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Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic method

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ABSTRACT

This paper deals with the nonlinear oscillations of a particle which moves on a rotating parabola. An analytic approximate technique, namely optimal homotopy asymptotic method (OHAM) is employed to propose an analytic approach to solve nonlinear oscillations. The validity of the OHAM is independent on whether or not there exist small or large parameters in the considered nonlinear equations. Our procedure provides us with a convenient way to optimally control the convergence of the approximate solutions. An example is given and the results reveal that this procedure is very effective, simple and accurate. This paper demonstrates the general validity and the great potential of the OHAM.

1. Introduction

The nonlinear problems are more difficult to solve than the linear ones. There exist some well-known analytical approaches applicable for nonlinear problems, such as the harmonic balance method [1], the multiple scales method [2], the Adomian decomposition method [3], the modified Lindstedt–Poincaré methods [4–6], the variational iteration method [7,8], the energy balance method [9], the δ method [10], or the homotopy perturbation method [11,12]. All of the above mentioned methods work very well for weakly nonlinear mechanical systems and some of them work even for strongly nonlinear problems.

In recent years, a growing interest towards the application of the homotopy techniques in nonlinear and strongly nonlinear problems has appeared in engineering practice. In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method [13]. This is in essence quite different from perturbation techniques. The HAM provides great freedom to use different base functions to express solutions of a nonlinear problem so that one can approximate a nonlinear problem more efficiently by means of base functions. This method has been successfully applied to solve many types of nonlinear problems [14–19].

Different from Liao's method, the homotopy perturbation method (HPM) proposed by He in 1998 [20] is in fact a new perturbation technique coupled with the homotopy technique. This method was also successfully applied in solving many types of nonlinear problems [21–23].

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In this paper, a different homotopy approach, namely the optimal homotopy asymptotic method (OHAM) is proposed to solve nonlinear problems. The efficiency of our procedure starts from the construction and the determination of the auxiliary function. Moreover, this method uses the principle of minimal sensitivity in order to achieve accurate results. The proposed method does not require a small parameter in the equation and provides a convenient way to optimally control the convergence of the solution.

Let us consider a nonlinear ODE of the form:

$$\ddot{X}(t)+k^2X(t)=f(X(t),\dot{X}(t),\ddot{X}(t)) \tag{1}$$

where the dot denotes the derivative with respect to time, k is a constant, f is in general a nonlinear term. The initial conditions are

$$X(0)=a,\dot{X}(0)=0 \tag{2}$$

where a is the amplitude of the oscillations. Note that it is unnecessary to assume the existence of any small or large parameter in Eq. (1). Thus, the proposed approach is rather general [24–29].

2. Formulation and solution approach

Eq. (1) describes a system oscillating with an unknown period T. We switch to a scalar time $\tau=2\pi t/T=\Omega t$. Under the transformations:

$$\tau=\Omega t, X(t)=ax(\tau) \tag{3}$$

the original Eq. (1) becomes

$$\Omega^2x''(\tau)+k^2x(\tau)=\frac{f(ax(\tau),a\Omega x'(\tau),a\Omega^2x''(\tau))}{a} \tag{4}$$

and the initial conditions become

$$x(0)=1,x'(0)=0 \tag{5}$$

where the prime denotes the derivative with respect to τ.

By the homotopy technique, we construct a homotopy in a more general form:

$$H(\phi(\tau,p),h(\tau,p))=(1-p)L(\phi(\tau,p))-h(\tau,p)N[\phi(\tau,p),\Omega(\lambda,p)]=0 \tag{6}$$

where L is a linear operator:

$$L(\phi(\tau,p))=\Omega_0^2\left[\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+\phi(\tau,p)\right] \tag{7}$$

while N is a nonlinear operator:

$$N[\phi(\tau,p),\Omega(\lambda,p)]=\Omega^2(\lambda,p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+(k^2+\lambda)\phi(\tau,p)-\frac{1}{a}f\left(a\phi(\tau,p),a\Omega(\lambda,p)\frac{\partial\phi(\tau,p)}{\partial\tau},a\Omega^2(\lambda,p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}\right)-p\lambda\phi(\tau,p) \tag{8}$$

where $p\in[0,1]$ is the embedding parameter, $h(\tau,p)$ is an auxiliary function so that $h(\tau,0)=0$, $h(\tau,p)\neq 0$ for $p\neq 0$, λ is an arbitrary parameter and Ω_0 will be given later. From Eqs. (2) and (3) we obtain the initial conditions:

$$\phi(0,p)=1,\frac{\partial\phi(\tau,p)}{\partial\tau}|_{\tau=0}=0 \tag{9}$$

Obviously when p=0 and 1 it holds:

$$\phi(\tau,0)=x_0(\tau),\phi(\tau,1)=x(\tau),\Omega(0)=\Omega_0,\Omega(1)=\Omega \tag{10}$$

where $x_0(\tau)$ is an initial approximation of $x(\tau)$. Therefore, as the embedding parameter p increases from 0 to 1, $\phi(\tau,p)$ varies from the initial approximation $x_0(\tau)$ to the solution $x(\tau)$, so does $\Omega(p)$ from the initial approximation Ω_0 to the exact frequency Ω.

