{12} = 0 \tag{33}$$

where $g_1 = f_1 + f_2/n^2h^2$, $g_j = f_j + f_{j+1}/n^2h^2 + f_{j+2}/n^4h^4$, $j = 3i$, $i = 1,2,3,4$ and f_i, i=0–14 are defined in Appendix.

The substitution $v_s=\omega_s/n$ turns the phase velocity Eq. (33) into frequency equation, which is given by

$$f_0\omega_s^{10}+g_1n^2\omega_s^8+g_3n^4\omega_s^6+g_6n^6\omega_s^4+g_9n^8\omega_s^2+n^{10}g_{12}=0 \tag{34}$$

Eq. (34) is the dispersion equation for the flexural wave motion of a plate in terms of ratio $\omega_s=\omega/\delta$.

Adopting the procedure discussed in Section 5, we obtain the group velocity of the flexural motion in terms of phase velocity v_s as given below.

$$c_g=-\frac{1}{v_s}\left(\frac{f_1v_s^8+g_1^*v_s^6+(g_6+g_2^*)v_s^4+(2g_9+g_3^*)v_s^2+(3g_{12}+g_4^*)}{5f_0v_s^8+4g_1v_s^6+3g_3v_s^4+2g_6v_s^2+2g_9}\right) \tag{35}$$

where $g_i^*=2f_j+f_{j+1}/n^2h^2$, $j=3i$, $i=1,2,3,4$.

The group velocity c_g can be expressed in terms of the frequency ω_s from relation (35) by using $v_s(n)=\omega_s(n)/n$.

The asymptotic differential equation for the flexural motion (on applying the procedure outlined in Section 5) of a thermoelastic plate is given by

$$\left\{\begin{array}{l}(f_5\partial_t^6+f_8\nabla^2\partial_t^4+f_{11}\nabla^4\partial_t^2+f_{14}\nabla^6)-h^2(f_2\partial_t^8+f_4\nabla^2\partial_t^6+f_7\partial^4\partial_t^4+f_{10}\nabla^6\partial_t^2+f_{13}\nabla^8)\\ +h^4(f_0\partial_t^{10}+f_1\nabla^2\partial_t^8+f_3\nabla^4\partial_t^6+f_6\nabla^6\partial_t^4+f_9\nabla^8\partial_t^2+f_{12}\nabla^{10}\partial_t^2)\end{array}\right\}W=0 \tag{36}$$

6.1. Long and short wavelength waves

The limiting forms of secular Eq. (34) in case of long wave length $(nh\to 0)$ and short wave length $(nh\to\infty)$ approximations are given by

$$\omega_s^6\left(f_0\omega_s^4+\frac{f_2}{h^2}\omega_s^2+\frac{f_4}{h^4}\right)=0 \tag{37}$$

$$f_0v_s^{10}+f_1v_s^8+f_3v_s^6+f_6v_s^4+f_9v_s^2+f_{12}=0 \tag{38}$$

Clearly Eq. (37) has one trivial root $(\omega_s^2=0)$ of multiplicity three with corresponding phase velocity equal to zero and two non-trivial roots. The solution of Eq. (38) gives the phase velocities v_s of five wave modes (in general complex) as a function of nh, δ and ε and hence the wave motion under consideration is dispersive in character and attenuating in space.

7. Equivalence with thermoelastic Rayleigh–Lamb wave equation

Consider the thermoelastic Rayleigh–Lamb frequency equations for a thermoelastic plate in the context of coupled theory of thermoelasticity [19],

$$\left[\frac{\tan m_1h}{\tan\beta_1h}\right]^{\pm 1}-\frac{m_1(\alpha^2-m_1^2)}{m_3(\alpha^2-m_3^2)}\left[\frac{\tan m_3h}{\tan\beta_1h}\right]^{\pm 1}=\frac{4\beta_1m_1n^2(m_3^2-m_1^2)}{(n^2-\beta_1^2)^2(\alpha^2-m_3^2)} \tag{39}$$

where $\alpha^2=n^2(c^2-1)$, $\beta_1^2=n^2(c^2/\delta^2-1)$, $m_1^2=n^2(a^2c^2-1)$, $m_3^2=n^2(b^2c^2-1)$, $a^2,b^2=\{1+\mathrm{i}\omega^{-1}(1+\varepsilon)\pm\sqrt{[1-\mathrm{i}\omega^{-1}(1-\varepsilon)]^2+4\omega^{-2}\varepsilon}\}/2$.

Here, the positive and negative powers correspond to the antisymmetric and symmetric waves modes, respectively.

