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Predicting optimal drive sweep rates for autoresonance in Duffing-type oscillators: A beat method using Teager–Kaiser instantaneous frequency

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ABSTRACT

Sustained resonance in a linear oscillator is achievable with a drive whose constant frequency matches the resonant frequency of the oscillator. But in oscillators with nonlinear restoring forces such as the pendulum, Duffing and Duffing–Van der Pol oscillator, the resonant frequency changes as the amplitude changes, so a constant frequency drive results in a beat oscillation instead of sustained resonance. Duffing-type nonlinear oscillators can be driven into sustained resonance, called autoresonance, when the drive frequency is swept in time to match the changing resonant frequency of the oscillator. We find that near-optimal drive linear sweep rates for autoresonance can be estimated from the beat oscillation resulting from constant frequency excitation. Specifically, a least squares estimate of the Teager–Kaiser instantaneous frequency versus time for the beat response to a stationary drive provides a near-optimal estimate of the nonstationary drive linear sweep rate needed to sustain resonance in the pendulum, Duffing and Duffing–Van der Pol oscillators. We confirm these predictions with model-based numerical simulations. An advantage of the beat method of estimating optimal drive sweep rates for maximal autoresonant response is that no model is required so experimentally generated beat oscillation data can be used for systems where no model is available.

1. Introduction

Control of nonlinear oscillations to achieve a desired state is important in many fields, from engineering to medicine. Sustained resonance is usually an undesirable state in mechanical systems where it can lead to mechanical breakdown, for example, in an airplane wing. However, sustained resonance may be desirable in electrical systems, for example, in electronic tuning to produce maximum response to an input signal.

The open-loop (feed-forward) control scheme of driving a linear oscillator at its resonant frequency achieves sustained resonance because linear oscillators such as

$$\ddot{x}+\omega^2 x = F\cos(\Omega t) \tag{1}$$

have fixed resonant frequencies that are independent of amplitude, so that, without damping, a drive of constant frequency results in a continually increasing amplitude. By contrast, oscillators with nonlinear restoring forces such as the pendulum

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(Eq. (2)), Duffing (Eq. (3)), and Duffing-van der Pol oscillators (Eq. (4)), which we call Duffing-type oscillators,

$$\ddot{x}+\sin(x)=F\cos(\Omega t) \tag{2}$$

$$\ddot{x}+x-\varepsilon x^3=F\cos(\Omega t) \tag{3}$$

$$\ddot{x}+x-\varepsilon x^3+\mu(1-x)\dot{x}=F\cos(\Omega t) \tag{4}$$

have resonant frequencies that depend on their instantaneous amplitudes. We call the instantaneous resonant frequency at lowest amplitude the low-amplitude oscillator frequency, ω_o, which we set equal to 1. The simple open-loop control of setting the drive frequency equal to the low-amplitude oscillator frequency will not sustain resonance in these oscillators. Driving a Duffing oscillator at its low-amplitude oscillator frequency produces a transient increase in amplitude. But since the oscillator frequency is amplitude dependent, as the amplitude increases the oscillator frequency shifts away from the drive frequency, thereby decreasing the oscillator amplitude. As the amplitude decreases, the oscillator frequency shifts back towards the drive frequency and the amplitude of the oscillator increases again. Therefore, driving a Duffing-type oscillator at its low-amplitude oscillator frequency results in a beat response consisting of a slow amplitude-modulation (AM) of a fast frequency-modulated (FM) carrier. The beat response is amplitude-modulated as the oscillator slips in and out of resonance. The beat response is frequency-modulated as the oscillator frequency changes due to amplitude changes. This is shown in Fig. 1(a)–(c) for the pendulum, softening Duffing oscillator and softening Duffing-Van der Pol oscillator, respectively, all with equivalent nonlinear restoring force terms.

Sustained resonance instead of a beat oscillation is possible for Duffing-type oscillators by starting in resonance and sweeping the drive frequency to match the shifting oscillator frequency, or, alternatively, sweeping an oscillator parameter that moves the oscillator frequency in the opposite direction of its amplitude-dependent frequency shift, effectively cancelling the amplitude dependence of the oscillator frequency. Owing to the practical importance of maximal resonant response and the pervasiveness of Duffing-type oscillators, the "secret" of starting in resonance and sweeping the drive frequency to sustain resonance has been discovered and rediscovered many times over the years under different names. The earliest example is centuries old, that of church bell-ringers decreasing their bell-ringing frequency as oscillation amplitude increases (a form of feedback control), now called the "bell-ringer mode" of the driven pendulum [1].

Sustained resonance can also be achieved by starting off-resonance and slowly passing through resonance leading to capture into resonance via phase-locking of drive and oscillator. This "phase stability principle" was discovered by Veksler [2] and McMillan [3] in 1944 and 1945 and used in the synchrotron and synchrocyclotron. The term autoresonance was first introduced by Kolomenski and Lebedev [4] in the context of cyclotron resonance stability. Friedland [5–7] has applied autoresonance to plasma physics, atomic physics, and planetary dynamics, and has described autoresonance as "a salient property of many nonlinear systems to stay in resonance with driving perturbations despite variation of system