Expanding $\phi(\tau,p)$ and $\Omega(p)$ in series with respect to the parameter p, one has, respectively

$$\phi(\tau,p)=x_0(\tau)+px_1(\tau)+p^2x_2(\tau)+\cdots \tag{11}$$

$$\Omega(p)=\Omega_0+p\Omega_1+p^2\Omega_2+\cdots \tag{12}$$

If the initial approximation $x_0(\tau)$ and the auxiliary function $h(\tau,p)$ are properly chosen so that the above series converges at $p=1$, one has

$$x(\tau)=x_0(\tau)+x_1(\tau)+x_2(\tau)+\cdots \tag{13}$$

$$\Omega=\Omega_0+\Omega_1+\Omega_2+\cdots \tag{14}$$

Notice that the series (11) and (12) contain the auxiliary function $h(\tau,p)$ which determines their convergence regions.

The results of the mth-order approximations are given by

$$\overline{x}(\tau)\approx x_0(\tau)+x_1(\tau)+\cdots+x_m(\tau) \tag{15}$$

$$\overline{\Omega}=\Omega_0+\Omega_1+\cdots+\Omega_m \tag{16}$$

We propose that the auxiliary function $h(\tau,p)$ to be of the form

$$h(\tau,p)=pK_1+p^2K_2+\cdots+p^mK_m(\tau) \tag{17}$$

where K_1, K_2,…,K_{m-1} can be constants and the last value $K_m(\tau)$ can be a function depending on the variable τ.

Substituting Eqs. (11) and (12) into Eq. (8) yields

$$N(\phi,\Omega)=N_0(x_0,\Omega_0,a,\lambda)+pN_1(x_0,x_1,\Omega_0,\Omega_1,a,\lambda)+p^2N_2(x_0,x_1,x_2,\Omega_0,\Omega_1,a,\lambda)+\cdots \tag{18}$$

If we substitute Eqs. (18) and (17) into Eq. (6) and we equate to zero the coefficients of various powers of p, we obtain the following linear equations:

$$L(x_0)=0,x_0(0)=1,x'(0)=0 \tag{19}$$

$$L(x_i)-L(x_{i-1})-\sum_{j=1}^{i}K_jN_{i-j}(x_0,x_1,\ldots,x_{i-j},\Omega_0,\Omega_1,\ldots,\Omega_{i-j},a,\lambda)=0,x_i(0)=x_i'(0)=0,\quad i=1,2,\ldots,m-1$$

$$L(x_m)-L(x_{m-1})-\sum_{j=1}^{m-1}K_jN_{m-1-j}-K_m(\tau)N_0=0,x_m(0)=x_m'(0)=0 \tag{20}$$

Note that Ω_k can be determined avoiding the presence of secular terms in Eq. (20).

The frequency Ω depends upon the arbitrary parameter λ and we apply the so-called "principle of minimal sensitivity" [30] in order to fix the value of λ. We do this imposing that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}=0 \tag{21}$$

This principle of the minimal sensitivity appears for the first time in the quantum field theory, $\lambda\phi^4$ theory or quantum chromodynamics [30]. In its original formulation a Lagrangian density $\mathcal{L}$ which is not exactly solvable, is interpolated with a solvable Lagrangian $\mathcal{L}_0(\lambda)$ depending upon one (ore more) parameter $\lambda:\mathcal{L}_\delta=\mathcal{L}_0(\lambda)+\delta(\mathcal{L}-\mathcal{L}_0(\lambda))$, δ being a parameter. We notice that the interpolation of the full Lagrangian with the solvable one, $\mathcal{L}_0(\lambda)$, brings an artificial dependence upon the arbitrary parameter λ. Such dependence, which would vanish if all perturbative orders were calculated, can be made weaker to a finite perturbative order, by requiring some physical observable P to be locally insensitive to λ, i.e. $\partial P/\partial\lambda=0$. In the above application this physical observable P is in fact the frequency Ω. This condition is known as the principle of minimal sensitivity and is normally seen to improve the convergence to the exact solution.