In these notations, after expanding all tangent terms into the power series (considering only positive exponent), Eq. (39) takes the form

$$\psi_0+\frac{1}{3}h^2\psi_2+\frac{2}{15}h^4\psi_4+\frac{17}{315}h^6\psi_6+\frac{62}{2835}h^8\psi_8+\cdots=0 \tag{40}$$

where

$$\psi_0(c,\delta)=1+\frac{4n^2\beta_1^2}{(n^2-\beta_1^2)^2}$$

$$\psi_2(c,\delta)=\alpha^2+\frac{4n^2\beta_1^4}{(n^2-\beta_1^2)^2}$$

$$\psi_4(c,\delta)=\alpha^2(m_1^2+m_3^2)-m_1^2m_3^2+\frac{4n^2\beta_1^6}{(n^2-\beta_1^2)^2}$$

$$\psi_6(c,\delta)=\alpha^2(m_1^4+m_1^2m_3^2+m_3^4)-m_1^2m_3^2(m_1^2+m_3^2)+\frac{4n^2\beta_1^8}{(n^2-\beta_1^2)^2}$$

$$\psi_8(c,\delta) = \alpha^2(m_1^4+m_3^4)(m_1^2+m_3^2) - m_1^2m_3^2(m_1^4+m_1^2m_3^2+m_3^4) + \frac{4n^2\beta_1^{10}}{(n^2-\beta_1^2)^2}$$

Upon retaining an appropriate number of terms, the model of any asymptotic order $O(h^N)$ may be obtained analytically in terms of the polynomial dispersion relations. In reality, with reasonable choice of accuracy, the value of N is limited by the associated labor consuming derivation of coefficients and the difficulties with numerical solution of high order polynomial equations due to the extremely high rate of variation in coefficients. On close inspection, it can be established that all coefficients of $\psi_n(c,\delta)$ are completely identical to those of P_{S_2} in operator P of Eq. (24) associated with the flexural motion in the operator plate model, i.e.,

$$\psi_n(c,\delta) \cong P_{S_2}^{(n)}(c,\delta), (n=0,2,4,6,\ldots) \tag{41}$$

This type of motion is governed by the dispersion relation $P_{S_2}(c,\delta)=0$, reproduced here with the corresponding order of approximation

$$v_s^2(P_{S_2}^{(0)}(c,\delta) + \tfrac{1}{3}\eta^2P_{S_2}^{(2)}(c,\delta) + \tfrac{2}{15}\eta^4P_{S_2}^{(4)}(c,\delta) + \tfrac{17}{315}\eta^6P_{S_2}^{(6)}(c,\delta) + \tfrac{62}{2835}\eta^8P_{S_2}^{(8)}(c,\delta) + \cdots) = 0 \tag{42}$$

where $\eta = nh$

$$\psi_0(c,\delta) = v_s^2$$

$$\psi_2(c,\delta) = \delta^2v_s^4 + (3-4\delta^2)v_s^2 - 4(1-\delta^2)$$

$$\psi_4(c,\delta) = A_2\delta^2v_s^6 + 2(2-\delta^2-2\delta^4A_2)v_s^4 + (8\delta^2-11+4\delta^4A_2)v_s^2 + 8(1-\delta^2)$$

$$\psi_6(c,\delta) = A_4\delta^4v_s^8 + (4-3\delta^2A_2-4\delta^6A_4)v_s^6 + (-16+3\delta^2+12\delta^4A_2+4\delta^6A_4)v_s^4 + (23-12\delta^2-12\delta^4A_2)v_s^2 + 12(\delta^2-1)$$

$$\psi_8(c,\delta) = A_6\delta^8v_s^{10} + 4[1-\delta^6(A_4+\delta^2A_6)]v_s^8 + 2[-10+\delta^4(3A_2+8\delta^2A_4+2\delta^4A_6)]v_s^6 + 4[10-\delta^2(1+6\delta^2A_2+4\delta^4A_4)]v_s^4 - [39-8\delta^2(2-3\delta^2A_2)]v_s^2 + 16(1-\delta^2)$$

where $A_2 = a^2(1-b^2)+b^2$, $A_4 = a^2(1-b^2)(a^2+b^2)+b^4$, and $A_6 = a^2(1-b^2)(a^4+a^2b^2+b^4)+b^6$.

The only difference between Eqs. (33) and (42) is in factors out side the braces. Eq. (42) has an extra trivial solution and the identity of expressions inside the braces guarantees the equivalence of non-trivial roots. The frequency spectrum for the antisymmetric motion of a plate is shown in Fig. 10. This has resemblance with that in Fig. 7.

8. Special cases

(a) *Elastic plate under thermal equilibrium*: In the case of uncoupled thermoelasticity (elastic plate), the coefficient of linear thermal expansion vanishes because the elastic and thermal fields are independent of each other so that $\beta=0$ and hence $\varepsilon=0$. The dispersion equation for flexural motion (34) remains similar in nature with the same degree polynomial in v_s, which gives five modes of wave propagation in a homogeneous isotropic elastic plate and is in agreement with Sharma and Kumar [9]. Various coefficients can be obtained by taking $\varepsilon=0$ in relevant relation. The Rayleigh–Lamb frequency Eq. (39) in this case gets reduced to Eq. (1) of Ref. [8].