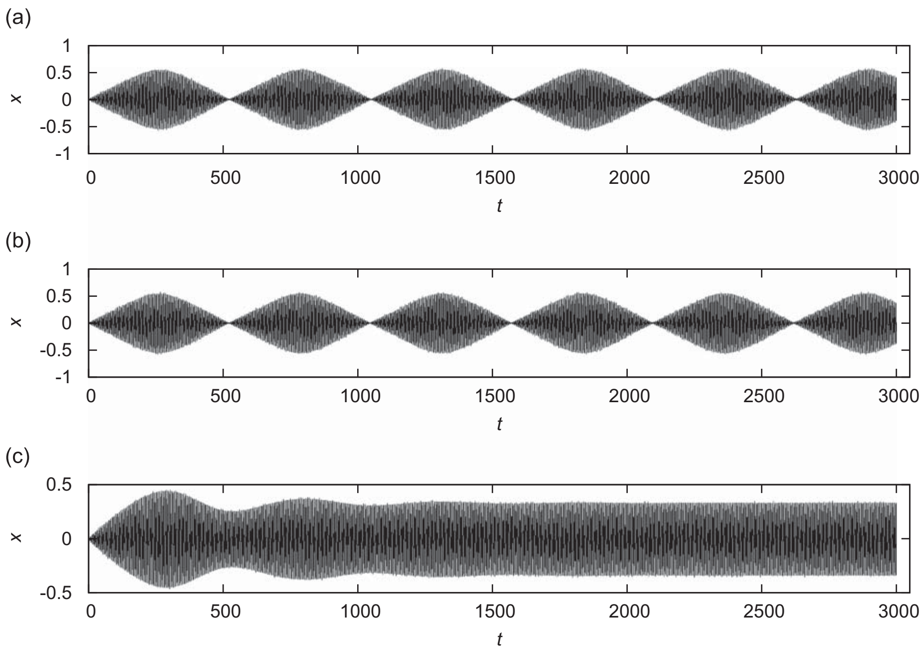

Fig. 1. Beat oscillation response for the (a) pendulum, (b) softening Duffing ($\varepsilon=\frac{1}{6}$) and (c) softening Duffing–Van der Pol oscillator ($\varepsilon=\frac{1}{6}, \mu=0.005$) to a drive ($F=0.005$) of constant frequency set to the low-amplitude oscillator frequency ($\omega=1$).

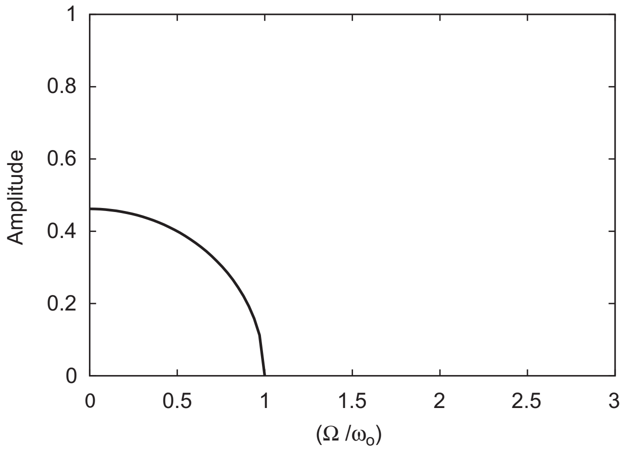

Fig. 2. The resonant frequency of the softening Duffing oscillator in Eq. (2) depends on its amplitude, plotted here as the nonlinear backbone ($F=0$) curve of amplitude versus frequency ratio.

parameters" [8]. An energy-based theory of autoresonance applicable to Duffing-type oscillators was developed by Chacon [9].

Hübler proposed an open-loop method of resonant stimulation of nonlinear oscillators with application to resonance spectroscopy and introduced the "principle of the dynamic key" that "optimal driving forces have the same dynamics as the time-reflected transient dynamics of the unperturbed system" [10,11]. In Hübler's method the system's differential equation is integrated starting at the desired highly stimulated state. By time-reversing the data, the ideal forcing function is obtained. An advantage of Hübler's method is that all resonances are taken into account in determining the ideal forcing function. However, the ideal forcing functions determined by Hübler's method may be difficult to generate and use in modeling. A major limitation of Hübler's method is the requirement of an accurate model.

Despite the importance of sustained resonance in nonlinear oscillators, there exists no simple way to determine optimal drive sweep rates that maximize autoresonant response in Duffing-type oscillators. Perfect frequency matching over time between drive and oscillator frequency for Duffing-type oscillators is not possible with a single swept periodic drive because the generation of harmonics in response to periodic excitation would require multiple drive sweeps for synchronization of drive and oscillator frequencies. Also, perfect frequency matching of drive and oscillator frequency, even for the fundamental oscillator frequency, would require a nonlinear sweep due to the nonlinear amplitude-frequency relation for Duffing-type oscillators. Fig. 2 shows the backbone curve for the softening Duffing oscillator [12].

The ease of generating linear sweeps and their convenience in modeling make it useful to consider the problem of predicting optimal drive linear sweep rates that maximize autoresonant response in Duffing-type oscillators. However, no method currently exists for predicting optimal drive linear sweep rates that maximize autoresonant response in Duffing-type oscillators. This paper introduces a simple beat method for estimating optimal linear sweep rates that maximize autoresonant response in Duffing-type oscillators. In the beat method, the oscillator is driven, either experimentally, or if a model is available, in numerical simulation, at its low-amplitude resonant frequency and a least squares estimate of the slope of the instantaneous frequency versus time curve for the beat response provides an estimate of the optimal sweep rate needed for the drive to track the changing oscillator frequency and sustain resonance.

This paper is organized as follows. Section 2 presents the steps used in applying the beat method to estimate optimal drive linear sweep rates that maximize autoresonant response in the pendulum, softening Duffing, and softening Duffing–Van der Pol oscillators. Section 3 uses the near-optimal sweep rate estimates provided by the beat method as starting points to numerically search for optimal sweep rates. Estimated and optimal sweep rates are compared. The existence of extremely sharp sweep rate transitions from maximal autoresonant response to loss of autoresonance is shown to occur and the implication to optimal sweep rate estimation is discussed. Section 4 presents conclusions of this investigation in predicting optimal drive linear sweep rates that maximize autoresonant response in Duffing-type oscillators.

2. A beat method for predicting optimal drive linear sweep rates that maximize autoresonant response in Duffing-type oscillators

The slow amplitude variation of the beat oscillation response of Duffing-type oscillators driven at their low-amplitude oscillator frequency results from the instantaneous frequency of the oscillator changing in response to amplitude changes, causing the oscillator to slip in and out of resonance. The slope of the oscillator's instantaneous frequency versus time curve is a linear approximation of the optimal drive linear sweep rate that tracks the changing oscillator frequency, thereby sustaining resonance.

