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Transient torsional vibration responses of finite, semi-infinite and infinite hollow cylinders

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ABSTRACT

Torsional guided waves are often used to detect the defects in a hollow cylinder. To realize the excitation of the torsional guided waves with high efficiency, the transient vibration responses of finite, semi-infinite and infinite hollow cylinders to external torsional forces must be clarified theoretically. In this study, the method of eigenfunction expansion is employed to solve the above problems. The exact analytical solutions derived by this method are not only explicit but also concise. Furthermore, the analytical solution of the transient torsional vibration of the finite hollow cylinder is numerically evaluated. The results obtained agree well with those simulated by the finite element method.

1. Introduction

Ultrasonic guided waves have a great potential to be employed as an NDE technique for pipelines because they can inspect an area simultaneously without detaching all the insulations from the outer wall of pipe. According to the study of Gazis [1], we know that there are numerous torsional, longitudinal and flexual guided wave modes propagating in a hollow cylinder axially. And they are designated by the symbols $T(0,m)$, $L(0,m)$ and $F(n,m)$ $(n,m=1,2,\ldots)$, respectively [2,3]. Generally, some single pure mode is intended to be excited for defect detection. The waves reflected from the defect may be too complex to analyze if multi-guided wave modes are excited. The usage of $L(0,2)$ mode was suggested by Lowe et al. [4–6] because it is the fastest mode in a weakly dispersive region of frequency and sensitive to circumferential defects. Unfortunately, this mode is insensitive to axial defects. In recent years, Demma et al. [7] and Kwun et al. [8] suggested the usage of the torsional wave mode $T(0,1)$ which is more sensitive to the axial defects than the $L(0,2)$ mode.

Though many apparatus have been developed to excite torsional wave modes [9,10], the exact analytical solution of the transient vibration responses of the hollow cylinder to external torsional forces has not been obtained yet. Soldatos [11] pointed out that exact dynamic analyses of elastic solids can provide valuable, accurate information in cases that, dealing with certain important mechanical properties of them, corresponding predictions based on approximate modelings are not satisfactory. Generally, two fundamental methods, the integral transform and eigenfunction expansion techniques, are employed to study the exact transient responses of the elastic solids. Folk et al. [12] used a double integral transform method to solve a problem of longitudinal strain propagation produced by the sudden application of a pressure to the end of a semi-infinite solid circular cylinder. Pan et al. [13] solved the three-dimensional transient response problem of an infinite solid cylinder by the method of integral transform. The response solutions obtained by the integral transform

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technique are complex and difficult for numerical evaluation. Though the problems of transient torsional responses of semi-infinite and infinite hollow cylinders can also be solved by the technique of integral transform, the eigenfunction expansion method is employed to deal with the above problems in this study. The method of eigenfunction expansion for three-dimensional elastodynamic problems with traction and displacement boundary conditions was developed by Reismann [14]. And it was described in detail by Eringen and Suhubi [15]. Tang and Cheng [16] extended it to deal with elastodynamic problems with mixed boundary conditions. Pao [17] thought that the eigenfunction expansion method is one of the most elegant methods for solving elastodynamic problems because the formula obtained by it is not only concise but also particularly suitable for analyzing the influence of body and surface forces on the transient elastodynamic responses. Tang and Cheng [18] employed the eigenfunction expansion method to obtain the three-dimensional transient responses solution of the finite hollow cylinder with rigid-smooth end boundary conditions. Then the three-dimensional transient response solution of the infinite one is derived based on the above solution. Note that the rigid-smooth boundary conditions are not of great practical importance. Up to now, the exact analytical solutions of the three-dimensional transient responses of the finite and semi-infinite hollow cylinders to arbitrary external forces have not been obtained.

The successful application of the eigenfunction expansion method in the elastodynamic problems depends on the finding of corresponding eigenfunctions. In this study, the torsional vibration eigenfunctions of a finite hollow cylinder with traction-free end and lateral boundaries are derived by the technique of variable separation. Then the eigenfunction expansion method is introduced to obtain the exact transient response of it to torsional surface and body forces. Furthermore, the transient torsional response solutions of semi-infinite and infinite hollow cylinders are derived based on the above solution. And the transient torsional response solution of the finite hollow cylinder is numerically evaluated. The results obtained agree very well with those simulated by the finite element method.

2. Transient vibration responses of a finite hollow cylinder, the ends of which are located at $z=0$ and $2l$, to axisymmetric torsional forces

2.1. Statement of the problem

When a finite hollow cylinder, the ends of which are located at $z=0$ and $2l$, is subjected to homogeneous axisymmetric torsional loads, as shown in Fig. 1, longitudinal and flexual waves cannot be excited, and its motion is governed by

$$\frac{1}{c_T^2}\frac{\partial^2 u_\theta}{\partial t^2}-f(r,z,t)=\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial z^2},\quad r\in[a,b],\ z\in[0,2l], \tag{1}$$

where c_T is the equivoluminal wave velocity, f is the density of the body force, a and b are inner and outer radii, respectively, and u_θ is the circumferential displacement. The solution of equation (1) must satisfy the boundary conditions

$$\sigma_{r\theta}|_{r=a}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\Big|_{r=a}=s_i(z,t),\quad z\in[0,2l], \tag{2}$$

$$\sigma_{r\theta}|_{r=b}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\Big|_{r=b}=s_o(z,t),\quad z\in[0,2l], \tag{3}$$

$$\sigma_{z\theta}|_{z=0}=\mu\frac{\partial u_\theta}{\partial z}\Big|_{z=0}=s_l(r,t),\quad r\in[a,b], \tag{4}$$

$$\sigma_{z\theta}|_{z=2l}=\mu\frac{\partial u_\theta}{\partial z}\Big|_{z=2l}=s_r(r,t),\quad r\in[a,b], \tag{5}$$

where $\sigma_{r\theta}$ and $\sigma_{z\theta}$ are two stress components, μ is Lamé's elastic constant, s_i and s_o are the densities of the forces applied on the inner and outer surfaces of the hollow cylinder, respectively, and s_l and s_r are the densities of the forces applied on the