(b) *Elastic plate under isentropic conditions*: In this case of thermoelasticity, the coefficient of thermal conductivity vanishes, so that $K=0$. The dispersion equation for flexural motion (33) remains as the same degree polynomial in v_s, which gives five modes of wave propagation in a homogenous isotropic plate. The phase and group velocity profiles have been shown in Figs. 2 and 4 with non-dimensional wavenumber.

9. Numerical result and discussion

For the purpose of numerical illustrations we consider the case of the transversal and flexural modes of wave propagation in an infinite homogenous isotropic thermoelastic plate of Aluminum-epoxy material, the physical data of which are given by [19]

$$\varepsilon = 0.073, \lambda = 7.59\times10^{10}\,\mathrm{N\,m^{-2}}, \mu = 1.89\times10^{10}\,\mathrm{N\,m^{-2}}, \rho = 2.19\times10^3\,\mathrm{Kg\,m^{-3}}$$

$$K = 2.508\,\mathrm{K\,m^{-1}\,s^{-1}\,{}^\circ C^{-1}}, C_e = 961.4\,\mathrm{J\,Kg^{-1}\,{}^\circ C^{-1}},$$

If we write

$$c^{-1} = V^{-1} + \mathrm{i}\omega^{-1}Q \tag{43}$$

So that $n = R + \mathrm{i}Q$, where $R=\omega/V$, V, Q, ω are real numbers.

The secular Eqs. (28) and (33) are in general complex polynomial equations and hence provide us complex phase velocities of transversal and flexural motions, respectively, at first instant. The real phase speeds (V_i) and attenuation

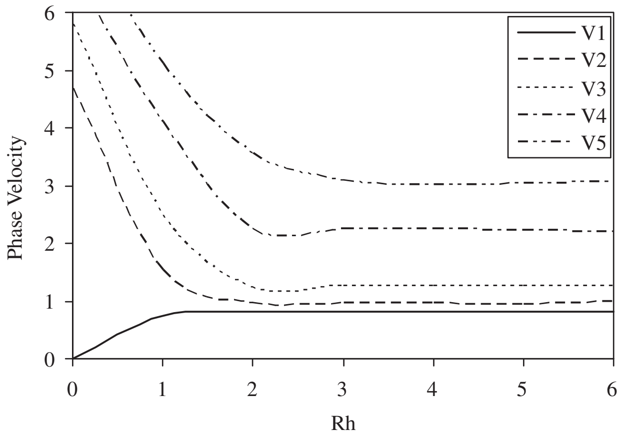

Fig. 1. Variations of phase velocity of flexural modes in isothermal elastic plate with *Rh*.

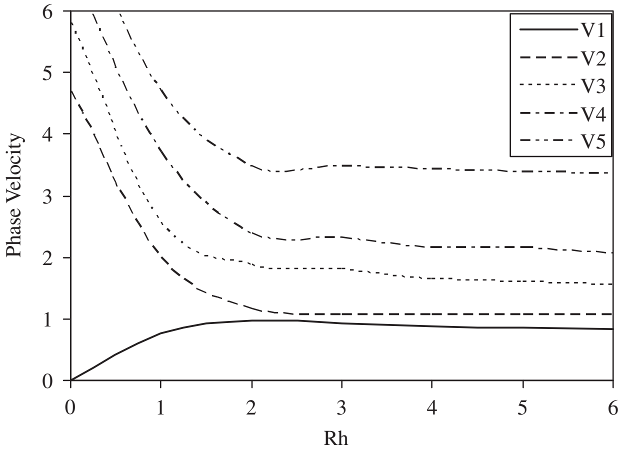

Fig. 2. Variations of phase velocity of flexural modes in non-conducting (isentropic) elastic plate with *Rh*.

coefficients (Q_i) can be obtained from the complex values of the phase velocity (c_i) by using representation (43). Graeffe's root squaring method [26, p. 58] has been used to solve the secular Eqs. (28) and (33). This method (i) requires no prior information about the roots of an equation; (ii) is capable of giving all the roots and hence has advantage over the other methods. The dispersion curves for phase velocity and attenuation coefficient for all modes of transversal and flexural motions have been computed from the secular Eqs. (28) and (33) with the help of MATHCAD software and are plotted graphically in Figs. 1–9 in case of elastic and thermoelastic plates. The phase velocity for Rayleigh–Lamb frequency Eq. (39) has been obtained by using FORTRAN programming and is illustrated graphically in Figs. 10 and 11. All the modes are found to be dispersive in character.