At this moment, the mth-order approximation given by Eq. (15) depends on the parameters (functions) K_1, K_2,…,K_m. The constants K_1, K_2,…,K_{m-1} and those constants which eventually appear in the expression of $K_m(\tau)$, can be identified via various methods, such as the least square method, the Galerkin method, the collocation method or by minimizing the square residual error.

Our procedure contains the auxiliary function $h(\tau,p)$, which provides us with a simple way to adjust and control the convergence of solution. It is very important to properly choose the last function $K_m(\tau)$, which appears in the mth-order approximation (15).

Unlike other homotopy methods, such as HAM or HPM, in the proposed procedure (OHAM) the construction of homotopy is quite different. In the frame of OHAM the linear operator L is well defined by Eq. (7) and the initial approximation is rigorously determined from Eq. (19), while in other homotopy procedures such as HAM these ones are arbitrarily chosen. Instead of an infinite series (as is the case of HAM), the OHAM searches for only a few terms (mostly two or three terms). The way to ensure the convergence in OHAM is quite different and more rigorous. Unlike other homotopy procedures, OHAM ensure a very rapid convergence since it needs only two iterations for achieving a very accurate solution. This is in fact the true power of the method. OHAM does not need a recurrence formula as other homotopy procedures such as HAM does. OHAM is an iterative procedure which converges to the exact solution after only two iterations. Iterations are performed in a very simple manner by identifying some coefficients. OHAM does not need high-order approximations, as HAM does. OHAM does not use the rules established in the frame of HAM, it is a self-sustained method which has no "open questions" as other homotopy procedures. OHAM does not need the restrictive condition

$A(1)$=1 as HAM does. Finally, OHAM provides an analytic solution for complicated nonlinear problems expressed on only two rows, unlike other homotopy procedures which need few pages to express an analytic solution.

3. The motion of a particle on a rotating parabola

We introduce the basic ideas of the proposed method by considering the motion of a particle on a rotating parabola, by considering the following nonlinear differential equation, mentioned by Nayfeh and Mook in [1]:

$$(1+4q^2X^2)\frac{d^2X}{dt^2}+\Lambda X+4q^2\left(\frac{dX}{dt}\right)^2X=0 \tag{22}$$

with the boundary conditions:

$$X(0)=a,\frac{dX}{dt}(0)=0 \tag{23}$$

where q and Λ are known constants and need not be small.

Under the transformations (3), Eqs. (22) and (23) become

$$\Omega^2x''+\omega_0^2x+4q^2a^2\Omega^2(x^2x''+x\dot{x}^2)=0 \tag{24}$$

respectively

$$x(0)=1,x'(0)=0 \tag{25}$$

where $\Lambda=\omega_0^2$ and $'=d/d\tau$.

The operators (7) and (8) are, respectively

$$L(\phi(\tau,p))=\Omega_0^2[\phi''(\tau,p)+\phi(\tau,p)] \tag{26}$$

$$N[\phi(\tau,p),\Omega(\tau,p)]=\Omega^2(p)\phi''(\tau,p)+(\omega_0^2+\lambda)\phi(\tau,p)+4q^2a^2\Omega^2[\phi^2(\tau,p)\phi''(\tau,p)+\phi(\tau,p)\phi'^2(\tau,p)]-p\lambda\phi(\tau,p) \tag{27}$$

where ϕ and Ω are given by Eqs. (11) and (12), respectively, and λ is an unknown parameter. From Eqs. (19) and (20), (m=2), we obtain the following three equations:

$$\Omega_0^2(x_0''+x_0)=0,x_0(0)=1,x'(0)=0 \tag{28}$$

$$\Omega_0^2(x_1''+x_1)-\Omega_0^2(x_0''+x_0)-K_1[\Omega_0^2x_0''+(\omega_0^2+\lambda)x_0+4q^2a^2\Omega_0^2(x_0x_0''+x_0'^2x_0)]=0,x_1(0)=x_1'(0)=0 \tag{29}$$

$$\begin{aligned}&\Omega_0^2(x_2''+x_2)-\Omega_0^2(x_1''+x_1)-K_1\{2\Omega_0\Omega_1x_0''+\Omega_0^2x_1''+(\omega_0^2+\lambda)x_1+4q^2a^2[\Omega_0^2(2x_0x_0''x_1+x_0^2x_1''+2x_0x_0'x_1'+2x_0x_0'x_1'\\&+x_0'^2x_1)+2\Omega_0\Omega_1(x_0^2x_0''+x_0'^2x_0)]-\lambda x_0\}-K_2(\tau)[\Omega_0^2x_0''+(\omega_0^2+\lambda)x_0+4q^2a^2\Omega_0^2(x_0^2x_0''+x_0'^2x_0)]=0,x_2(0)=x_2'(0)=0\end{aligned} \tag{30}$$