The beat method of estimating optimal drive linear sweep rates that maximize autoresonant response consists of six steps:

1. The beat response of a Duffing-type oscillator to a stationary drive set to the low-amplitude oscillator frequency is obtained either experimentally or, if a model is available, by numerical simulation.
2. Instantaneous frequencies during the first half-cycle of the beat response are calculated using the Energy Separation Algorithm [13] based on the Teager–Kaiser energy operator [14].
3. Initial transient instantaneous frequency estimates are ignored until they stabilize to the oscillator's starting frequency.
4. A simple local minima moving filter is applied to the instantaneous frequency data to smooth the instantaneous frequency versus time curve.
5. A least squares estimate of the slope of the smoothed instantaneous frequency versus time curve provides a near-optimal estimate of the drive sweep rate that maximizes autoresonant response.
6. The estimated near-optimal drive sweep rate may be sufficient for some purposes or may be used to provide a starting value to numerically search for the optimal sweep rate.

In the following sections we apply the beat method to estimate optimal drive linear sweep rates that maximize autoresonant response in the pendulum, softening Duffing and softening Duffing–Van der Pol oscillators described by Eqs. (2)–(4).

2.1. Beat method step 1: Obtain the beat response of a Duffing-type oscillator to a stationary drive set to the low-amplitude oscillator frequency

Beat responses were obtained from numerical simulations of Duffing-type oscillators driven at their low-amplitude oscillator frequency nondimensionalized to equal 1. Numerical simulations were conducted using the high-level Dynast system simulation software that uses "a stiff-stable implicit multi-step backward-differentiation formula" [15,16]. Beat responses for the pendulum, softening Duffing oscillator and softening Duffing–Van der Pol oscillator are shown in Fig. 1(a)–(c), respectively.

2.2. Beat method step 2: Estimate instantaneous frequencies during the first half-cycle of the beat response

We use the continuous Energy Separation Algorithm (ESA) based on the continuous Teager–Kaiser (TK) energy operator for instantaneous frequency estimation and incorporate it in a high-level system numerical simulation language. While the TK energy operator and ESA were developed primarily to be used as discrete algorithms, their continuous forms were used here because high-level simulation languages like Dynast more easily accommodate differential operators than difference operators. An alternative continuous approach to calculating instantaneous frequencies based on the Hilbert transform is inconvenient for incorporating into high-level simulation languages because time-domain convolution integration is required. The TK–ESA has been shown to provide accurate instantaneous frequency estimates for frequency swept signals [17].

The TK differential energy operator

$$\Psi(u) = [\dot{u}(t)]^2 - u(t)\ddot{u}(t) \tag{5}$$

was developed to track instantaneous oscillator energy and defined to yield a result proportional to energy $a^2\omega^2$ when applied to the signal (i.e., solution) $u(t) = a\cos(\omega t + \phi)$ of a harmonic oscillator [14].

From the frequency, amplitude and energy relationships for a harmonic oscillator, Kaiser, Maragos and Quatieri showed that instantaneous frequency and instantaneous amplitude can be extracted from signals having combined amplitude modulation and frequency modulation by applying the TK differential energy operator to the signal and its first derivative (the Energy Separation Algorithm) as follows [13]:

$$\Psi(\dot{u}) = [\ddot{u}(t)]^2 - \dot{u}(t)\dddot{u}(t) \tag{6}$$

$$a(t) = \frac{\Psi(u)}{\sqrt{\Psi(\dot{u})}} \tag{7}$$

$$\omega(t) = \sqrt{\frac{\Psi(\dot{u})}{\Psi(u)}} \tag{8}$$

The TK–ESA instantaneous frequency versus time for the beat response of a pendulum, softening Duffing oscillator, and softening Duffing–Van der Pol oscillator, driven at their low-amplitude oscillator frequencies is shown in Fig. 3(a)–(c), respectively for the first half-cycle of the beat. Only the first half-cycle of the beat is used because the resonant increase in beat amplitude occurs in this portion of the cycle.

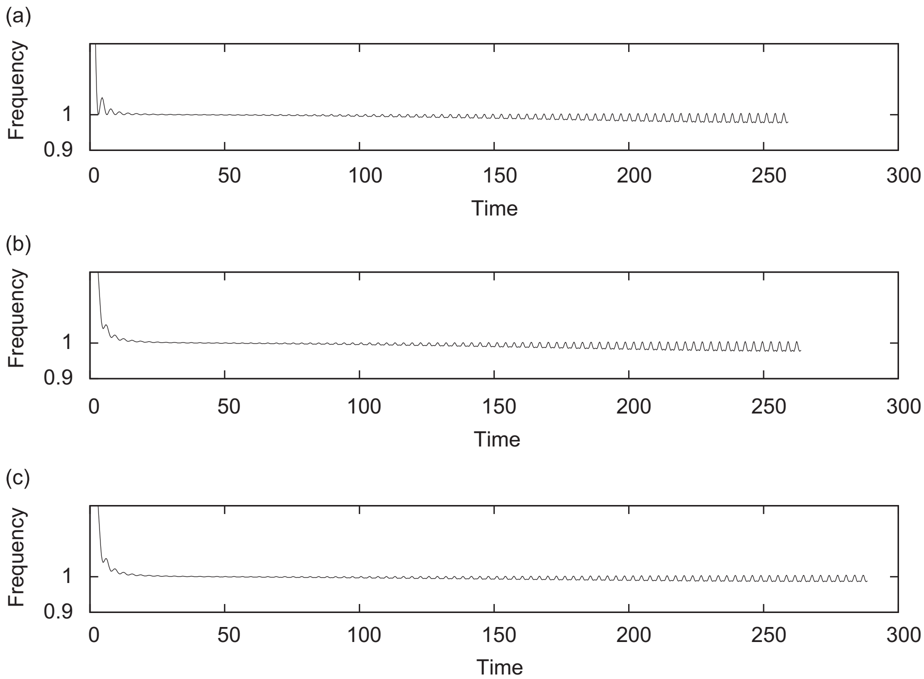

Fig. 3. TK–ESA instantaneous frequency versus time for the first half-cycle of the beat response for the (a) pendulum ($\omega = 1, F = 0.005$), (b) softening Duffing oscillator ($\varepsilon = \frac{1}{6}, \omega = 1, F = 0.005$), (c) softening Duffing–Van der Pol oscillator ($\varepsilon = \frac{1}{6}, \mu = 0.005, \omega = 1, F = 0.005$), all driven at their low-amplitude oscillator frequency. Initial numerical simulation transients and later high-frequency noise are visible.