According to Sharma et al. [27] at low frequencies mechanical energy transfer is more effective than thermal conduction and that conditions locally are therefore nearly isentropic (constant entropy); whereas at high frequencies, thermal energy transfer is a more predominant process and the prevailing conditions are nearly isothermal. Thus at low frequency limits the wave like modes are identified with the small amplitude waves in elastic material that does not conduct heat and may be regarded as inherent in the classical elastodynamics derived strictly from mechanical principles. However, here we have analyzed flexural wave motion under isentropic conditions as well as at isothermal one for completion purpose in Figs. 1, 2 and 3, 4, respectively. Figs. 1 and 2 show the variations of phase velocities ($V_i,\ i=1,2,3,4,5$) of various flexural modes versus non-dimensional wavenumber (Rh) in an elastic plate under isothermal and isentropic conditions. It is noticed that phase velocity of fundamental mode (V_1) starts from zero value at vanishing wavenumber ($Rh\to 0$), increases steadily in the wavenumber range $0\le Rh\le 1$ to ultimately become asymptotically close to Rayleigh wave velocity for $Rh\ge 2$. The profiles of all other modes observe cut-off frequencies at extremely small wavenumbers which decrease monotonically in the interval $0\le Rh\le 2$ to become almost dispersionless and asymptotically close to that of shear wave modes for $Rh\ge 2$. While the isentropic and isothermal values of phase velocity of fundamental mode (V_1) are comparably equal in the interval $0\le Rh\le 1$ but former is grater than latter for $Rh\ge 1$. The isentropic phase velocities V_2 and V_3 have small magnitudes in the range

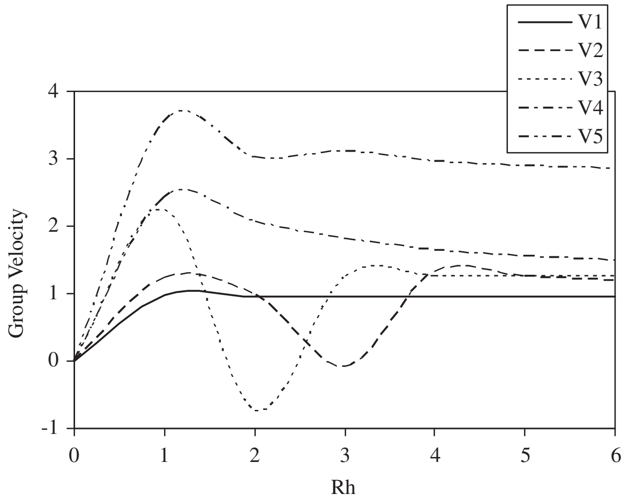

Fig. 3. Variations of group velocity of flexural modes in isothermal elastic plane with *Rh*.

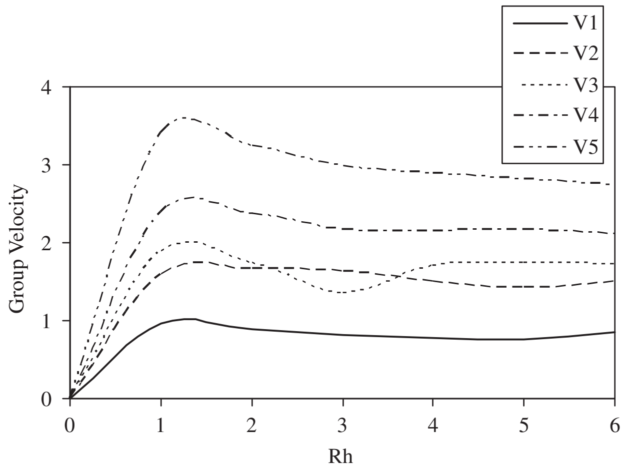

Fig. 4. Variations of group velocity of flexural modes in non-conducting (isentropic) thermoelastic plate with *Rh*.

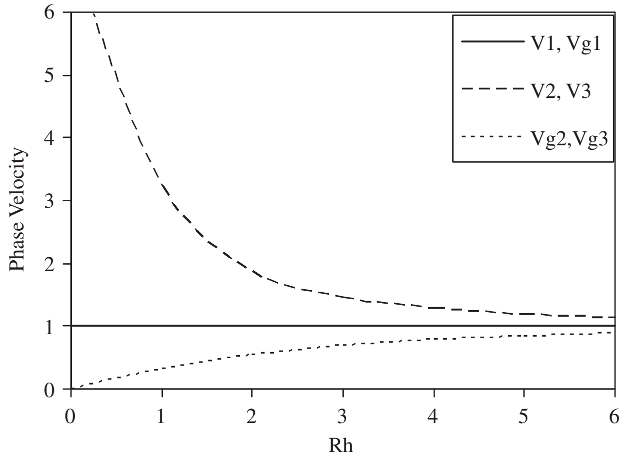

Fig. 5. Variations of phase velocity of transversal modes in thermoelastic plate with *Rh*.