Eq. (28) has the following solution:

$$x_0(\tau)=\cos\tau \tag{31}$$

If this result is substituted into Eq. (29) and assuming that $K_1=C_1$=constant, we obtain the following equation:

$$\Omega_0^2(x_1''+x_1)-C_1[(\omega_0^2+\lambda-\Omega_0^2-2q^2a^2\Omega_0^2)\cos\tau-2q^2a^2\Omega_0^2\cos 3\tau]=0,x_1(0)=x_1'(0)=0 \tag{32}$$

where C_1 is an unknown constant at this moment. Avoiding the presence of a secular term needs:

$$\Omega_0^2=\frac{\omega_0^2+\lambda}{1+2q^2a^2} \tag{33}$$

with this requirement, the solution of Eq. (32) is

$$x_1(\tau)=\frac{1}{4}C_1q^2a^2(\cos 3\tau-\cos\tau) \tag{34}$$

If we substitute Eqs. (31), (33) and (34) into Eq. (30), we obtain the equation in x_2:

$$\begin{aligned}&\Omega_0^2(x_2''+x_2)+\frac{2C_1q^2a^2(\omega_0^2+\lambda)}{1+2q^2a^2}\cos 3\tau+C_1\left\{\left[\frac{C_1q^4a^4(\omega_0^2+\lambda)}{2(1+2q^2a^2)}+2\Omega_0\Omega_1(1+2q^2a^2)+\lambda\right]\cos\tau\right.\\&\left.+\left[\frac{(\omega_0^2+\lambda)C_1q^2a^2(3q^2a^2+16)}{2(1+2q^2a^2)}+2\Omega_0\Omega_1q^2a^2\right]\cos 3\tau+\frac{9C_1q^4a^4(\omega_0^2+\lambda)}{2(1+2q^2a^2)}\cos 5\tau\right\}\\&+K_2(\tau)\left[\frac{2q^2a^2(\omega_0^2+\lambda)}{1+2q^2a^2}\cos 3\tau\right]=0,x_2(0)=x_2'(0)=0\end{aligned} \tag{35}$$

No secular term in $x_2(\tau)$ requires that

$$2\Omega_0\Omega_1 = -\frac{\lambda}{1+2q^2a^2} - \frac{C_1q^4a^4(\omega_0^2+\lambda)}{2(1+2q^2a^2)^2} \tag{36}$$

From Eqs. (36) and (14), we obtain the frequency in the form

$$\Omega = \Omega_0 - \frac{\lambda}{\Omega_0(1+2q^2a^2)} - \frac{C_1q^4a^4\Omega_0}{4(1+2q^2a^2)} \tag{37}$$

where Ω_0 is given by Eq. (33).

The parameter λ can be determined applying the "principle of minimal sensitivity" (21) and thus we obtain

$$\lambda = \frac{C_1\omega_0q^4a^4}{2+4q^2a^2-C_1q^4a^4} \tag{38}$$

This result is substituted into Eq. (37) and we have

$$\Omega = \frac{\omega_0}{1+2q^2a^2}\sqrt{1+2q^2a^2-\frac{1}{2}C_1q^4a^4} \tag{39}$$

Substituting Eqs. (37)–(39) into Eq. (35), we obtain

$$x_2'' + x_2 + 2C_1q^2a^2\cos 3\tau + \frac{C_1^2q^2a^2(5q^4a^4+7q^2a^2+2)}{1+2q^2a^2}\cos 3\tau + \frac{9}{2}C_1^2q^4a^4\cos 5\tau + 2K_2(\tau)q^2a^2\cos 3\tau = 0$$

$$x_2(0) = x_2'(0) = 0 \tag{40}$$

There are many possibilities to choose the function $K_2(\tau)$. The convergence of the solution $x_2(\tau)$ and consequently the convergence of the approximate solution $\tilde{x}(\tau)$ depend on the auxiliary function $K_2(\tau)$. Basically, the shape of $K_2(\tau)$ must follow the terms appearing in Eq. (35), which are $\cos\tau$, $\cos 3\tau$, $\cos 5\tau$ (odd-order harmonics). Therefore we try to choose $K_2(\tau)$ so that in Eq. (35) the product

$$K_2\left[\frac{2q^2a^2(\omega_0^2+\lambda)}{1+2q^2a^2}\cos 3\tau\right]$$

be of the same shape with the other terms (a combination of functions $\cos\tau$, $\cos 3\tau$, $\cos 5\tau, \ldots$).

