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On the existence of longitudinal or flexural waves in rods at nonlinear higher harmonics

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ABSTRACT

This article theoretically studies the conditions for existence of longitudinal or flexural waves in nonlinear, isotropic rods and presents numerical simulations corroborating the theoretical results. It has been known that the existence of guided waves at nonlinearity induced double harmonics is subject to constraints which arise from the potential of power flux transfer from the primary generating mode to the generated higher order modes. The knowledge about the behavior of waves in rods at harmonics higher than double is still limited. This gap was addressed here by the method of perturbation coupled with wavemode orthogonality and forced response. This reduces the nonlinear problem to a forced linear problem which was subsequently investigated to formulate an angular order-based constraint as the condition of existence/nonexistence of nonlinearity-driven higher harmonics of longitudinal and flexural waves in rods.

1. Introduction

In an earlier article [1], the authors conducted theoretical studies on the symmetry characteristics of Rayleigh–Lamb guided waves in nonlinear, isotropic plates. Considering weak nonlinearity, the problem was reduced to a set of forced linear problems following the earlier works by de Lima and Hamilton [2] and Auld [3]. The problem was developed to formulate an energy level constraint as the defining factor for the absence of antisymmetric Lamb waves at any order of even higher harmonics. The energy constraint also indicated the potential presence of both antisymmetric and symmetric Lamb waves at any order of odd harmonics. This result is the plate analogue of Gol'dberg's [4] results on nonlinear bulk waves where he concluded that at the double harmonic, transverse waves get generated by longitudinal waves but longitudinal waves are not generated by transverse waves.

The purpose of the present article is to extend the study of plate waves of Ref. [1] to the case of rod waves. Guided waves combine the sensitivity of nonlinear parameters with large inspection ranges [5]. Therefore, their application to non-destructive evaluation and structural health monitoring has drawn considerable research interest [6–8]. As in the case of plate waves, it is shown in this paper that there are generation laws in the case of rod waves in uniform homogeneous waveguides. A deviation from these laws in experiments is indicative of the presence of local inhomogeneities like cracks, notches, etc., in the waveguide. Hence, it is of practical importance in the field of rod wave NDE and SHM to discriminate which wave-modes have the possibility of generation at higher harmonics and which cannot be generated.

Following Murnaghan [9], strain energy is expressed as a summation series of powers of strain components. The analysis is similar to that by de Lima and Hamilton [10] who solved the problem for the first-order nonlinearity. It was

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assumed that the nonlinear displacement caused at the higher harmonics is small compared to the primary excitation. This assumption of weak nonlinearity is a common assumption which pertains to solid mechanics of metal waveguides under elastic loads. de Lima and Hamilton's analysis is extended here by decomposing a problem consisting of several orders of nonlinearity to several forced problems, each corresponding to a single order of nonlinearity. It was found that the angular order of the primary generating mode determines which modes have the potential of being generated at either even or odd higher harmonic.

The results of the analytical study were finally corroborated with numerical results from the nonlinear extension of the semi-analytical finite element method [11,12].

2. Statement of the nonlinear problem

The equation of motion for nonlinear elasticity in a stress free rod is given by (Fig. 1)

$$(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla\times(\nabla\times\mathbf{u})+\mathbf{f}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1}$$

with stress-free boundary conditions:

$$[\mathbf{S^L}(\mathbf{u})-\overline{\mathbf{S}}(\mathbf{u})]\cdot\mathbf{n_r}=\mathbf{0}\quad \text{on } \Gamma \tag{2}$$

where $\mathbf{u}$ is the particle displacement, λ and μ are the Lamé constants, ρ_0 is the initial density of the body, $\mathbf{f}$ is the body force, $\mathbf{n_r}$ is the unit vector normal to the surface of the waveguide Γ, $\mathbf{S^L}$ and $\overline{\mathbf{S}}$ are the linear and nonlinear parts of the second Piola–Kirchoff stress tensor, respectively.