2.3. Beat method step 3: Ignore initial TK–ESA instantaneous frequency estimates until transients settle

The first TK–ESA instantaneous frequency estimates were ignored until transients settled, i.e., when instantaneous frequency estimates started matching the known starting frequency ($\omega = 1$). The initial transients in the TK–ESA instantaneous frequency estimates can be seen in Fig. 3.

2.4. Beat method step 4: Smooth the instantaneous frequency versus time curve using a local minima moving filter

A simple local minima moving filter was applied to the instantaneous frequency data to smooth the instantaneous frequency envelope. The TK–ESA is sensitive to high-frequency noise because the TK energy operator uses differential operators. Therefore the instantaneous frequency value at each simulation time point was replaced by the lower bound over a forward window of 20 values for softening oscillators and replaced by the higher bound for hardening oscillators. Fig. 4(a)–(c) shows the results of this smoothing applied to the data originally plotted in Fig. 3(a)–(c).

2.5. Beat method step 5: Perform a least squares slope estimate of the smoothed instantaneous frequency versus time curve

MS Excel was used to provide a least squares estimate of the slope of instantaneous frequency versus time data for the first half-cycle of the beat response. The slope of the instantaneous frequency versus time curve for the first half-cycle provides an estimate of the optimal autoresonance drive linear sweep rate.

2.6. Beat method step 6: Use the near-optimal sweep rate estimate provided by the beat method as a starting value to numerically search for the optimal sweep rate

To use the estimated sweep rates to search for optimal sweep rates, the constant frequency drive on the right-hand side of Eqs. (2)–(4) was replaced by a drive with linear frequency sweep $\Omega = \omega_o + \alpha t$:

$$F\cos\left(\omega_o t + \frac{(\alpha t^2)}{2}\right) \tag{9}$$

since $\theta = \int \Omega\, dt$.

Negative sweep rates, corresponding to downward frequency sweeps, are needed to track the changing oscillator frequency and sustain resonance in softening oscillators (backbone curve bends to the left), while positive sweep rates are needed for hardening oscillators (backbone curve bends to the right).

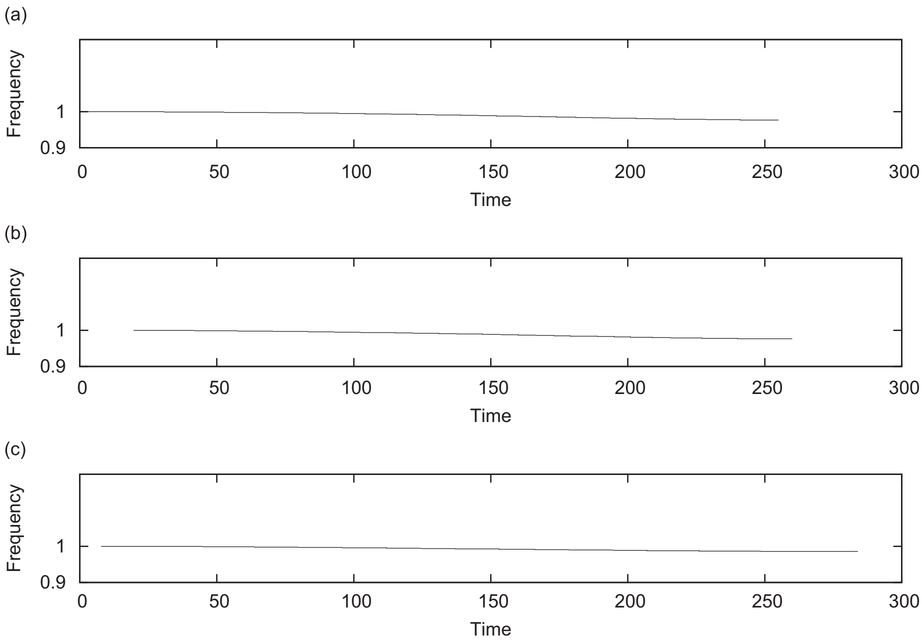

Fig. 4. TK–ESA instantaneous frequency versus time for the first half-cycle of the beat response for the (a) pendulum ($\omega = 1, F = 0.005$), (b) softening Duffing oscillator ($\varepsilon = \frac{1}{6}, \omega = 1, F = 0.005$), (c) softening Duffing–Van der Pol oscillator ($\varepsilon = \frac{1}{6}, \mu = 0.005, \omega = 1, F = 0.005$), all driven at their low-amplitude oscillator frequency, ignoring initial numerical simulation transients and applying a local minima moving filter ($\varepsilon = \frac{1}{6}, \omega = 1, F = 0.005$).

Table 1
Comparison of estimated and optimal drive linear sweep rates for maximal autoresonant response in the pendulum, softening Duffing oscillator and softening Duffing–Van der Pol oscillator.

Oscillator	Estimated sweep rate	Optimal sweep rate	% diff.
Pendulum	−0.0001142	−0.0001043	9.1
Softening Duffing	−0.0001149	−0.0001052	8.8
Softening Duffing–Van der Pol	−0.0000616	−0.0000653	5.8

Estimated sweep rates were calculated using the beat method while optimal sweep rates were determined by parameter search starting at the estimated sweep rate.