In other applications, such as those presented in [26–28], the function $K_2(\tau)$ (or $h(\tau,p)$) could be chosen as exponential function, polynomial function and so on, depending on the shape of the terms already present in the specific iteration.

All three cases presented in the paper demonstrate the importance of the function $K_2(\tau)$ on the accuracy of the solution. In the same time, a bigger number of constants in $K_2(\tau)$ lead to a better accuracy of the results. If the error obtained using a certain $K_2(\tau)$ is unsatisfactory, one can choose other shapes for this function.

We will consider three cases:

Case A: We consider the function K_2 of the form

$$K_2(\tau) = C_2' \tag{41}$$

where C_2' is a constant.

Substituting Eq. (41) into Eq. (40), we obtain the equation in x_2:

$$x_2'' + x_2 + \left[2(C_1+C_2')q^2a^2 + \frac{C_1^2q^2a^2(5q^4a^4+7q^2a^2+2)}{1+2q^2a^2}\right]\cos 3\tau + \frac{9}{2}C_1^2q^4a^4\cos 5\tau = 0, x_2(0) = x_2'(0) = 0 \tag{42}$$

The solution of Eq. (42) becomes

$$x_2(\tau) = \left[\frac{C_1+C_2'}{4} + \frac{C_1^2q^2a^2(5q^4a^4+7q^2a^2+2)}{8(1+2q^2a^2)}\right](\cos 3\tau - \cos\tau) + \frac{3}{16}C_1^2q^4a^4(\cos 5\tau - \cos\tau) \tag{43}$$

The second-order approximate solution is

$$\overline{x}(\tau) = x_0(\tau) + x_1(\tau) + x_2(\tau)$$

where x_0, x_1 and x_2 are given by Eqs. (31), (34) and (43). Using the transformations (3), the second-order approximate solution of Eq. (22) becomes

$$\overline{x}(t) = A\cos\Omega t + B\cos 3\Omega t + C\cos 5\Omega t \tag{44}$$

where Ω is given by Eq. (39) and

$$A = a - \frac{2C_1+C_2'}{4}q^2a^3 - \frac{C_1^2q^2a^3(16q^4a^4+17q^2a^2+4)}{16(1+2q^2a^2)}$$

$$B=\frac{2C_1+C_2'}{4}q^2a^3+\frac{C_1^2q^2a^3(5q^4a^4+7q^2a^2+2)}{8(1+2q^2a^2)}$$

$$C=\frac{3}{16}C_1^2q^4a^5 \tag{45}$$

Case B: We consider the function $K_2(\tau)$ if the form

$$K_2(\tau)=C_2+C_3\cos 2\tau+C_4\cos 4\tau \tag{46}$$

where C_2, C_3 and C_4 are constants.

Substituting Eq. (46) into Eq. (40) and avoiding the presence of a secular term, we obtain

$$C_4=-C_3 \tag{47}$$

respectively

$$x_2''+x_2+\left[2(C_1+C_2)q^2a^2+\frac{C_1^2q^2a^2(5q^4a^4+7q^2a^2+2)}{1+2q^2a^2}\right]\cos 3\tau+\left[\frac{9}{2}C_1^2q^4a^4+C_3q^2a^2\right]\cos 5\tau-C_3q^2a^2\cos 7\tau=0$$

$$x_2(0)=x_2'(0)=0 \tag{48}$$

With these requirements, the solution of Eq. (48) becomes

$$x_2(\tau)=\left[\frac{C_1+C_2}{4}q^2a^2+\frac{C_1^2q^2a^2(5q^4a^4+7q^2a^2+2)}{8(1+2q^2a^2)}\right](\cos 3\tau-\cos\tau)+\left[\frac{3}{16}C_1^2q^4a^4+\frac{1}{24}C_3q^2a^2\right](\cos 5\tau-\cos\tau)$$
$$-\frac{1}{48}C_3q^2a^2(\cos 7\tau-\cos\tau) \tag{49}$$

The second-order approximate solution in this case is

$$\overline{x}=\tilde{A}\cos\Omega t+\tilde{B}\cos 3\Omega t+\tilde{C}\cos 5\Omega t+\tilde{D}\cos 7\Omega t \tag{50}$$

where Ω is given by Eq. (39) and the coefficients are

$$\tilde{A}=a-\frac{2C_1+C_2}{4}q^2a^3-\frac{C_1^2q^2a^3(16q^4a^4+17q^2a^2+4)}{16(1+2q^2a^2)}-\frac{1}{48}C_3q^2a^3$$

$$\tilde{B}=\frac{2C_1+C_2}{4}q^2a^3+\frac{C_1^2q^2a^3(5q^4a^4+7q^2a^2+2)}{8(1+2q^2a^2)}$$

$$\tilde{C}=\frac{3}{16}C_1^2q^4a^5+\frac{1}{24}C_3q^2a^3$$

$$\tilde{D}=-\frac{1}{48}C_3q^2a^3 \tag{51}$$

Case C: We consider the function $K_2(\tau)$ of the form

$$K_2(\tau)=C_2^*+C_3^*\cos 2\tau+C_4^*\cos 4\tau+C_5^*\cos 6\tau+C_6^*\cos 8\tau \tag{52}$$

where C_2^*, C_3^*, C_4^*, C_5^* and C_6^* are constants.