Energy is written in Murnaghan potentials [9,13]:

$$E=\phi_2+\phi_3+\phi_4\ldots \tag{3}$$

where ϕ_n corresponds to the set of terms in the energy expression which are of degree n in strain multiples. Strain can be expressed in terms of covariant differentials of displacements:

$$\varepsilon_{ij}=\tfrac{1}{2}(u_{i;j}+u_{j;i}+u_{k;i}u^k_{;j}) \tag{4}$$

Stress and body force are given by

$$\sigma^{ij}=\frac{\partial E}{\partial\varepsilon_{ij}},\quad f^i=\sigma^{ij}_{;j} \tag{5}$$

3. Solution to the nonlinear problem

Following Auld [3] and de Lima and Hamilton [10] and using the method of perturbation, the first-order nonlinear solution is written as linear combination of the existing guided wavemodes at 2ω:

$$\mathbf{v}(\mathbf{r},z,t)=\frac{1}{2}\sum_{m=1}^{\infty}A_m(z)\mathbf{v}_m(\mathbf{r})e^{-i2\omega t}+c.c. \tag{6}$$

where $c.c.$ denotes complex conjugate, $\mathbf{v}_m=\partial\mathbf{u}_m/\partial t$ is the particle velocity of the mth mode at 2ω, and A_m is the higher order modal amplitude given by

$$A_m(z)=\overline{A}_m(z)e^{i(2\kappa z)}-\overline{A}_m(0)e^{i\kappa_n^* z} \tag{7}$$

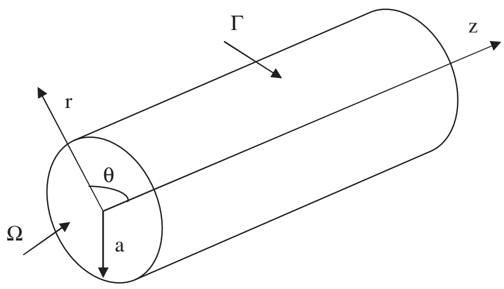

Fig. 1. Schematic of a stress free rod.

where

$$\overline{A}_m(z) = \mathrm{i}\,\frac{(f_n^{\mathrm{vol}} + f_n^{\mathrm{surf}})}{4P_{mn}[\kappa_n^* - 2\kappa]}, \quad \kappa_n^* \neq 2\kappa \quad \text{(asynchronous solution)} \tag{8}$$

$$\overline{A}_m(z) = \frac{(f_n^{\mathrm{vol}} + f_n^{\mathrm{surf}})}{4P_{mn}} z, \quad \kappa_n^* = 2\kappa \quad \text{(synchronous solution)} \tag{9}$$

$$P_{mn} = -\frac{1}{4}\int_\Omega (\mathbf{v_n^*} \cdot \mathbf{S_m} + \mathbf{v_m} \cdot \mathbf{S_n^*}) \cdot \mathbf{n}_z \, \mathrm{d}\Omega \tag{10}$$

$$f_n^{\mathrm{surf}}(z) = \int_\Gamma \mathbf{v_n^*} \cdot \overline{\mathbf{S}} \cdot \mathbf{n_r} \, \mathrm{d}\Gamma \tag{11}$$

$$f_n^{\mathrm{vol}}(z) = \int_\Omega \mathbf{v_n^*} \cdot \overline{\mathbf{f}} \, \mathrm{d}\Omega \tag{12}$$

κ is the wavenumber of the primary generating mode, κ_n is the wavenumber of the wave that is not orthogonal to the mth mode at the higher harmonic and κ_n^* is its complex conjugate. $\mathbf{S}_m$ is the stress tensor for the mth mode, $\mathbf{n}_z$ is the unit vector in the wave propagation direction z. $\overline{\mathbf{S}}$ and $\overline{\mathbf{f}}$ are the nonlinear surface traction and body force, respectively, as given by the primary wave (Eq. (5)). Ω, Γ are the rod cross-sectional area and the rod surface, respectively (Fig. 1).