The near-optimal sweep rate estimates provided by the beat method may be sufficient for generating a maximal autoresonant response. If this is not the case, and if a model is available, sweep rate estimates provided by the beat method can be used as starting values to numerically search for optimal drive sweep rates. To find optimal drive sweep rates, sweep rates were systematically varied above and below the starting sweep rate estimated by the beat method. The drive sweep rate that resulted in a maximal autoresonant response within a fixed simulation time ($t = 3000\,\mathrm{s}$) for each oscillator type was identified as the optimal drive sweep rate.

A test of the beat method is provided by a comparison of estimated and optimal sweep rates for generating maximal autoresonant responses. Table 1 shows estimated and optimal periodic drive linear sweep rates for the pendulum, softening Duffing oscillator, and softening Duffing–Van der Pol oscillator. Within a parameter space spanning many orders of magnitude, the differences between estimated and optimal sweep rates were 9.1 percent, 8.8 percent and 6.2 percent, respectively.

The beat method was also tested for hardening versions of the Duffing and Duffing–Van der Pol oscillators, where the cubic restoring force term is positive in Eqs. (3) and (4). Table 2 shows estimated and optimal periodic drive linear sweep rates for the hardening Duffing oscillator and hardening Duffing–Van der Pol oscillator. Percent differences between estimated optimal sweep rates were 3.2 percent and 5.6 percent, respectively. As the pendulum is an inherently softening oscillator, only the Duffing and Duffing–Van der Pol oscillators are included in this category.

Time responses for the pendulum, softening Duffing oscillator and softening Duffing–Van der Pol oscillator driven with optimal sweep rates are shown in Fig. 5(a)–(c), respectively. Time responses for the same oscillators driven with estimated sweep rates are shown in Fig. 6(a)–(c), respectively. It was, at first, surprising, with single-digit percent differences between estimated and optimal sweep rates, that in only one of the three Duffing-type oscillators, the Duffing–Van der Pol oscillator, did the estimated sweep rate actually result in autoresonance. The explanation lies in the extremely sharp sweep

Table 2
Comparison of estimated and optimal drive linear sweep rates for maximal autoresonant response in the hardening Duffing oscillator and hardening Duffing–Van der Pol oscillator.

Oscillator	Estimated sweep rate	Optimal sweep rate	% diff.
Hardening Duffing	0.0001028	0.0000996	3.2
Hardening Duffing–Van der Pol	0.0000520	0.0000550	5.6

Estimated sweep rates were calculated using the beat method while optimal sweep rates were determined by parameter search starting at the estimated sweep rate.

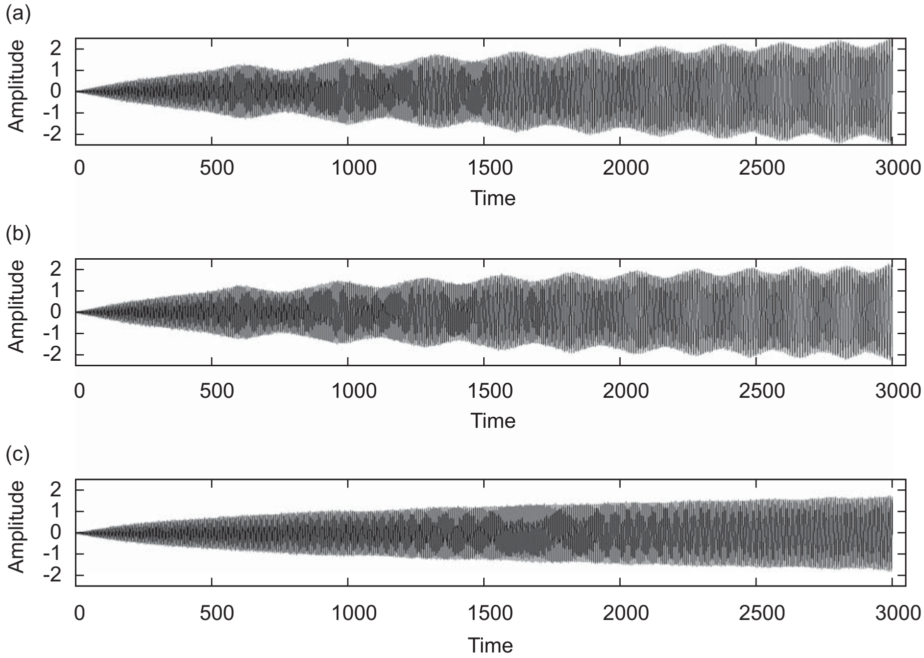

Fig. 5. Time responses using optimal drive sweep rates obtained from a numerical parameter search for the (a) pendulum ($\omega=1, \alpha=-0.0001043$), (b) softening Duffing oscillator ($\varepsilon=\frac{1}{6}, \omega=1, \alpha=-0.0001052$) and (c) softening Duffing–Van der Pol oscillator ($\varepsilon=\frac{1}{6}, \omega=1, \mu=0.005, \alpha=-0.0000653$) using the downward swept drive ($F\cos(\omega t-0.5(\alpha t^2)), F=0.005$).

rate transition from maximal autoresonant response to loss of autoresonance that occurs immediately above the optimal sweep rate and discussed in the next section.

3. Extremely sharp sweep rate transitions from maximal autoresonant response to loss of autoresonance immediately above optimal sweep rates

For all oscillators tested, numerical simulations show a gradual increase in autoresonant response with increasing sweep rate up to a maximal autoresonant response at the optimal sweep rate. This is followed by an extremely sharp transition from maximal autoresonant response to loss of autoresonance as the sweep rate is increased above the optimal sweep rate. Table 3 quantifies the extremely sharp transitions by listing the difference in sweep rate between optimal autoresonant response and loss of autoresonance for all oscillators tested.