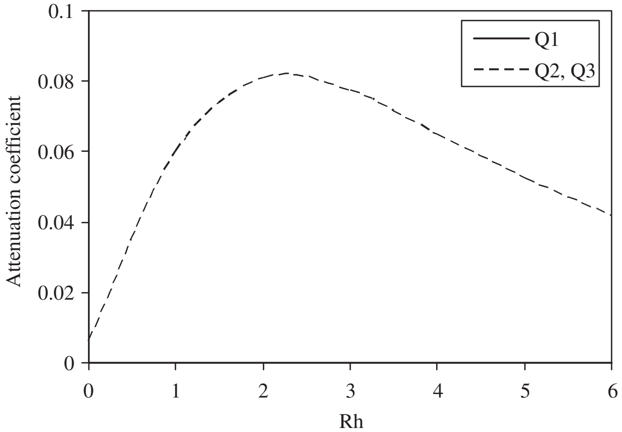

Fig. 6. Variations of attenuation coefficient of transversal modes in thermoelastic plate with *Rh*.

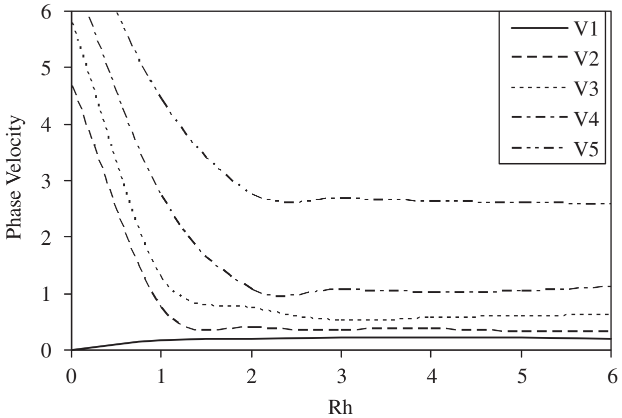

Fig. 7. Variations of phase velocity of flexural modes in thermoelastic plate with *Rh*.

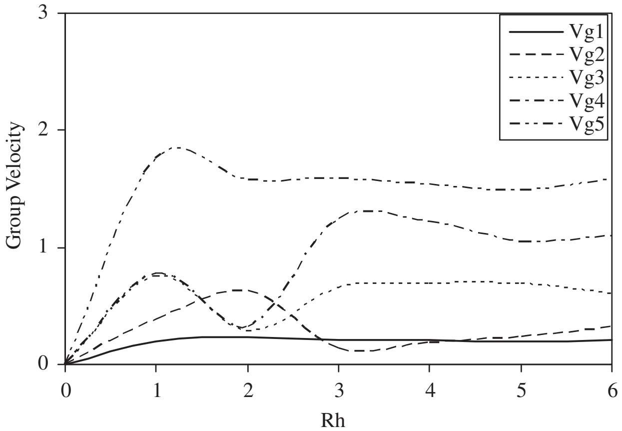

Fig. 8. Variations of group velocity of flexural modes in thermoelastic plate with *Rh*.

$0 \leq Rh \leq 1$ as compared to their isothermal values; however, their trends get reversed for $Rh > 1$. Moreover, the trend of variations of the magnitude of isentropic and isothermal phase velocity V_4 is exactly opposite to that of V_2 and V_3 at all wavenumbers. The magnitude of isothermal value of phase velocity V_5 is greater than that of its isothermal value at all wavenumbers.

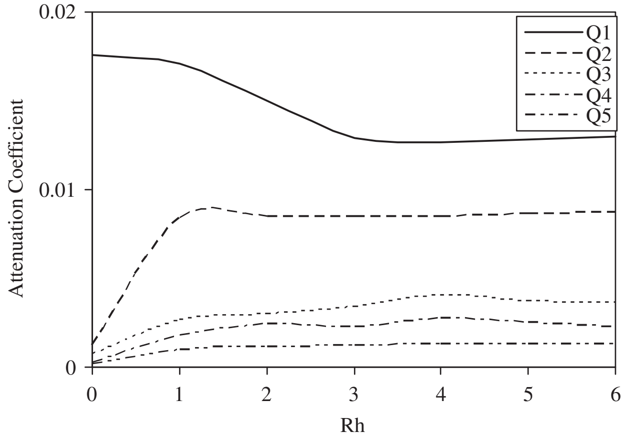

Fig. 9. Variations of attenuation coefficient flexural modes in thermoelastic plate with *Rh*.

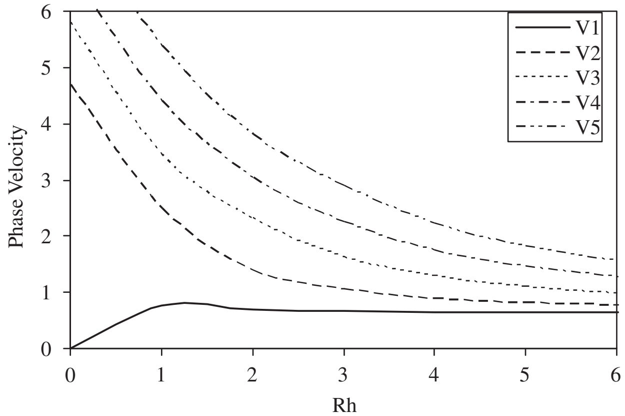

Fig. 10. Variations of phase velocity of flexural modes in thermoelastic plate with *Rh*. (Rayleigh–Lamb type equation).