4. Analysis of solution

The first-order nonlinear solution (Eq. (7)) can be extended to higher orders by using appropriate $\overline{\mathbf{S}}$ and $\overline{\mathbf{f}}$ in Eqs. (11) and (12) and by using normal mode expansion at the appropriate higher harmonic. Since the method of perturbation reduces the nonlinear problem to a forced linear problem, for the sake of simplicity it can be assumed that the energy expression (Eq. (3)) consists of any one single order of nonlinearity. Consequently, $\overline{\mathbf{S}}$ and $\overline{\mathbf{f}}$ are due to that particular order of nonlinearity alone. For an $(n-1)$th order nonlinearity, these are denoted by $\overline{\mathbf{S}}^{\mathbf{n}}$ and $\overline{\mathbf{f}}^{\mathbf{n}}$. The subscripts for f^{surf} and f^{vol} are, therefore, changed to 'l' for the sake of clarity.

For the specific case of a cylindrical rod of radius a, Eqs. (11) and (12) become [10]

$$f_l^{\mathrm{surf}} = -\frac{a}{2}\int_0^{2\pi} \mathbf{v}_l^*(a,\theta) \cdot \overline{\mathbf{S}}^{\mathbf{n}}(a,\theta) \cdot \mathbf{n_r} \, \mathrm{d}\theta \tag{13}$$

$$f_l^{\mathrm{vol}} = \frac{1}{2}\int_0^a \int_0^{2\pi} \mathbf{v}_l^*(r,\theta) \cdot \overline{\mathbf{f}}^{\mathbf{n}}(r,\theta) r \, \mathrm{d}\theta \, \mathrm{d}r \tag{14}$$

where the superscript n refers to the nonlinear effect of the primary excitation mode (at frequency $n\omega$) and the subscript l refers to the potential higher-harmonic generation. The particle velocity for the l th Pochhammer Chree wave in rods at frequency $n\omega$ is [14]

$$v_r = V_r(r)\cos(q\theta)\mathrm{e}^{\mathrm{i}(\kappa_l z - n\omega t)} \tag{15}$$

$$v_\theta = V_\theta(r)\sin(q\theta)\mathrm{e}^{\mathrm{i}(\kappa_l z - n\omega t)} \tag{16}$$

$$v_z = V_z(r)\cos(q\theta)\mathrm{e}^{\mathrm{i}(\kappa_l z - n\omega t)} \tag{17}$$

where q is an integer related to the family of modes. $q=0$ for longitudinal modes, and $q \geq 1$ for flexural modes. Torsional modes do not depend upon θ and will not be dealt with here. Substituting the expressions for f_l^{surf} and f_l^{vol} and ignoring the exponential harmonic term yields:

$$f_l^{\mathrm{surf}} = -\frac{a}{2}\int_0^{2\pi} [V_r\overline{S}_{rr}^n\cos(q\theta) + V_\theta\overline{S}_{\theta r}^n\sin(q\theta) + V_z\overline{S}_{zr}^n\cos(q\theta)] \, \mathrm{d}\theta \tag{18}$$

$$f_l^{\mathrm{vol}} = \frac{1}{2}\int_0^a \int_0^{2\pi} [V_r\overline{f}_r^n\cos(q\theta) + V_\theta\overline{f}_\theta^n\sin(q\theta) + V_z\overline{f}_z^n\cos(q\theta)] \mathrm{r} \, \mathrm{d}\theta \, \mathrm{dr} \tag{19}$$

For an $(n-1)$th order nonlinearity the energy expression, Eq. (3), contains terms which have $(n+1)$ multiples of strains. Therefore from the velocity expressions, Eqs. (15)–(17), the corresponding nonlinear stress tensor $\overline{\mathbf{S}}^{\mathbf{n}}$ and body force vector $\overline{\mathbf{f}}^{\mathbf{n}}$ contain n multiples of strains (Eq. (5)).

Hence for a $(n-1)$th order nonlinearity, any generic term in the stress ($\overline{\mathbf{S}}^{\mathbf{n}}$) and body force ($\overline{\mathbf{f}}^{\mathbf{n}}$) can be expressed as

$$T^n = f(r)\sin^t(p\theta)\cos^s(p\theta), \quad t+s=n \tag{20}$$

where p is related to the family of primary excitation mode, $f(r)$ is an arbitrary function of the radius r and either t or s can be equal to 0.