Two types of plots help visualize the extremely sharp transitions from maximal autoresonant response to loss of autoresonance due to small changes in sweep rate. Fig. 7(a)–(c) display the maximum amplitude reached in a fixed simulation time as a function of sweep rate for the pendulum, the softening Duffing oscillator, and softening Duffing–Van der Pol oscillator, respectively. Figs. 8–10(a)–(c) show time responses for each oscillator, respectively, at three different sweep rates: (a) slightly below optimal, (b) optimal and (c) slightly above optimal.

The observed extremely sharp sweep rate transitions from autoresonance to loss of autoresonance are consistent with the finding of sharp sweep rate transitions in general nonstationary responses reported by Agrawal and Evan-Iwanowski [18]: "Detailed analysis of nonstationary responses leads to a startling situation. The separation between …categories…is very sharp. For instance the small difference in the transition (sweep) rates between 1.269 and 1.268—i.e., only 0.001—produces a marked difference in the nonstationary response."

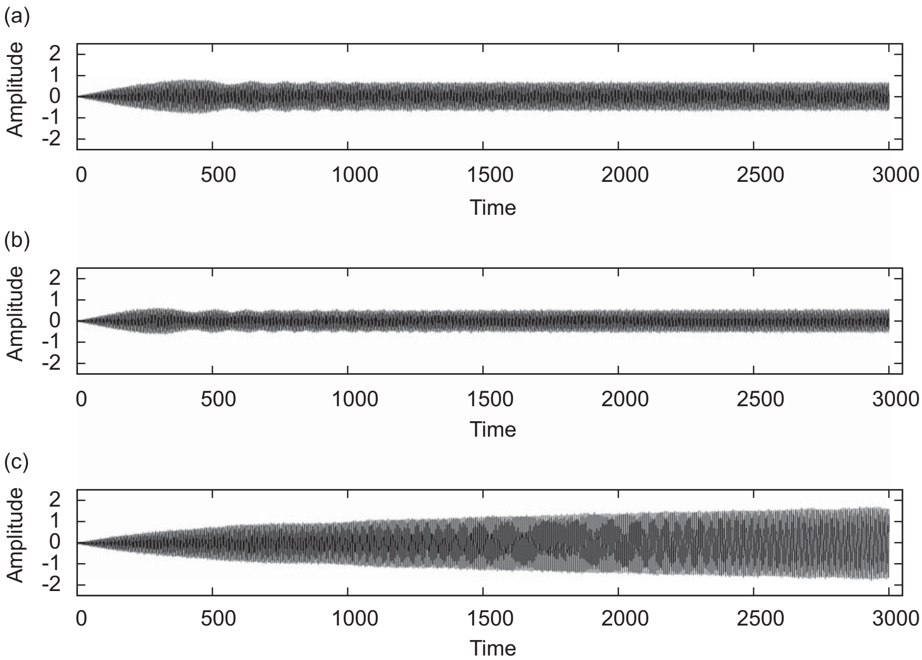

Fig. 6. Time responses using estimated drive sweep rates obtained from the beat method for the (a) pendulum ($\omega = 1, \alpha = -0.0001059$), (b) softening Duffing oscillator ($\varepsilon = \frac{1}{6}, \omega = 1, \alpha = -0.0001149$) and (c) softening Duffing–Van der Pol oscillator ($\varepsilon = \frac{1}{6}, \omega = 1, \mu = 0.005, \alpha = -0.0000640$) using the downward swept drive ($F\cos(\omega t - 0.5(\alpha t^2)), F = 0.005$).

Table 3
Sweep rate differences resulting in transitions between maximal autoresonant response and loss of autoresonance in the pendulum, softening Duffing oscillator, softening Duffing–Van der Pol oscillator, hardening Duffing oscillator and hardening Duffing–Van der Pol oscillator.

Oscillator	Sweep rate differences for autoresonance transitions
Pendulum	0.0000006
Softening Duffing	0.0000006
Softening Duffing–Van der Pol	0.0000002
Hardening Duffing	0.0000006
Hardening Duffing–Van der Pol	0.0000001

Estimated sweep rates that are near-optimal but lower than the optimal sweep rate result in autoresonance. However, due to the extremely sharp sweep rate transitions from maximal autoresonant response to loss of autoresonance occurring immediately above the optimal sweep rate, estimated sweep rates that are near-optimal but higher than the optimal sweep rate will generally not result in autoresonance. The value of the beat method in these cases is that the beat method is the only method currently available for estimating near-optimal sweep rates that can then be used as starting values to numerically or experimentally search for optimal sweep rates that generate maximal autoresonance response.

4. Conclusions

The possibility of predicting optimal drive linear sweep rates for autoresonance in Duffing-type oscillators was investigated in this paper. The main results of this investigation are:

1. Optimal drive linear sweep rates that generate maximal autoresonant response in softening and hardening Duffing-type oscillators can be estimated from the beat response to fixed frequency excitation at the oscillator's low-amplitude resonant frequency. Interestingly, this means that, for Duffing-type oscillators, the beat response to a stationary drive can be used in estimating an optimal nonstationary drive parameter (i.e., sweep rate). Duffing-type oscillators driven at their low-amplitude resonant frequency develop an amplitude-modulated and frequency-modulated beat oscillation. A least squares estimate of the smoothed slope of the instantaneous frequency versus time curve for the beat response during the first half-cycle is a near-optimal estimate of the optimal drive linear sweep rate needed to sustain resonance