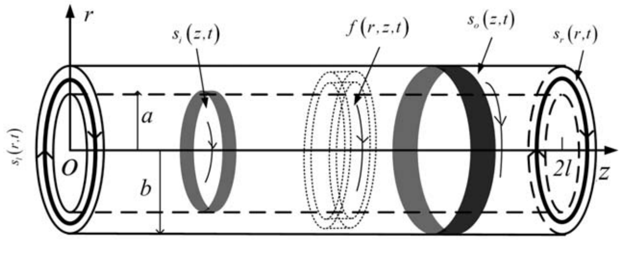

Fig. 1. A finite hollow cylinder with inner radius a and outer radius b, two ends of which are located at $z=0$ and $2l$. It is subjected to the torsional body force with density $f(r,z,t)$. The left and right end boundaries of it are subjected to torsional surface forces $s_l(r,t)$ and $s_r(r,t)$, respectively. The inner and outer lateral boundaries of it are subjected to torsional surface forces $s_i(z,t)$ and $s_o(z,t)$, respectively.

left and right ends of it, respectively. Furthermore, the solution of equation (1) must satisfy the initial conditions

$$u_\theta|_{t=0}=0,\quad \left.\frac{\partial u_\theta}{\partial t}\right|_{t=0}=0,\quad r\in[a,b],\;\; z\in[0,2l]. \tag{6}$$

2.2. Torsional vibration eigenfunctions

The eigenvalue problem corresponding to Eqs. (1)–(6) is formulated as

$$\frac{\partial^2 u^\theta_{mk}}{\partial r^2}+\frac{1}{r}\frac{\partial u^\theta_{mk}}{\partial r}-\frac{u^\theta_{mk}}{r^2}+\frac{\partial^2 u^\theta_{mk}}{\partial z^2}+\frac{\omega^2_{mk}}{c_T^2}u^\theta_{mk}=0,\quad r\in[a,b],\;\; z\in[0,2l], \tag{7}$$

$$\left.\left(\frac{\partial u^\theta_{mk}}{\partial r}-\frac{u^\theta_{mk}}{r}\right)\right|_{r=a,b}=0,\quad z\in[0,2l], \tag{8}$$

and

$$\left.\frac{\partial u^\theta_{mk}}{\partial z}\right|_{z=0,2l}=0,\quad r\in[a,b]. \tag{9}$$

Setting

$$u^\theta_{mk}=R_{mk}(r)Z_{mk}(z), \tag{10}$$

then substituting Eq. (10) into Eq. (7), we have

$$Z''_{mk}+\xi^2_{mk}Z_{mk}=0 \tag{11}$$

and

$$R''_{mk}+\frac{R'_{mk}}{r}+\left(\frac{\omega^2_{mk}}{c_T^2}-\xi^2_{mk}-\frac{1}{r^2}\right)R''_{mk}=0, \tag{12}$$

where m and k denote the orders of discrete angular frequency ω and wavenumber ξ, respectively. Solving Eq. (11), we obtain

$$Z_{mk}(z)=A_{mk}\cos\xi_{mk}z+B_{mk}\sin\xi_{mk}z, \tag{13}$$

where A_{mk} and B_{mk} are arbitrary constants. Now, we discuss the solution of equation (12) for three different cases.

Case I: $\omega^2_{mk}/c_T^2<\xi^2_{mk}$. The solution of Eq. (12) is

$$R_{mk}(r)=C_{mk}I_1(\beta_{mk}r)+D_{mk}K_1(\beta_{mk}r), \tag{14}$$

where

$$\beta^2_{mk}=\xi^2_{mk}-\frac{\omega^2_{mk}}{c_T^2}, \tag{15}$$

C_{mk} and D_{mk} are arbitrary constants, I_1 is the first-order modified Bessel function of first kind, and K_1 is the first-order modified Bessel function of second kind. Substituting Eq. (14) into Eq. (8), we obtain

$$-\frac{1}{a}C_{mk}I_2(\beta a)-\frac{1}{a}D_{mk}K_2(\beta a)=0 \tag{16}$$

and

$$-\frac{1}{b}C_{mk}I_2(\beta b)-\frac{1}{b}D_{mk}K_2(\beta b)=0. \tag{17}$$

Eqs. (16) and (17) form a system of equations with variables C_{mk} and D_{mk}. The coefficient determinant of it is

$$\Delta=\frac{1}{ab}\begin{vmatrix} I_2(\beta a) & K_2(\beta a)\\ I_2(\beta b) & K_2(\beta b)\end{vmatrix}=\frac{1}{ab}[I_2(\beta a)K_2(\beta b)-I_2(\beta b)K_2(\beta a)]. \tag{18}$$

It is well known that $I_2(x)$, i.e., the second-order modified Bessel function of first kind, is a monotonic increasing function, $K_2(x)$, i.e., the second-order modified Bessel function of second kind, is a monotonic decreasing function, and both of them are positive when $x>0$. Then we have

$$0<I_2(\beta a)<I_2(\beta b) \tag{19}$$

and

$$K_2(\beta a)>K_2(\beta b)>0, \tag{20}$$

because $\beta b > \beta a > 0$. Eqs. (19) and (20) lead to

$$I_2(\beta a)K_2(\beta b) - I_2(\beta b)K_2(\beta a) < 0. \tag{21}$$

We know from Eqs. (18) and (21) that

$$\Delta < 0, \tag{22}$$

which means that the system of equations formed by Eqs. (16) and (17) has no non-zero solutions, i.e.,

$$C_{mk} = 0, \quad D_{mk} = 0. \tag{23}$$

Case II: $\omega_{mk}^2/c_T^2 = \xi_{mk}^2$. Eq. (12) becomes an Euler equation and its solution is

$$R_{mk}(r) = \frac{C_{mk}}{r} + D_{mk}r. \tag{24}$$

Substitution of Eq. (24) into Eq. (8) leads to

$$C_{mk} = 0. \tag{25}$$

Then we obtain after substituting Eq. (25) into Eq. (24) that

$$R_{mk}(r) = D_{mk}r. \tag{26}$$

Case III: $\omega_{mk}^2/c_T^2 > \xi_{mk}^2$. The solution of Eq. (12) is

$$R_{mk}(r) = C_{mk}J_1(\beta_{mk}r) + D_{mk}Y_1(\beta_{mk}r), \tag{27}$$

where

$$\beta_{mk}^2 = \frac{\omega_{mk}^2}{c_T^2} - \xi_{mk}^2, \tag{28}$$

J_1 and Y_1 are the first-order Bessel and Neumann functions, respectively.