Substituting Eq. (52) into Eq. (40), we obtain

$$C_4^*=-C_3^* \tag{53}$$

$$x_2(\tau)=\left[\frac{2C_1+2C_2^*+C_5^*}{8}q^2a^2+\frac{C_1^2q^2a^2(5q^4a^4+7q^2a^2+2)}{8(1+2q^2a^2)}\right](\cos 3\tau-\cos\tau)+\left[\frac{3}{16}C_1^2q^4a^4+\frac{1}{24}(C_3^*+C_6^*)q^2a^2\right](\cos 5\tau$$
$$-\cos\tau)+\frac{1}{48}C_3^*q^2a^2(\cos\tau-\cos 7\tau)+\frac{1}{80}C_5^*q^2a^2(\cos 9\tau-\cos\tau)+\frac{C_6^*}{120}q^2a^2(\cos 11\tau-\cos\tau) \tag{54}$$

The second-order approximate solution of Eq. (22) becomes

$$\overline{x}(t)=\overline{A}\cos\Omega t+\overline{B}\cos 3\Omega t+\overline{C}\cos 5\Omega t+\overline{D}\cos 7\Omega t+\overline{E}\cos 9\Omega t+\overline{F}\cos 11\Omega t \tag{55}$$

where

$$\overline{A}=a-\frac{q^2a^3}{240}(120C_1+60C_2^*+5C_3^*+33C_4^*+12C_5^*)-\frac{C_1^2q^2a^3(16q^4a^4+17q^2a^2+4)}{16(1+2q^2a^2)}$$

$$\overline{B}=\frac{4C_1+2C_2^*+C_4^*}{8}q^2a^3+\frac{C_1^2q^2a^3(5q^4a^4+7q^2a^2+2)}{8(1+2q^2a^2)}$$

$$\overline{C}=\frac{C_3^*+C_5^*}{24}q^2a^3+\frac{3}{16}C_1^2q^4a^5$$

$$\overline{D}=-\frac{1}{48}C_3^2a^3$$

$$\overline{E}=\frac{1}{80}C_4^*q^2a^3$$

$$\overline{F}=\frac{1}{120}C_5^*q^2a^3 \tag{56}$$

3.1. Numerical examples

We will show that the error of the solutions decreases when the number of terms in the auxiliary function $h(\tau,p)$ increases. In Eqs. (22) and (23), we consider $\Lambda=\omega_0=1$, a=1 and two cases for q in every of the cases A, B and C. The constants C_i are obtained using the least square method.

(a) For q=1 in the case A, it is obtained

$$C_1=-0.401483291, C_2'=-0.065781508 \tag{57}$$

The second-order approximate solution (44) becomes in this case

$$\overline{x}(t)=1.092937297\cos\Omega t-0.123160203\cos 3\Omega t+0.030222906\cos 5\Omega t \tag{58}$$

where Ω is obtained from Eq. (39): Ω=0.596353888.

(b) For q=1 in the case B, it is obtained

$$C_1=-0.398431527; C_2=-0.052485317; C_3=0.0341786762 \tag{59}$$

$$\overline{x}(t)=1.089257032\cos\Omega t-0.119734278\cos 3\Omega t+0.031189301\cos 5\Omega t-0.00712055\cos 7\Omega t \tag{60}$$

where Ω=0.596211722.

(c) For q=1 in the case C, we obtain the following results:

$$C_1=-0.395753003, C_2^*=-0.24453992, C_3^*=0.396618201, C_4^*=-0.396618201,$$
$$C_5^*=0.534493194, C_6^*=-0.497490133 \tag{61}$$

$$\overline{x}(t)=1.08140204\cos\Omega t-0.100837908\cos 3\Omega t+0.025163334\cos 5\Omega t$$
$$-0.008262879\cos 7\Omega t+0.006681164\cos 9\Omega t-0.004145751\cos 11\Omega t \tag{62}$$

where Ω=0.596087918.