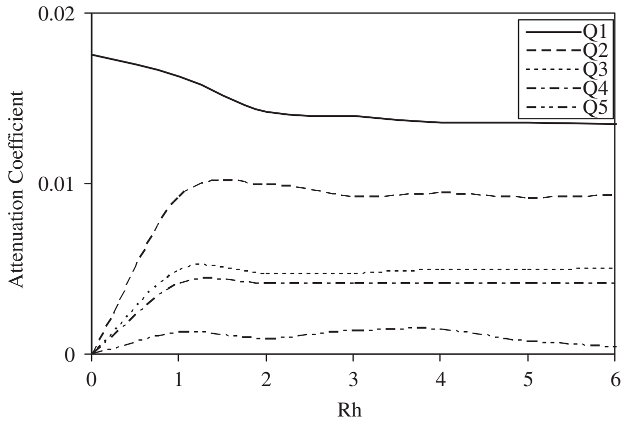

Fig. 11. Variations of attenuation coefficient of flexural modes in thermoelastic plate with *Rh*. (Rayleigh–Lamb type equation).

Figs. 3 and 4 present the variations of group velocities ($V_{g_i},\ i = 1, 2, 3, 4, 5$) of flexural modes in an elastic plate, both at isentropic and isothermal conditions, versus non-dimensional wavenumber, respectively. The isentropic and isothermal values of group velocities of various modes are found to be zero at vanishing wavenumbers which correspond to the condition of zero energy transmission in horizontal direction. The profiles of group velocity, both isentropic and isothermal, increase monotonically in the interval $0 \leq Rh \leq 1$ and tend to phase velocity profiles of respective modes at high frequency limits for all modes. The appearance of the negative group velocity V_{g_2} and V_{g_3} in the intervals $1.75 \le Rh \le 2.75$ and $2.75 \le Rh \le 3.75$ is also noticed in case of isothermal elastic plate which agrees with Tolstoy and Usdin [28]. In these intervals, the group velocity is in opposite direction to that of phase velocity and energy transportation took place in a direction opposite to that of the wave vector.

It is observed from Fig. 5 that the phase velocity profiles of transversal wave modes V_2 and V_3 is coincident and decrease monotonically in the wavenumber range $0 \le Rh \le 2$ and then vary steadily almost in linear manner to become asymptotic to shear horizontal modes. The profiles of fundamental mode (V_1) observe linear variations and remain dispersionless with magnitude equal to δ at all values of wavenumber (Rh). It has also been observed that phase velocity profiles of all the three modes become asymptotically close to that of shear horizontal wave velocity with magnitude δ. The phase velocity of fundamental mode (V_1) is dispersionless and hence its group and phase velocities coincide with each other. The group velocity profiles V_{g_2} and V_{g_3} is also coincident and increase monotonically from zero value at vanishing wavenumber to become asymptotically close to phase velocities of respective modes. The phase velocity of second and third modes of transversal motion being complex lead to attenuating nature and the variations of whose attenuation coefficients Q_2 and Q_3 are plotted in Fig. 6 versus wavenumber. The profiles of these attenuation coefficients are also coincident which increases in the range $0 \le Rh \le 2$ before it starts decreasing steadily with increasing wavenumber afterwards to ultimately get stabilized.