Substituting Eq. (13) into Eq. (10), then into Eq. (9) and setting $z = 0$, we obtain

$$B_{mk} = 0. \tag{29}$$

Then we know from Eqs. (10), (13), (26)–(29) that

$$u_{mk}^{\theta} = \begin{cases} D_{mk}r\cos\xi_{mk}z, & \omega_{mk}^2/c_T^2 = \xi_{mk}^2, \\ [C_{mk}J_1(\beta_{mk}r) + D_{mk}Y_1(\beta_{mk}r)]\cos\xi_{mk}z, & \omega_{mk}^2/c_T^2 > \xi_{mk}^2. \end{cases} \tag{30}$$

Substituting Eq. (30) into Eq. (9) and setting $z = 2l$, we have

$$\xi_{mk} = \frac{k\pi}{2l}, \quad k = 1, 2, 3, \ldots\ . \tag{31}$$

Eq. (30) is just the torsional vibration eigenfunctions of the finite hollow cylinder, the ends of which are located at $z = 0$ and $2l$. And it is easy to prove that the eigenfunctions form an orthogonal set [15,19], then we have

$$\int_a^b \int_0^{2l} \rho u_{mk}^{\theta} u_{ij}^{\theta} r\,\mathrm{d}r\,\mathrm{d}z = \begin{cases} 0, & m \neq i \text{ or } k \neq j, \\ M_{mk}, & m = i \text{ and } k = j, \end{cases} \tag{32}$$

where ρ is the density of the cylinder, and

$$M_{mk} = \int_a^b \int_0^{2l} \rho [u_{mk}^{\theta}(r, z)]^2 r\,\mathrm{d}r\,\mathrm{d}z. \tag{33}$$

2.3. *Method of eigenfunction expansion*

The method of eigenfunction expansion for elastodynamic problems has been discussed in detail by Eringen and Suhubi [15]. And it is reviewed briefly here. The motion of an isotropic elastic body of volume Ω enclosed by a surface $\Sigma = \Sigma_1 + \Sigma_2$ is governed by

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u}(\mathbf{x}, t) + \mu\nabla^2\mathbf{u}(\mathbf{x}, t) + \rho\mathbf{f}(\mathbf{x}, t) = \rho\ddot{\mathbf{u}}(\mathbf{x}, t), \quad \mathbf{x} \text{ in } \Omega, \tag{34}$$

and boundary conditions

$$\mathbf{u}(\mathbf{x}, t) = \overline{\mathbf{u}}, \quad \mathbf{x} \text{ on } \Sigma_1, \tag{35}$$

$$\boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n} = \overline{\mathbf{T}}, \quad \mathbf{x} \text{ on } \Sigma_2, \tag{36}$$

and initial conditions

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0, \quad \mathbf{x} \text{ in } \Omega, \tag{37}$$

$$\dot{\mathbf{u}}(\mathbf{x},0)=\mathbf{v}_0, \quad \mathbf{x} \text{ in } \Omega, \tag{38}$$

where $\mathbf{f}$ is body force, $\boldsymbol{\sigma}$ is stress tensor, $\mathbf{u}$ is displacement vector, ρ is density, λ and μ are Lamé's constants, $\mathbf{u}_0$, $\mathbf{v}_0$ and $\overline{\mathbf{u}}$, $\overline{\mathbf{T}}$ are prescribed quantities.

The eigenvalue problem corresponding to Eqs. (34)–(36) can be formulated as

$$(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}^{(m)}(\mathbf{x})+\mu\nabla^2\mathbf{u}^{(m)}(\mathbf{x})+\rho\omega_m^2\mathbf{u}^{(m)}(\mathbf{x})=0, \quad \mathbf{x} \text{ in } \Omega, \tag{39}$$

$$\mathbf{u}^{(m)}(\mathbf{x})=0, \quad \mathbf{x} \text{ on } \Sigma_1, \tag{40}$$

and

$$\boldsymbol{\sigma}^{(m)}(\mathbf{x})\cdot\mathbf{n}=0, \quad \mathbf{x} \text{ on } \Sigma_2, \tag{41}$$

where $\mathbf{u}^{(m)}$ is the eigenfunctions and $\boldsymbol{\sigma}^{(m)}$ is the corresponding stress tensor. The eigenvalues ω_m^2 are real and non-negative [15,17]. For self-adjoint boundary conditions, the eigenfunctions form an orthogonal set with the weighting function ρ, i.e.,

$$\int_\Omega \rho\mathbf{u}^{(m)}\cdot\mathbf{u}^{(n)}\,\mathrm{d}V=0, \quad m\neq n, \tag{42}$$

and the norm of the eigenfunctions is given by

$$M^{(m)}=\int_\Omega \rho\mathbf{u}^{(m)}\cdot\mathbf{u}^{(n)}\,\mathrm{d}V. \tag{43}$$

If the initial displacement and velocity vectors, $\mathbf{u}_0$ and $\mathbf{v}_0$, are equal to zero, the solution to Eqs. (34)–(38) is

$$\mathbf{u}(\mathbf{x},t)=\sum_m\left[\int_0^t \Phi_m(\tau)\sin\omega_m(t-\tau)\,\mathrm{d}\tau\right]\mathbf{u}^{(m)}(\mathbf{x}), \tag{44}$$

where

$$\Phi_m(t)=\frac{1}{M^{(m)}\omega_m}\left\{\int_\Omega \rho\mathbf{f}\cdot\mathbf{u}^{(m)}\,\mathrm{d}\mathbf{V}-\int_{S_1}\overline{\mathbf{u}}\cdot(\boldsymbol{\sigma}^{(m)}\cdot\mathbf{n})\,\mathrm{d}S+\int_{S_2}\overline{\mathbf{T}}\cdot\mathbf{u}^{(m)}\,\mathrm{d}S\right\}. \tag{45}$$