(d) For q=2 in the case A we obtain

$$C_1=-0.167434521, C_2'=-0.02382096 \tag{63}$$

$$\overline{x}(t)=1.081827345\cos\Omega t-0.165930301\cos 3\Omega t+0.084102956\cos 5\Omega t \tag{64}$$

where Ω=0.357278398.

(e) For q=2 in the case B it is obtained

$$C_1=-0.164357411, C_2=0.017447955, C_3=-0.073610524 \tag{65}$$

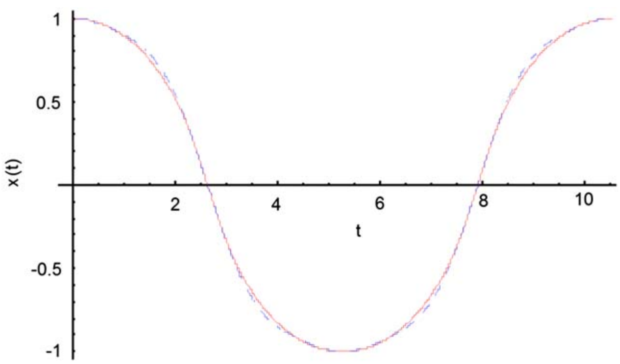

Fig. 1. Comparison between the approximate results in case (a), Eq. (58) and numerical results of Eq. (22) for $\Lambda=\omega_0=a=q=1$: ——— numerical solution; _ _ _ _ approximate solution.

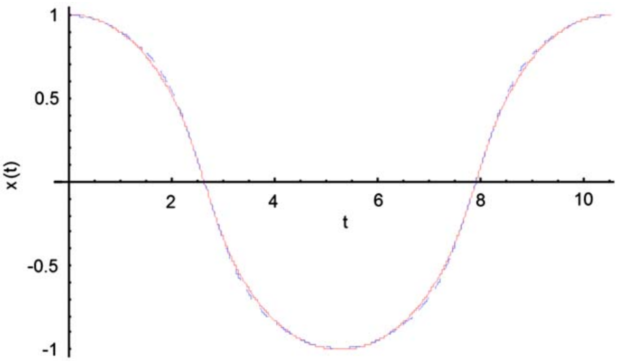

Fig. 2. Comparison between the approximate results in case (b), Eq. (60) and numerical results of Eq. (22) for $\Lambda=\omega_0=a=q=1$: ——— numerical solution; _ _ _ _ approximate solution.

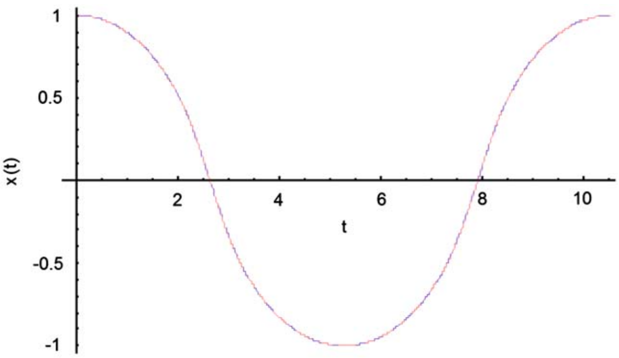

Fig. 3. Comparison between the approximate results in case (c), Eq. (62) and numerical results of Eq. (22) for $\Lambda=\omega_0=a=q=1$: ——— numerical solution; _ _ _ _ approximate solution.

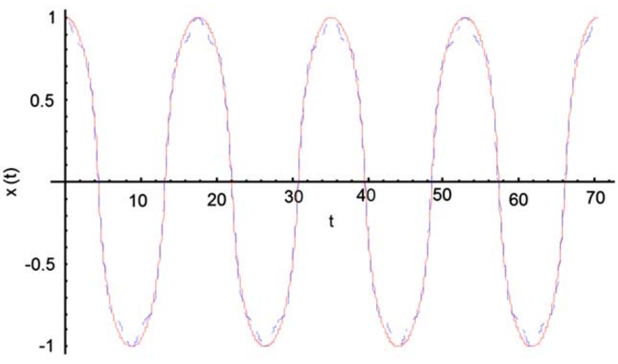

Fig. 4. Comparison between the approximate results in case (d), Eq. (64) and numerical results of Eq. (22) for $\Lambda=\omega_0=a=1$, $q=2$: ——— numerical solution; _ _ _ _ approximate solution.

$$\overline{x}(t)=1.071279364\cos\Omega t-0.146185231\cos 3\Omega t+0.068771659\cos 5\Omega t+0.00613421\cos 7\Omega t \tag{66}$$

where Ω=0.356852829.