From Eqs. (18) and (19) it can be seen that each term in the expressions for f_l^{surf} and f_l^{vol} involves an integral of the form:

$$I^n = \int_0^{2\pi} F(r)\sin^t(p\theta)\cos^s(p\theta)\sin(l\theta)\,\mathrm{d}\theta \tag{21}$$

or

$$I^n = \int_0^{2\pi} F(r)\sin^t(p\theta)\cos^s(p\theta)\cos(l\theta)\,\mathrm{d}\theta \tag{22}$$

For the ease of analysis, we denote $\sin^t(p\theta)\cos^s(p\theta) = E^n$.

4.1. Case 1: n is odd (odd harmonics)

Since n is odd and $t+s=n$, either t is odd or s is odd. Assuming that t is odd, we have the following expansion (using De Moivre's formula, Euler's formula and binomial expansion):

$$E^n = \left(\sum_{k_1=0}^{(t-1)/2} B_{k_1}\sin\{(t-2k_1)p\theta\}\right)\left(X + \sum_{k_2=0}^{s/2-1} C_{k_2}\cos\{(s-2k_2)p\theta\}\right) \tag{23}$$

$$= \left(X\sum_{k_1=0}^{(t-1)/2} B_{k_1}\sin\{(t-2k_1)p\theta\}\right) + \sum_{k_1=0}^{(t-1)/2}\sum_{k_2=0}^{s/2-1}(B_{k_1}\sin\{(t-2k_1)p\theta\}C_{k_2}\cos\{(s-2k_2)p\theta\}) \tag{24}$$

where X, B_{k1}, C_{k2} are independent of θ. Further,

$$S = \sum_{k_1=0}^{(t-1)/2}\sum_{k_2=0}^{s/2-1} B_{k_1}\sin\{(t-2k_1)p\theta\}C_{k_2}\cos\{(s-2k_2)p\theta\} = \sum_{k_1=0}^{(t-1)/2}\sum_{k_2=0}^{s/2-1}\frac{1}{2}B_{k_1}C_{k_2}(\sin\{(t+s-2k_1-2k_2)p\theta\}+\sin\{(t-2k_1-s+2k_2)p\theta\})$$

The term $(t+s-2k_1-2k_2)$ assumes all odd numbers between 3 ($k_1=(t-1)/2$, $k_2=s/2-1$) and $t+s$ ($k_1=k_2=0$). Similarly, $(t-2k_1-s+2k_2)$ assumes only odd values. It assumes a value of -1 ($k_1=(t-1)/2$, $k_2=s/2-1$) which is equivalent to 1 if the negative sign is taken out of the sine term. In other words, S can be expressed as

$$S = \sum_{k=1,3\ldots}^{t+s} B_k\sin(kp\theta) \tag{25}$$

where B_k are constants. Substituting this in Eq. (23), after some algebraic manipulations:

$$E^n = \sum_{k_1=1,3,\ldots}^{t} B_{k_1}\sin(k_1 p\theta) + \sum_{k_2=1,3\ldots}^{t+s} B_{k_2}\sin(k_2 p\theta) = \sum_{k=1,3,\ldots}^{t+s} E_k\sin(kp\theta) \tag{26}$$

where E_k depend only on k. Similarly, it can be shown that if t is even, we have the following:

$$E^n = \sum_{k=1,3\ldots}^{t+s} E_k\cos(kp\theta) \tag{27}$$

If Eq. (21) holds, we have

$$I^n = \int_0^{2\pi} F(r)\left(\sum_{k=1,3\ldots}^{t+s} E_k\sin(kp\theta)\right)\sin(l\theta)\,\mathrm{d}\theta, \quad t\text{ odd} \tag{28}$$

or

$$I^n = \int_0^{2\pi} F(r)\left(\sum_{k=1,3\ldots}^{t+s} E_k\cos(kp\theta)\right)\sin(l\theta)\,\mathrm{d}\theta, \quad t\text{ even} \tag{29}$$

Because of the special behavior of sine and cosine integrals between limits 0 and 2π, the integral in Eq. (29) is always 0 whereas the integral in Eq. (28) is nonzero if and only if $l=kp$ for some value of k. Similarly it can be shown that even if Eq. (22) holds, I^n is nonzero iff $l=kp$ for some $k=1,3,\ldots,n$.