2.4. Transient torsional vibration response of a finite hollow cylinder

According to the method of eigenfunction expansion presented by Reismann [14,15], we know the transient torsional vibration response of the above finite hollow cylinder can be expressed as

$$u_\theta(r,z,t)=\sum_{mk}\frac{1}{M_{mk}\omega_{mk}}\left[\int_0^t \Psi_{mk}(\tau)\sin\omega_{mk}(t-\tau)\,\mathrm{d}\tau\right]u_{mk}^\theta(r,z), \tag{46}$$

where

$$\begin{aligned}\Psi_{mk}(\tau)=&\int_a^b s_l(r,\tau)u_{mk}^\theta(r,0)r\,\mathrm{d}r+\int_a^b s_r(r,\tau)u_{mk}^\theta(r,2l)r\,\mathrm{d}r+\int_0^{2l} s_i(z,\tau)u_{mk}^\theta(a,z)a\,\mathrm{d}z\\&+\int_0^{2l} s_o(z,\tau)u_{mk}^\theta(b,z)b\,\mathrm{d}z+\int_0^{2l}\int_a^b \rho f(r,z,\tau)u_{mk}^\theta(r,z)r\,\mathrm{d}r\,\mathrm{d}z.\end{aligned} \tag{47}$$

3. Transient torsional vibration responses of a semi-infinite hollow cylinder to axisymmetric torsional forces

The dynamic response problem of a semi-infinite hollow cylinder subjected to external axisymmetric torsional forces, as shown in Fig. 2, can be formulated as the wave equation

$$\frac{1}{c_T^2}\frac{\partial^2 u_\theta}{\partial t^2}-f(r,z,t)=\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial z^2}, \quad r\in[a,b],\ z\in[0,\infty], \tag{48}$$

the boundary conditions

$$\sigma_{r\theta}|_{r=a}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\bigg|_{r=a}=s_i(z,t), \quad z\in[0,\infty], \tag{49}$$

$$\sigma_{r\theta}|_{r=b}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\bigg|_{r=b}=s_o(z,t), \quad z\in[0,\infty], \tag{50}$$

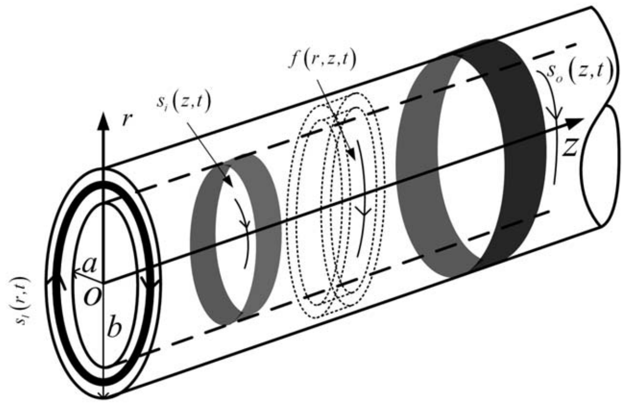

Fig. 2. A semi-finite hollow cylinder with inner radius a and outer radius b. It is subjected to the torsional body force with density $f(r,z,t)$. The end boundary of it is subjected to torsional surface force $s_l(r,t)$. The inner and outer lateral boundaries of it are subjected to torsional surface forces $s_i(z,t)$ and $s_o(z,t)$, respectively.

$$\sigma_{z\theta}|_{z=0} = \mu \frac{\partial u_\theta}{\partial z}\bigg|_{z=0} = s_l(r,t), \quad r \in [a,b], \tag{51}$$

and the initial conditions

$$u_\theta|_{t=0} = 0, \quad \frac{\partial u_\theta}{\partial t}\bigg|_{t=0} = 0, \quad r \in [a,b], \; z \in [0,\infty]. \tag{52}$$

The solution of equations (48)–(52) can be derived from Eq. (46). The derivation is as follows. We can obtain from Eq. (33) after invoking Eqs. (30) and (31) that

$$M_{mk} = l \int_a^b \rho [R_{mk}(r)]^2 r \, \mathrm{d}r, \tag{53}$$

where $R_{mk}(r)$ is shown as Eqs. (26) and (27). Setting

$$M_{mk}^{\infty} = \int_a^b \rho [R_{mk}(r)]^2 r \, \mathrm{d}r, \tag{54}$$

we have

$$M_{mk} = l M_{mk}^{\infty}. \tag{55}$$

It is easy to know from Eq. (31) that the interval between two successive wavenumber is

$$\Delta\xi_{mk} = \frac{\pi}{2l}. \tag{56}$$

Then we can derive from Eqs. (55) and (56) that

$$M_{mk} = \frac{\pi}{2\Delta\xi_{mk}} M_{mk}^{\infty}. \tag{57}$$

Substitution of Eq. (57) into Eq. (46) gives

$$u_\theta(r,z,t) = \sum_{mk} \frac{2\Delta\xi_{mk}}{\pi M_{mk}^{\infty}\omega_{mk}} \left[\int_0^t \Psi_{mk}(\tau) \sin\omega_{mk}(t-\tau) \, \mathrm{d}\tau \right] u_{mk}^{\theta}(r,z). \tag{58}$$

As the hollow cylinder is of semi-infinite length, namely, $l \to \infty$, interval between two successive roots $\Delta\xi_{mk}$ approaches zero. Therefore, in the limit of l approaching infinity, the summation over the index k in Eq. (58) will be replaced by the integral over the continuous eigenvalue ξ_m, then we have

$$u_\theta(r,z,t) = \sum_m \int_0^\infty \frac{2}{\pi M_m^{\infty}\omega_m} \left[\int_0^t \Psi_m(\tau) \sin\omega_m(t-\tau) \, \mathrm{d}\tau \right] u_m^{\theta}(r,z) \, \mathrm{d}\xi_m, \tag{59}$$

where

$$\Psi_m(\tau) = \int_a^b s_l(r,\tau) u_m^{\theta}(r,0) r \, \mathrm{d}r + \int_0^\infty s_i(z,\tau) u_m^{\theta}(a,z) a \, \mathrm{d}z + \int_0^\infty s_o(z,\tau) u_m^{\theta}(b,z) b \, \mathrm{d}z + \int_0^\infty \int_a^b \rho f(r,z,\tau) u_m^{\theta}(r,z) r \, \mathrm{d}r \, \mathrm{d}z. \tag{60}$$

Note that the term corresponding to the right end boundary condition in Eq. (47), i.e.,

$$\int_a^b s_r(r,\tau)u_{mk}^{\theta}(r,2l)r\,\mathrm{d}r \tag{61}$$

should be omitted when $l\to\infty$. Eqs. (59) and (60) are just the exact solutions of the transient torsional vibration response of the semi-infinite hollow cylinder shown in Fig. 2.