Fig. 7 shows the phase velocity profiles of flexural wave motion for isotropic thermoelastic plate versus wavenumber. The trends of variations of phase velocity profiles $(V_i,\ i = 1, 2, 3, 4, 5)$ in thermoelastic plate are similar to that in the elastic plate as can be observed from Figs. 1, 2 and 7, but with the exception of significant modifications in their magnitude due to thermal effects at all values of wavenumber (Rh). It is noticed that, the magnitude of phase velocities of thermoelastic plate have small values as compared to their respective values in elastic plate, both isothermal and isentropic, at all values of wavenumber (Rh). Because the roots of secular Eq. (33) are complex for all considered values of wavenumber (Rh), therefore the wave modes are attenuated in space. The phase speed profile of fundamental mode (V_1) increases from a zero value as $Rh \to 0$ in the wavenumber range $0 \le Rh \le 1$, which varies steadily thereafter to become asymptotically close to thermoelastic Rayleigh wave velocity at extremely large wavenumbers $(Rh \to \infty)$. This is attributed to the fact that Ref. [21] a finite thickness plate appears to be half-space in such situations and the vibration energy is mainly transmitted through the surface of the plate. The free surfaces admit a Rayleigh-type surface wave with complex wavenumber and hence phase velocity. Consequently, the surface wave propagates with attenuation due to the radiation of energy into the medium. This radiated energy will be reflected back by the lower and upper surfaces. Thus the attenuated surface wave on the free surface is enhanced by this reflected energy to form a propagation wave. In fact, the multiple reflections between the upper and lower surfaces of the plate form caustics at one of the free surface and a strong stress concentration arises due to which wave field becomes unbounded in the limit $h \to \infty$. It has also been observed that as the thickness of the plate increases, the phase velocity decreases. This can be explained by the fact that as the thickness of the plate increases, the coupling effect of various interacting fields also increases resulting in lower phase velocity. It can also be observed that the Rayleigh wave velocity is reached at lower wavenumber as the thickness increases, because the transportation of energy mainly takes place in the neighborhood of the free surfaces of the plate in this case. The magnitude of velocity of this mode is noticed to be one-half to that in the elastic plate. The behavior and trends of the variations of the profiles of higher modes $(V_i,\ i = 2, 3, 4, 5)$ are more or less similar to that of their counterparts in elastic plate with the exception that their magnitudes get significantly reduced in the thermoelastic plate due to thermal variations. Fig. 8 shows group velocity profiles of all the modes of wave propagation in thermoelastic plate versus wavenumber. It is observed that the group velocity of all the modes get reduced at least 50 percent to that in elastic plate and no negative values of group velocity is noticed for any mode in the thermoelastic plate. The variations of attenuation coefficients $(Q_i,\ i = 1, 2, 3, 4, 5)$ versus wavenumber have been plotted in Fig. 9. It is revealed that the profiles of attenuation coefficients $(Q_i,\ i = 2, 3, 4, 5)$ in contrast to that of Q_1 increase from small values close to almost zero at vanishing wavenumber in the wavenumber range $0 \le Rh \le 1$ to become steady and stable thereafter for $Rh \ge 1$. The profile of attenuation coefficient Q_1 has a finite non-zero value which is significantly large as compared to that of $(Q_i,\ i = 2, 3, 4, 5)$ at vanishing wavenumbers. This decreases in the range $0 \le Rh \le 3$ and becomes steady and stable for $Rh \ge 3$. Figs. 10 and 11 show the variations of phase velocity and attenuation coefficient profiles of first five flexural modes versus wavenumber, which are obtained from Rayleigh–Lamb frequency equation for thermoelastic plate [19], respectively. It is observed that all modes are approximated very well and practically coincide with their counterparts in Figs. 7 and 9 obtained by employing operator plate model here except small variations in the magnitude of these quantities, especially phase velocity. The comparison of various profiles of phase velocity and attenuation in Figs. 7, 10 and Figs. 9, 11 reveal that operator method approximates thin and thick plate situations more effectively and accurately than the other approaches.

10. Conclusions

The asymptotic operator plate model for free vibrations; both flexural and transversal, in a homogenous thermoelastic plate leads to fifth degree and cubic polynomial secular equations, respectively, that governs frequency and phase velocity of various possible modes of wave propagation at all wavelengths. The infinite power series expansions of classical thermoelastic Rayleigh–Lamb frequency equation and secular equations obtained with operator plate model are found to be in close agreement up to approximations of order $o(\eta^{10})$. It is also observed that in case of both techniques the non-trivial roots are almost same. The negative values of group velocity of some modes are noticed in elastic plate at isothermal conditions, which is in agreement with Tolstoy and Usdin [28]. Phase velocity of fundamental flexural wave modes in thermoelastic and elastic, both isentropic and isothermal, plates', respectively, approach to thermoelastic Rayleigh wave and classical Rayleigh wave velocity, respectively, at large wavenumbers ($Rh \to \infty$). Phase and group velocity profiles of transversal wave modes V_2 and V_3 approach to that of shear horizontal mode. However, the fundamental mode V_1 remains dispersionless at all wavelengths in this case. The group velocity of all the wave modes in thermoelastic and elastic, both isentropic and isothermal, plates approach to phase velocity of respective mode at short wavelengths. The phase and group velocities have same magnitudes in case of non-dispersive wave modes. The thermal variations result in the reduction of phase and group velocities of the wave modes in addition to their attenuating character. Operator plate model approximates thin and thick plate structures more accurately than the other methods.

Acknowledgment

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Appendix

The various coefficients in Eq. (33) are obtained as follows:

$$f_0 = \delta^2(a_8 b_{10} - 10\xi\delta^2)$$

$$f_1 = a_6 b_3 + a_2 b_6 + a_4 b_8 + \delta^2[\alpha_8 b_{10} + a_9 b_9 + 10\delta^2\xi'(3a_1 - 2)]$$

$$f_2 = 20\delta^2[3a_9(1-\xi) + \delta^2(b_{10} + 90\xi')]$$

$$f_3 = 4\mathrm{i}\omega^{-1} b_6\delta - (a_5 b_3 + a_6 b_2 + a_2 b_5 + a_4 b_7 + a_3 b_8) - \delta^2[\alpha_7 b_{10} + \alpha_8 b_9 + (8+16\gamma)a_9 + 10\delta^2\xi'(2 + 12\gamma + a_1)]$$