The physical manifestation of the above result lies in restricting the families of higher order modes that can be generated at odd harmonics by a particular primary mode. Specifically, a primary flexural mode (designated by an angular order $p\neq 0$) cannot generate a longitudinal mode ($l=0$) at an odd higher harmonic (n odd). Vice versa, a primary longitudinal mode ($p=0$) cannot generate a flexural mode ($l\neq 0$) at an odd higher harmonic (n odd). Further, a primary flexural mode ($p\neq 0$) can only generate at the n th harmonic, those modes for which $l=kp$ where $k=1,3,\ldots,n$. No higher order modes can be generated at that harmonic.

4.2. Case 2: n is even (even harmonics)

Since n is even and $t+s=n$, either both t and s are odd or both are even. Assuming that both t and s are odd, we have the following expansion:

$$E^n=\left(\sum_{k_1=0}^{(t-1)/2}B_{k_1}\sin\{(t-2k_1)p\theta\}\right)\left(\sum_{k_2=0}^{s-1/2}A_{k_2}\cos\{(s-2k_2)p\theta\}\right)=\sum_{k_1=0}^{(t-1)/2}\sum_{k_2=0}^{s-1/2}B_{k_1}\sin\{(t-2k_1)p\theta\}A_{k_2}\cos\{(s-2k_2)p\theta\} \quad (30)$$

It can be shown that the above expression reduces to

$$E^n=\sum_{k=2,4,\ldots}^{t+s}E_k\sin(kp\theta)=\sum_{k=0,2,\ldots}^{t+s}E_k\sin(kp\theta) \quad (31)$$

Similarly, when t and s are even, it can be shown that

$$E^n=\sum_{k=0,2,\ldots}^{t+s}E_k\cos(kp\theta) \quad (32)$$

As shown in the earlier case, I^n (if either Eq. (21) or (22) holds) is nonzero iff $l=kp$ for some $k=0,2,4\ldots$.

As opposed to the case of odd harmonics, the above result shows that even a primary flexural mode (designated by an angular order $p\neq 0$) can generate a longitudinal mode ($l=0$ for $k=0$) at an even higher harmonic (n even). On the other hand, a primary longitudinal mode ($p=0$) still cannot generate a flexural mode ($l\neq 0$) at an even higher harmonic (n even).

4.3. Discussion

It can be seen from the analysis above that the behavior of the flexure modes of angular order 1 in rods is analogous to the behavior of antisymmetric Lamb modes and that longitudinal rod modes behave in a manner similar to symmetric Lamb modes. Both the set of first-order flexure modes and the set of antisymmetric Lamb modes are absent at even harmonics whereas they are present along with longitudinal and symmetric modes at odd harmonics in rods and plates, respectively. This is not surprising since the angular variation of the modeshape of a first-order flexure mode bears a relation with that of a longitudinal mode which is similar to the thickness variation of modeshapes of antisymmetric and symmetric Lamb modes. The physical behavior of higher order flexure modes is more difficult to explain because they are special to rods and do not have counterparts in plate waves.

5. Semi-analytical finite element simulations

The SAFE method was used to calculate the guided wave modeshapes and eigenvalues for a rod waveguide. Details of the method can be found in Bartoli et al. [12]. The modal solutions of the SAFE analysis were used to calculate double harmonic amplitudes for longitudinal and flexural modes for various primary modal generation conditions.

The geometry of the rod problem for SAFE is shown in Fig. 2. A circular cross-section of radius .01 m is discretized by using 512 triangular elements. The material properties used in this example are given in Table 1.