4. Transient vibration responses of the finite hollow cylinder, the ends of which are located at $z=-l$ and l, to axisymmetric torsional forces

The transient torsional vibration problem of the finite hollow cylinder, the ends of which are located at $z=0$ and $2l$, has been solved in Section 2. In this section, we will derive the transient torsional response solutions of the finite hollow cylinder, the ends of which are located at $z=-l$ and l. Though the problem studied in this section are same to that studied in Section 2 from the view of physics, the solutions obtained have different mathematical forms. And the transient torsional response solution of an infinite hollow cylinder can be derived from the results obtained in this section but cannot be derived from those given in Section 2.

Now the motion of the finite hollow cylinder, as shown in Fig. 3, is governed by the elastodynamic equation

$$\frac{1}{c_T^2}\frac{\partial^2 u_\theta}{\partial t^2}-f(r,z,t)=\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial z^2},\quad r\in[a,b],\ z\in[-l,l], \tag{62}$$

the boundary conditions

$$\sigma_{r\theta}|_{r=a}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\bigg|_{r=a}=s_i(z,t),\quad z\in[-l,l], \tag{63}$$

$$\sigma_{r\theta}|_{r=b}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\bigg|_{r=b}=s_o(z,t),\quad z\in[-l,l], \tag{64}$$

$$\sigma_{z\theta}|_{z=-l}=\mu\frac{\partial u_\theta}{\partial z}\bigg|_{z=-l}=s_l(r,t),\quad r\in[a,b], \tag{65}$$

$$\sigma_{z\theta}|_{z=l}=\mu\frac{\partial u_\theta}{\partial z}\bigg|_{z=l}=s_r(r,t),\quad r\in[a,b], \tag{66}$$

and the initial conditions

$$u_\theta|_{t=0}=0,\quad \frac{\partial u_\theta}{\partial t}\bigg|_{t=0}=0,\quad r\in[a,b],\ z\in[-l,l]. \tag{67}$$

4.1. Torsional vibration eigenfunctions

The solutions to Eqs. (62)–(67) are

$$u_{mk}^{\theta(1)}(r,z)=R_{mk}^{(1)}(r)\cos\xi_{mk}z, \tag{68}$$

where

$$R_{mk}^{(1)}(r)=\begin{cases}D_{mk}r, & \omega_{mk}^2/c_T^2=\xi_{mk}^2,\\ C_{mk}J_1(\beta_{mk}r)+D_{mk}Y_1(\beta_{mk}r), & \omega_{mk}^2/c_T^2>\xi_{mk}^2,\end{cases} \tag{69}$$

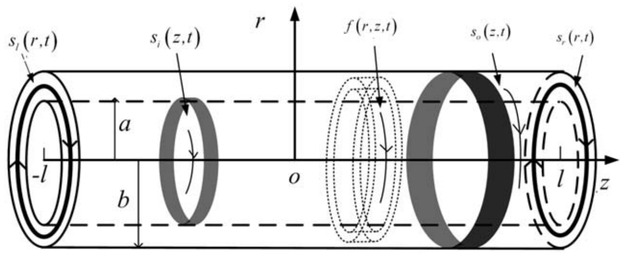

Fig. 3. A finite hollow cylinder with inner radius a and outer radius b, two ends of which are located at $z=-l$ and l. It is subjected to the torsional body force with density $f(r,z,t)$. The left and right end boundaries of it are subjected to torsional surface forces $s_l(r,t)$ and $s_r(r,t)$, respectively. The inner and outer lateral boundaries of it are subjected to torsional surface forces $s_i(z,t)$ and $s_o(z,t)$, respectively.

$$\xi_{mk} = \frac{k\pi}{l}, \quad k = 1, 2, 3, \ldots \tag{70}$$

and

$$u_{mk}^{\theta(2)}(r,z) = R_{mk}^{(2)}(r)\sin\xi_{mk}z, \tag{71}$$

where

$$R_{mk}^{(2)}(r) = \begin{cases} D_{mk}r, & \omega_{mk}^2/c_T^2 = \xi_{mk}^2, \\ C_{mk}J_1(\beta_{mk}r) + D_{mk}Y_1(\beta_{mk}r), & \omega_{mk}^2/c_T^2 > \xi_{mk}^2, \end{cases} \tag{72}$$

$$\xi_{mk} = \frac{(2k+1)\pi}{2l}, \quad k = 1, 2, 3, \ldots, \tag{73}$$

β_{mk} in Eqs. (69) and (72) is shown as Eq. (28). It is easy to prove that the eigenfunctions $u_{mk}^{\theta(1)}$ and $u_{mk}^{\theta(2)}$, as shown in Eq. (68) or (71), form an orthogonal set [15,19], then we have

$$\int_a^b \int_{-l}^{l} \rho u_{mk}^{\theta(p)}(r,z) u_{ij}^{\theta(q)}(r,z) r\,\mathrm{d}r\,\mathrm{d}z = \begin{cases} 0, & p \neq q, \\ 0, & p = q,\ m \neq i \text{ or } k \neq j, \\ M_{mk}^{(p)}, & p = q,\ m = i \text{ and } k = j, \end{cases} \tag{74}$$

where

$$M_{mk}^{(p)} = \int_a^b \int_{-l}^{l} \rho [u_{mk}^{\theta(p)}(r,z)]^2 r\,\mathrm{d}r\,\mathrm{d}z, \quad p = 1, 2. \tag{75}$$