$$f_4 = 20\delta^2[3\alpha_8(1-\xi) + 6(1+\gamma)a_9 + (4-5\gamma)a_6 + (1-2\gamma)b_{10} - \delta^2\{\beta_9 + 3\delta\varepsilon a_4 - 30(1+6\gamma)\delta^2\xi'\}] + 60[\xi b_3 + (1+\gamma)\delta\varepsilon b_8]$$

$$f_5 = 120[a_9 + b_{10} + 10\delta^2(1 - \xi + 3\xi')]$$

$$f_6 = \delta^2[\alpha_7 b_9 + 2(1+2\gamma)\alpha_8 + a_7 b_{10} - 10\delta^2\xi'\{a_1 - 2(1+4\gamma)\}] - 4\mathrm{i}\omega^{-1} b_5\delta - b_3 - (a_6 b_1 - a_5 b_2 + a_2 b_4 + a_4 b_6 - a_3 b_7)$$

$$f_7 = 20\delta^2[3(1-\xi)\alpha_7 + 6(1+\gamma)\alpha_8 + (1-2\gamma)\beta_9 - (4+5\gamma)a_5 + \delta^2(8 + 16\gamma + b_9 + \delta^2 a_6/5)] - 60[b_3 - b_2\xi + \varepsilon\delta\{\delta\alpha_3 - (1+\gamma)b_7\} + 10(3+4\gamma)\xi'\delta^4] - 40\mathrm{i}\omega^{-1}\delta[3b_5 - \delta^2(a_2 - 2\mathrm{i}\omega^{-1}\delta)]$$

$$f_8 = 120\delta^2[\alpha_8 + \beta_9 + a_6 - b_3\delta^{-2} + 10\delta^2\{3(1+\gamma)\delta^2\varepsilon^2 + 2(2 + \gamma + \xi - \xi' + \omega^{-2})\}]$$

$$f_9 = b_2 + a_5 b_1 + a_3 b_6 - 4\mathrm{i}\omega^{-1} b_4\delta - \delta^2[a_7 b_9 + 8(1+2\gamma)\alpha_7 + 10(1+4\gamma)\delta^2\xi'],$$

$$f_{10} = 60[b_2 + b_1\xi + \varepsilon\delta(1+\gamma)b_6] - 20\delta^2[(5\gamma - 4) + 6(1+\gamma)\alpha_7 + (1-2\gamma)\beta_9 + 3(1-\xi)a_7 - 3\varepsilon\delta^3 a_3 + 2\delta^2\{2a_5 + (1+2\gamma)(4 + 15\xi')\}] - 40\mathrm{i}\omega^{-1}\delta(3b_4 - 4\mathrm{i}\omega^{-1}\delta^3)$$

$$f_{11} = 120\delta^2[-8(1+2\gamma) - (\alpha_7 + a_5 + b_9 - b_2\delta^{-2}) + 10(1-2\gamma)\xi - 10\delta^2\{5 + 6\gamma - \xi' + 4(\xi + \omega^{-2}) - 3\delta^2\varepsilon^2(1+\gamma)\}]$$

$$f_{12} = -8\delta^2(3\delta^4 - 6\delta^2 + 4)$$

$$f_{13} = 40\delta^2(32\delta^4 - 81\delta^2 + 30)$$

$$f_{14} = 480\delta^2(5 - 12\delta^2)$$

$$\alpha_i = a_i + a_{i+1}$$

$$\beta_i = b_i + b_{i+1}, i = 7, 8, 9$$

$$b_1 = -3 + (3+4\gamma)\gamma + (53 - 100\gamma + 40\delta^4)\delta^2$$

$$b_2 = \gamma a_7 + [-68\gamma + \xi'(1+5\delta^2) + 10(3+5\gamma)\delta^2]\delta^2$$

$$b_3 = \gamma a_8 + [23\gamma - \xi'(1+10\delta^2) - 2(8+5\gamma)\delta^2]\delta^2$$

$$b_4 = -\mathrm{i}\omega^{-1}\delta[3-(3+4\gamma)+20(1-2\delta^2)\delta^2]$$

$$b_5 = \mathrm{i}\omega^{-1}\delta[a_7 + 30\delta^2(1-2\delta^2) - \mathrm{i}\omega^{-1}\delta^4]$$

$$b_6 = -10\varepsilon\delta^3[\gamma - 2(1+\gamma)\delta^2]$$

$$b_7 = 5\varepsilon\delta^3[2\gamma - (7+5\gamma-\xi)\delta^2]$$

$$b_8 = -5\varepsilon\delta^5(3+\gamma-\xi)$$

$$b_9 = 6 - a_1 + 2(\xi' + 2\xi) - 12\mathrm{i}\omega^{-1}\delta^2(2-\delta^2)$$

$$b_{10} = 1 - 6\xi - a_6$$

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