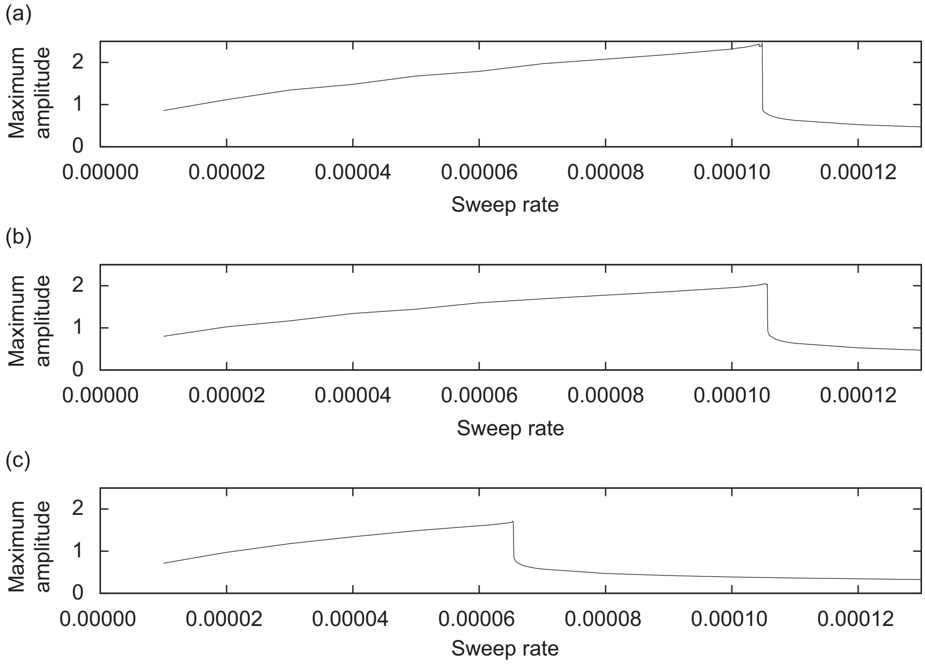

Fig. 7. Maximum amplitude as a function of sweep rate during a fixed simulation time ($t = 3000$ s) for the (a) pendulum, (b) softening Duffing oscillator and (c) softening Duffing–Van der Pol oscillator, showing extremely sharp sweep rate transitions from maximum autoresonance response to loss of autoresonance immediately above the optimal sweep rate.

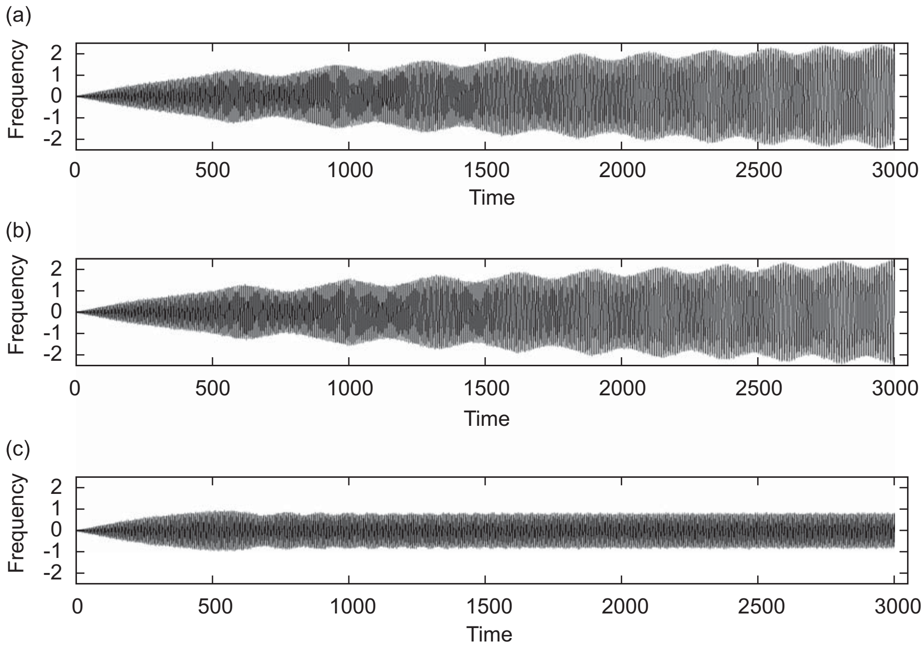

Fig. 8. Time responses at three drive sweep rates for the pendulum: (a) below optimal ($\alpha = -0.0001037$), (b) at optimal ($\alpha = -0.0001043$) and (c) above optimal ($\alpha = -0.0001049$), showing an extremely sharp sweep rate transition from maximum autoresonance response to loss of autoresonance immediately above the optimal sweep rate.

in Duffing-type oscillators. The near-optimal sweep rates estimated by the beat method can also be used as starting values to experimentally or numerically search for optimal sweep rates that generate maximal autoresonance response.

2. There are extremely sharp sweep rate transitions from maximal autoresonant response to loss of autoresonance as drive sweep rate increases immediately past the optimal sweep rate. This is consistent with, but even more pronounced, than earlier findings by Agrawal and Evan-Iwanowski [18] of the existence of such sharp transitions for