4.2. Transient torsional vibration responses

According to the method of eigenfunction expansion presented by Reismann [14], we know the transient solution which satisfies Eqs. (62)–(67) can be expressed as

$$u_\theta(r,z,t) = \sum_{p=1}^{2} \sum_{mk} \frac{1}{M_{mk}^{(p)}\omega_{mk}} \left[\int_0^t \Phi_{mk}^{(p)}(\tau)\sin\omega_{mk}(t-\tau)\,\mathrm{d}\tau\right] u_{mk}^{\theta(p)}(r,z), \tag{76}$$

where

$$\begin{aligned} \Phi_{mk}^{(p)}(\tau) = {} & \int_a^b s_l(r,\tau) u_{mk}^{\theta(p)}(r,-l) r\,\mathrm{d}r + \int_a^b s_r(r,\tau) u_{mk}^{\theta(p)}(r,l) r\,\mathrm{d}r + \int_{-l}^{l} s_i(z,\tau) u_{mk}^{\theta(p)}(a,z) a\,\mathrm{d}z \\ & + \int_{-l}^{l} s_o(z,\tau) u_{mk}^{\theta(p)}(b,z) b\,\mathrm{d}z + \int_{-l}^{l}\int_a^b \rho f(r,z,\tau) u_{mk}^{\theta(p)}(r,z) r\,\mathrm{d}r\,\mathrm{d}z, \quad p = 1, 2. \end{aligned} \tag{77}$$

5. Transient torsional vibration responses of an infinite hollow cylinder to axisymmetric torsional forces

Fig. 4 is an infinite hollow cylinder subjected to external torsional forces. The dynamic response problem of it can be formulated as

$$\frac{1}{c_T^2}\frac{\partial^2 u_\theta}{\partial t^2} - f(r,z,t) = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2}, \quad r \in [a,b],\ z \in (-\infty, +\infty), \tag{78}$$

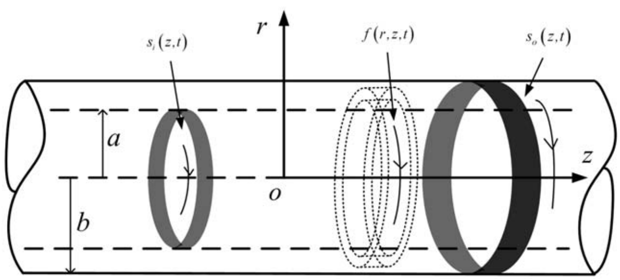

Fig. 4. A finite hollow cylinder with inner radius a and outer radius b. It is subjected to the torsional body force with density $f(r,z,t)$. The inner and outer lateral boundaries of it are subjected to torsional surface forces $s_i(z,t)$ and $s_o(z,t)$, respectively.

under the boundary conditions

$$\sigma_{r\theta}|_{r=a} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\Big|_{r=a} = s_i(z,t), \quad z \in (-\infty, +\infty), \tag{79}$$

$$\sigma_{r\theta}|_{r=b} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\Big|_{r=b} = s_o(z,t), \quad z \in (-\infty, +\infty), \tag{80}$$

and initial conditions

$$u_\theta|_{t=0} = 0, \quad \frac{\partial u_\theta}{\partial t}\Big|_{t=0} = 0, \quad r \in [a,b], \ z \in (-\infty, +\infty). \tag{81}$$

We can derived from Eqs. (68)–(73) and (75) that

$$M_{mk}^{(p)} = \frac{\pi}{\Delta\xi_{mk}} M_{mk}^{\infty(p)}, \quad p = 1, 2, \tag{82}$$

where

$$\Delta\xi_{mk} = \frac{\pi}{l} \tag{83}$$

and

$$M_{mk}^{\infty(p)} = \int_a^b \rho[R_{mk}^{(p)}(r)]^2 r\,\mathrm{d}r, \quad p = 1, 2. \tag{84}$$

Substituting Eq. (82) into Eqs. (76) and (77), then repeating the derivation in Section 3, we have

$$u_\theta(r,z,t) = \sum_{p=1}^{2}\sum_m \int_0^\infty \left\{\frac{1}{\pi M_m^{\infty(p)}\omega_m}\left[\int_0^t \Phi_m^{(p)}(\tau)\sin\omega_m(t-\tau)\,\mathrm{d}\tau\right] u_m^{\theta(p)}(r,z)\right\}\mathrm{d}\xi_m, \tag{85}$$

where

$$\Phi_m^{(p)}(\tau) = \int_{-\infty}^{\infty} s_i(z,\tau)u_m^{\theta(p)}(a,z)a\,\mathrm{d}z + \int_{-\infty}^{\infty} s_o(z,\tau)u_m^{\theta(p)}(b,z)b\,\mathrm{d}z + \int_{-\infty}^{\infty}\int_a^b \rho f(r,z,\tau)u_m^{\theta(p)}(r,z)r\,\mathrm{d}r\,\mathrm{d}z, \quad p = 1, 2. \tag{86}$$

Note that the terms corresponding to the end boundary conditions in Eq. (77), i.e.,

$$\int_a^b s_l(r,\tau)u_{mk}^{\theta(p)}(r,-l)r\,\mathrm{d}r \text{ and } \int_a^b s_r(r,\tau)u_{mk}^{\theta(p)}(r,l)r\,\mathrm{d}r \tag{87}$$

should be omitted when $l \to \infty$. Eqs. (85) and (86) are just the transient torsional response solutions of the infinite hollow cylinder.

6. Numerical examples

Here, we consider the excitation and propagation of the torsional waves in a finite steel hollow cylinder with length $2l = 0.3\,\mathrm{m}$, outer radius $b = 0.04\,\mathrm{m}$ and wall thickness $h = 0.02\,\mathrm{m}$, as shown in Fig. 1. The material parameters of the hollow cylinder are density $\rho = 7.8 \times 10^3\,\mathrm{kg/m^3}$, Young's module $E = 215.04\,\mathrm{GPa}$ and Poisson coefficient $\gamma = 0.28$. The left end surface force density is

$$s_l(r,t) = \begin{cases} G_1T_1(t) & \text{for } r \in \left[a + \frac{h}{3}, a + \frac{2h}{3}\right], \\ 0 & \text{otherwise,} \end{cases} \tag{88}$$

no external force is applied on the right end surface, the inner lateral surface density is

$$s_i(z,t) = 0, \quad z \in [0, 2l], \tag{89}$$

the outer lateral surface density is

$$s_o(z,t) = \begin{cases} G_2T_2(t) & \text{for } z \in [0.12\,\mathrm{m}, 0.124\,\mathrm{m}], \\ 0 & \text{otherwise,} \end{cases} \tag{90}$$