(f) For q=2 in the case C it is obtained

$$C_1=-0.16124603, C_2^*=-0.103086948, C_3^*=0.254864679, C_4^*=-0.254864679, C_5^*=0.236002415, C_6^*=-0.345719606 \tag{67}$$

$$\begin{aligned}\overline{x}(t)=&1.1067914118\cos\Omega t-0.148687186\cos 3\Omega t+0.062858358\cos 5\Omega t\\&-0.021238723\cos 7\Omega t+0.01180012\cos 9\Omega t-0.011523986\cos 11\Omega t\end{aligned} \tag{68}$$

where Ω=0.356422004.

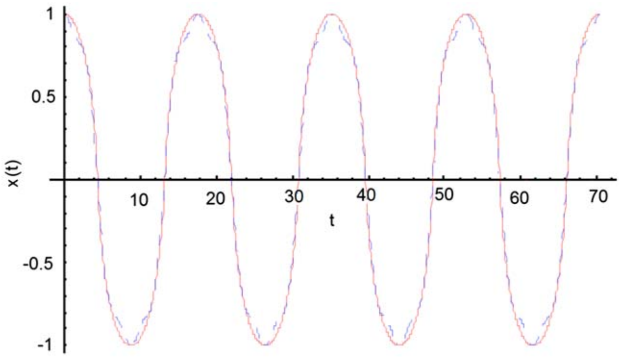

Fig. 5. Comparison between the approximate results in case (e), Eq. (66) and numerical results of Eq. (22) for $\Lambda=\omega_0=a=1$, $q=2$: ——— numerical solution; _ _ _ _ approximate solution.

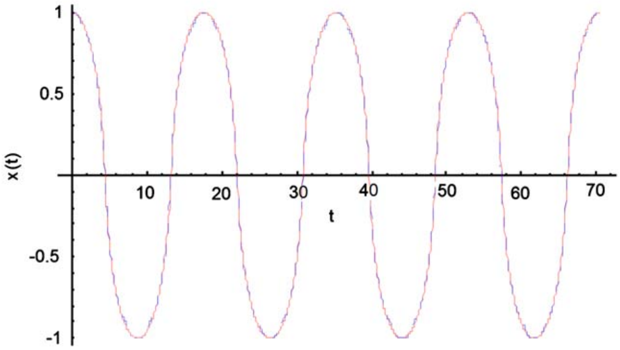

Fig. 6. Comparison between the approximate results in case (f), Eq. (68) and numerical results of Eq. (22) for $\Lambda=\omega_0=a=1$, $q=2$: ——— numerical solution; _ _ _ _ approximate solution.

It is easy to verify the accuracy of the obtained solutions if we graphically compare these analytical solutions with the numerical ones. Figs. 1–6 show the comparison between the present solutions and the numerical integration results obtained by a fourth-order Runge–Kutta method.

It can be seen from Figs. 1–6 that the solutions obtained by OHAM are nearly identical with the solutions obtained by a fourth-order Runge–Kutta method. Moreover, the analytical solutions obtained by our procedure prove to be more accurate along with an increased number of terms in the auxiliary function $h(\tau,p)$.

4. Conclusions

In this paper, the optimal homotopy asymptotic method (OHAM) is employed to propose a new analytic approximate solution for some nonlinear oscillations. The validity of the method is illustrated on the motion of a particle on a rotating parabola. Our procedure is valid even if the nonlinear equation does not contain any small or large parameters.

Our construction of homotopy is different from classical HAM, especially referring to the parameter λ (determined using the principle of minimal sensitivity), the auxiliary function $h(\tau,p)$, the operator L (unlike HAM the linear operator and the initial approximation are not arbitrarily chosen) and the presence of some constants $C_1,C_2,\ldots$ which ensure a fast convergence of the solution. The examples presented in this paper lead to the conclusion that the accuracy of the obtained results is growing along with increasing the number of constants in the auxiliary function. Unlike HAM, which employ the so-called $\hbar$-curves in order to ensure the convergence of the solution using a convergence-control parameter $\hbar$, the OHAM provides us with a simple and rigorous way to control and adjust the convergence of a solution through the auxiliary functions $h(\tau,p)$ involving a number of constants C_i which are optimally determined. Unlike HAM which needs recurrence formulas, OHAM is an iterative procedure and iterations are performed in a very simple manner by identifying some coefficients and therefore very good approximations are obtained in few terms. Actually the capital strength of OHAM is its fast convergence, since after only two iterations it converges to the exact solution, which proves that this method is very efficient in practice.

In this work we proposed a new approach, and this version of the method proves to be very rapid and effective and this is proved by comparing the solutions obtained through the proposed method with the solutions obtained via numerical simulations. This paper shows one step in the attempt to develop a new nonlinear analytical technique in the absence of small or large parameters.

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