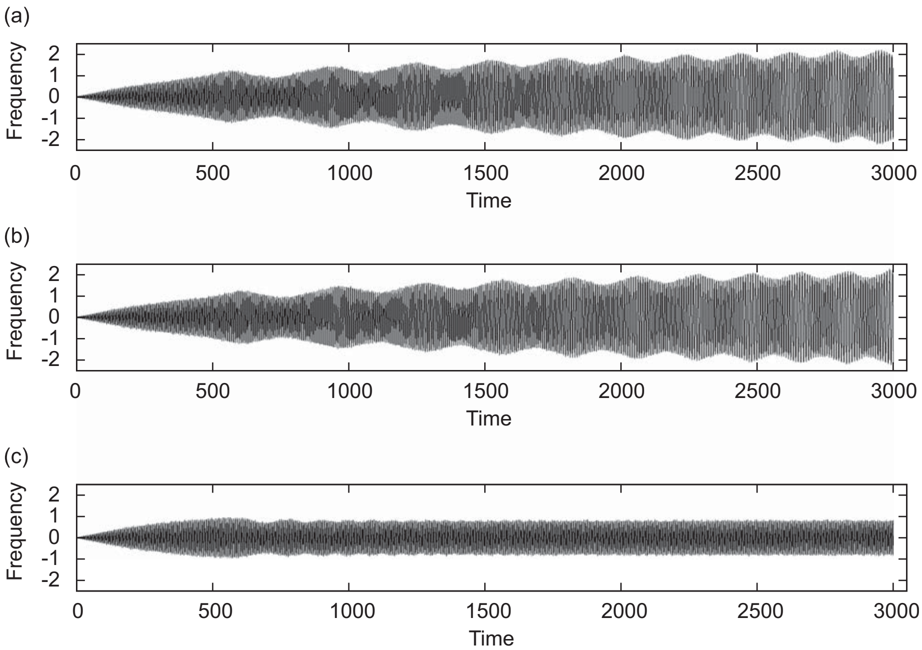

Fig. 9. Time responses at three drive sweep rates for the softening Duffing oscillator: (a) below optimal ($\alpha = -0.0001046$), (b) at optimal ($\alpha = -0.0001052$) and (c) above optimal ($\alpha = -0.0001058$), showing an extremely sharp sweep rate transition from maximum autoresonance response to loss of autoresonance immediately above the optimal sweep rate.

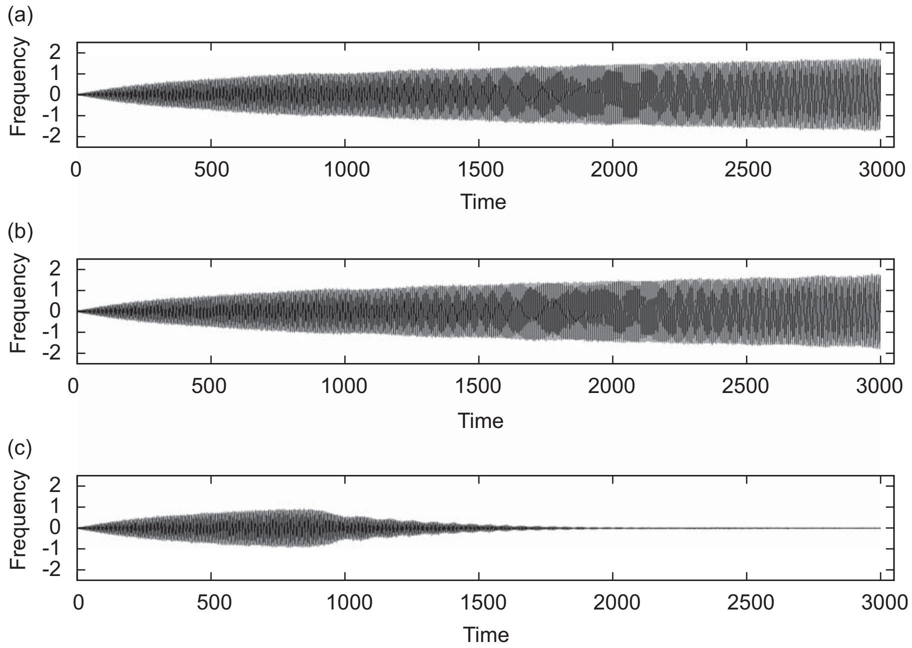

Fig. 10. Time responses at three drive sweep rates for the softening Duffing–Van der Pol oscillator: (a) below optimal ($\alpha = -0.0000651$), (b) at optimal ($\alpha = -0.0000653$) and (c) above optimal ($\alpha = -0.0000655$), showing an extremely sharp sweep rate transition from maximum autoresonance response to loss of autoresonance immediately above the optimal sweep rate.

general nonstationary responses. In those cases where sweep rate estimates provided by the beat method were close to but above the optimal sweep rate and therefore did not generate autoresonant responses, the estimates provided useful starting values to numerically search for optimal sweep rates that did generate maximal autoresonant response.

3. The beat method offers two advantages over Hübler's method of predicting optimal forcing functions for resonant stimulation of nonlinear oscillations. First, the beat method provides a near-optimal estimate of a single parameter, the

linear drive sweep rate, that makes it possible to use a linear sweep (an easily generated forcing function) to achieve maximal autoresonant response. This contrasts with the more complicated optimal forcing functions obtained from Hübler's method that may be more accurate but more difficult to experimentally generate or use in modeling. Second, unlike Hubler's method, the beat method does not require a model. Therefore, in addition to using models in numerical simulations, the beat method can also use experimental beat oscillation data obtained by driving a Duffing-type oscillator at its low-amplitude oscillator frequency. A least squares estimate of the slope of the smoothed TK–ESA instantaneous frequency versus time curve for the experimental beat response to a stationary drive at the oscillator's low-amplitude resonant frequency provides an estimate of the optimal drive linear sweep rate for generating maximal autoresonant response.

4. Autoresonance can be achieved by starting the drive sweep in resonance using the low-amplitude oscillator frequency instead of by ‘‘passage through resonance.’’ Friedland[8] suggests starting autoresonant sweeps off-resonance, slowly passing through resonance to assure phase-locking, instead of starting the sweep in resonance which ‘‘may require fine tuning of parameters.’’ The beat method accomplishes this fine tuning of the sweep rate parameter, at least in low-dimensional Duffing-type oscillators, so that autoresonance can be achieved by drive sweeps that start in resonance at the oscillator's low-amplitude resonant frequency. Starting drive sweeps in resonance has the advantage that resonance can be maintained throughout the drive sweep rather than only after capture into resonance. As noted by Friedland, as the number of degrees of freedom of the driven system increases, fine tuning of parameters becomes more difficult. Hence, in multidimensional systems, autoresonance via ‘‘passage through resonance’’ may be preferable to starting in resonance.
5. The continuous forms of the TK energy operator and ESA were found to be suitable for determining the instantaneous frequency versus time curve needed to estimate optimal drive linear sweep rates for autoresonance in Duffing-type oscillators. One advantage of the continuous forms of these algorithms over other instantaneous frequency algorithms is the relative ease of incorporating them in high-level system simulation languages.

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