the body force density is

$$f(r,z,t) = \begin{cases} G_3T_3(t) & \text{for } r \in \left[a + \frac{h}{3}, a + \frac{2h}{3}\right], \quad z \in [0.06\,\mathrm{m}, 0.064\,\mathrm{m}], \\ 0 & \text{otherwise,} \end{cases} \tag{91}$$

where $G_1 = 1.0$, $G_2 = 1.0$, $G_3 = 0.05$,

$$T_k(t) = \begin{cases} \sin(2\pi f_k t)\left[0.5 - 0.5\cos\left(\frac{2\pi t}{8/f_k}\right)\right], & t \le 8/f_k \\ 0, & t > 8/f_k \end{cases} \quad (k = 1,2,3) \tag{92}$$

and

$$f_k = \begin{cases} 100\,\text{kHz}, & k = 1, \\ 120\,\text{kHz}, & k = 2, \\ 60\,\text{kHz}, & k = 3. \end{cases} \tag{93}$$

Fig. 5 is the group velocity dispersion curves of first four-order torsional wave modes. Obviously, only modes $T(0,1)$ and $T(0,2)$ can be excited by the above surface forces, and only mode $T(0,1)$ can be excited by the above body force. The general purpose commercial finite element software called Abaqus is used. Eight-noded three-dimensional solid elements are employed in the FE simulation. The element size along z-axis is 1.0×10^{-3} m. And the hollow cylinder is radially divided into 18 uniform parts and circumferentially divided into 270 uniform parts. The time-step is 0.04 μs in the explicit algorithm. Figs. 6, 7 and 8 show the transient torsional displacements of outer surface at $z = 0.2$ m, which are excited by $s_l(r,t)$, $s_o(z,t)$ and $f(r,z,t)$, respectively. The solid lines in them are computed from the analytical solution, i.e., Eq. (59), while the black dots are simulated by the finite element method (FEM). Fig. 9 shows the total transient torsional displacement waveforms obtained by the two different methods mentioned above. Apparently, the results computed from the analytical solution agree well with those simulated by FEM.

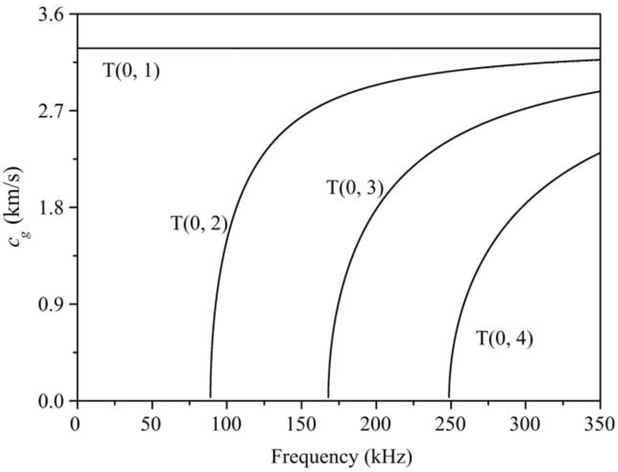

Fig. 5. The group velocity dispersion curves of first four branches of $T(0,m)$.

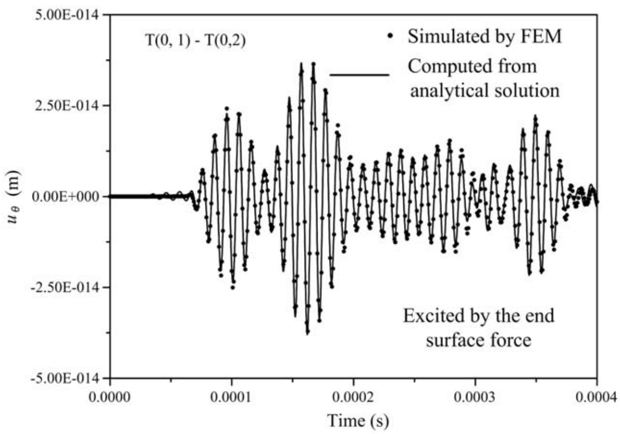

Fig. 6. The transient displacement of the outer surface at $z = 0.2$ m, which is exited by the torsional end surface force. The solid line is computed from the analytical solution. The black dot is simulated by the FEM.

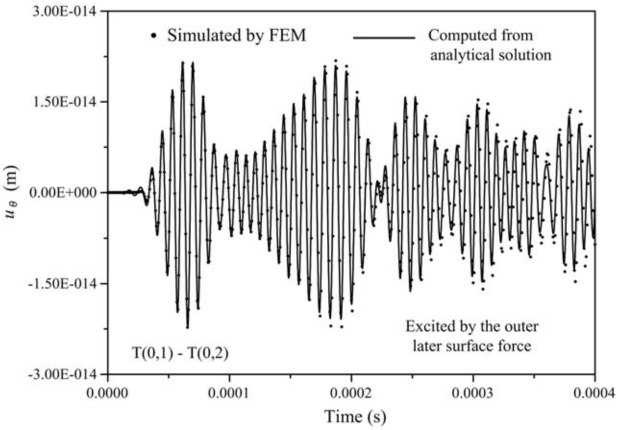

Fig. 7. The transient displacement of the outer surface at $z = 0.2$ m, which is exited by the torsional outer lateral surface force. The solid line is computed from the analytical solution. The black dot is simulated by the FEM.

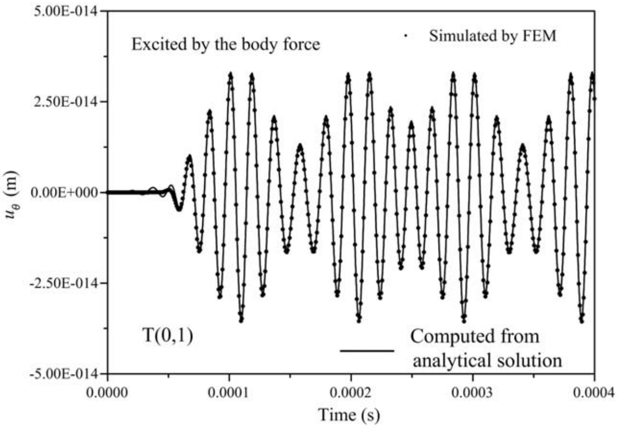

Fig. 8. The transient displacement of the outer surface at $z = 0.2$ m, which is exited by the torsional body force. The solid line is computed from the analytical solution. The black dot is simulated by the FEM.

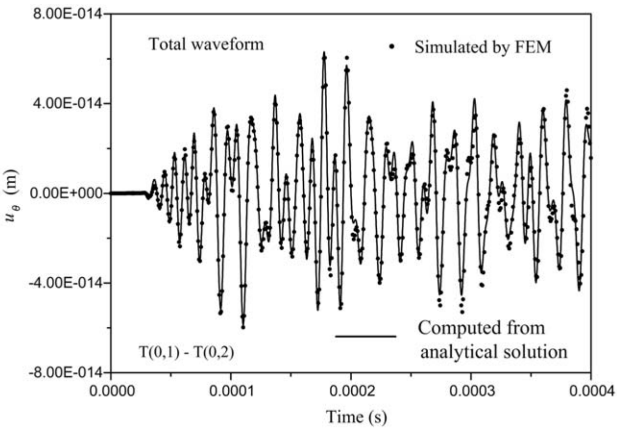

Fig. 9. The total transient displacement of the outer surface at $z = 0.2$ m. The solid line is computed from the analytical solution. The black dot is simulated by the FEM.

7. Conclusions

Note that two kinds of transient torsional response solutions of the finite hollow cylinder with different eigenfunctions are presented in Sections 2 and 4. Apparently, the solution presented in Section 2 is more suitable for numerical computation. The transient torsional response solutions of the semi-infinite and infinite hollow cylinders can be obtained based on the results presented in Sections 2 and 4, respectively.

Compared to the integral transform technique and the finite element method, the eigenfunction expansion method has two main advantages: (1) it is easy to numerically evaluate the solution derived by the eigenfunction expansion method, and (2) the contribution of each guided mode to total responses can be easily analyzed. Folk et al. [12] pointed out that the solution derived by the integral transform technique is too complex to evaluate it by simple means. And it is worth mentioning that on our AMD Athlon 3800 PC, the finite element method took over 30 h to produce the results shown in Figs. 6–9, while the eigenfunction expansion method used less than 15 min to complete the computation. But it should be admitted that the interaction between the guided waves and the defects in elastic guides can be analyzed by the finite element method, but the eigenfunction expansion method cannot do this.

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References

[1] D.C. Gazis, Three-dimensional investigation of the propagation of waves in hollow circular cylinders. I. Analytical foundation, *Journal of the Acoustical Society of America* 31 (1959) 568–573.
[2] A.H. Meitzler, Mode coupling occurring in the propagation of elastic pulses in wires, *Journal of the Acoustical Society of America* 33 (1961) 435–445.
[3] J. Zemanek, An experimental and theoretical investigation of elastic wave propagation in a cylinder, *Journal of the Acoustical Society of America* 51 (1971) 265–283.
[4] M.J.S. Lowe, D.N. Alleyne, P. Cawley, Defect detection in pipes using guided waves, *Ultrasonics* 36 (1998) 147–154.
[5] D.N. Alleyne, M.J.S. Lowe, P. Cawley, The reflection of guided waves from circumferential notches in pipes, *Journal of Applied Mechanics* 65 (1998) 635–641.
[6] D.N. Alleyne, B. Pavlakovic, M.J.S. Lowe, P. Cawley, Rapid, long range inspection of chemical plant pipework using guided waves, *Review of Progress in Quantitative Nondestructive Evaluation* 20 (2001) 180–187.
[7] A. Demma, P. Cawley, M.J.S. Lowe, The reflection of the fundamental torsional mode from cracks and notches in pipes, *Journal of the Acoustical Society of America* 114 (2003) 611–625.
[8] H. Kwun, S.Y. Kim, H. Matsumoto, S. Vinogradov, Detection of axial cracks in tube and pipe using torsional guided waves, *Review of Progress in Quantitative Nondestructive Evaluation* 27 (2008) 193–199.
[9] H. Kwun, C.M. Teller, Magnetostrictive generation and detection of longitudinal, torsional, and flexural waves in a steel rod, *Journal of the Acoustical Society of America* 96 (1994) 1202–1204.
[10] C.I. Park, S.H. Cho, Y.Y. Kim, Z-shaped magnetostrictive patch for efficient transduction of a torsional wave mode in a cylindrical waveguide, *Applied Physics Letters* 89 (2006) 174103.
[11] K.P. Soldatos, Review of three dimensional dynamic analysis of circular cylinders and cylindrical shells, *Applied Mechanics Review* 47 (1994) 501–506.
[12] R.T. Folk, G. Fox, C.A. Shook, C.W. Curtis, Elastic strain produced by sudden application of pressure to one end of a cylindrical bar. I. Theory, *Journal of the Acoustical Society of America* 30 (1958) 552–558.
[13] Y.D. Pan, C. Rossignol, B. Audoin, Acoustic waves generated by a laser point source in an isotropic cylinder, *Journal of the Acoustical Society of America* 116 (2004) 814–820.
[14] H. Reismann, On the forced motion of elastic solids, *Applied Science Research* 18 (1967) 156–165.
[15] A.C. Eringen, *Elastodynamics*, Vol. 2, Academic Press, New York, 1975.
[16] L.G. Tang, J.C. Cheng, An eigenfunction expansion method for the elastodynamic response of an elastic solid with mixed boundary surfaces, *Progress in Natural Science* 18 (2008) 1063–1068.
[17] Y.H. Pao, Elastic waves in solids, *Journal of Applied Mechanics* 50 (1983) 1152–1164.
[18] L.G. Tang, J.C. Cheng, Numerical analysis on laser-generated guided elastic waves in a hollow cylinder, *Journal of Nondestructive Evaluation* 21 (2002) 45–53.
[19] R.L. Weaver, Y.H. Pao, Axisymmetric elastic waves excited by a point source in a plate, *Journal of Applied Mechanics* 49 (1982) 821–836.