Fig. 3 shows the SAFE results for the phase velocity dispersion curve for the rod under consideration. The fundamental longitudinal (L(0,1)), torsional (T(0,1)), and first-order flexural (F(1,1)) modes are marked. Theoretical analysis has shown that the double harmonic in rod waves does not support the first-order flexure mode. SAFE was used to calculate the amplitudes of higher harmonic modes for a primary longitudinal (L(0,1)) and flexural (F(1,1)) generation, respectively. Fig. 4 shows the variation of A_m for different conversion sets as a function of frequency. The frequency vector is that of the

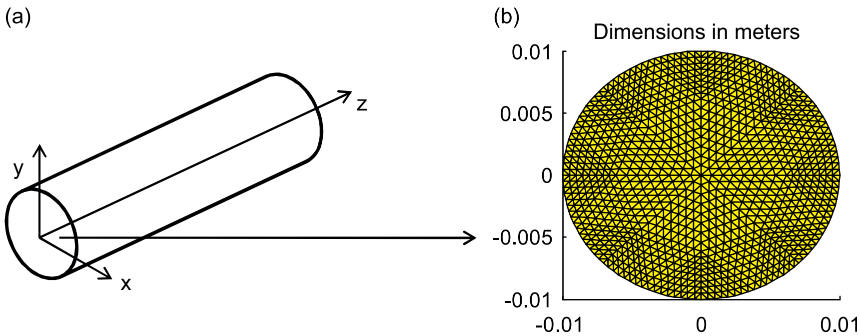

Fig. 2. (a) Schematic of the isotropic rod, (b) finite element discretization of the rod cross-section.

Table 1
Material properties for the rod.

ρ_0 (kg/m^3)	c_l (m/s)	c_t (m/s)	λ[a]	μ[a]	A[a,b]	B[a,b]	C[a,b]
7932	5960	3230	116.2	82.7	−320	−200	−190

[a] In GPa.
[b] 1st-order nonlinear constants.

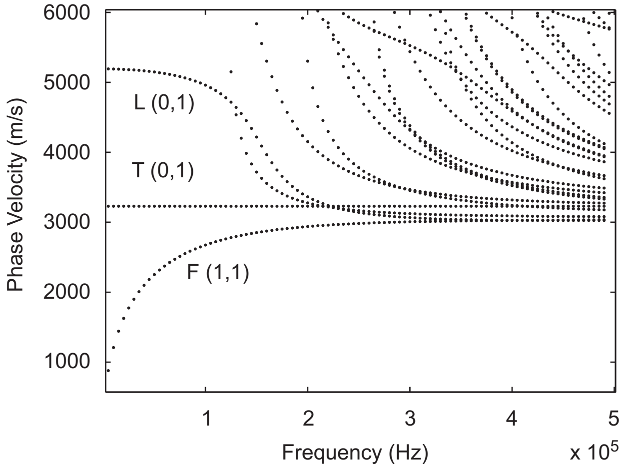

Fig. 3. Phase velocity dispersion curve for a steel rod .02 m in diameter.

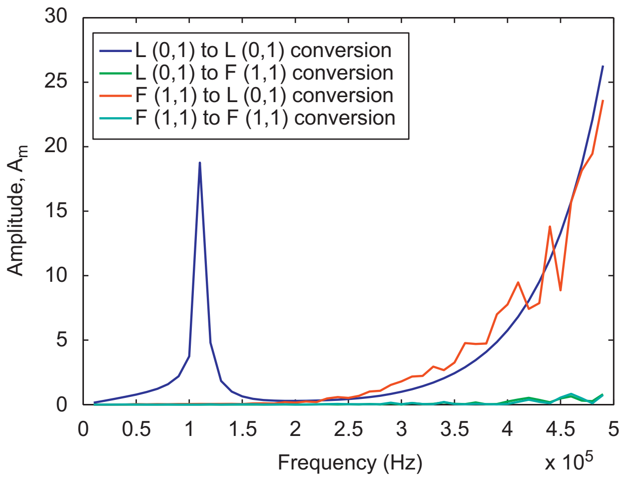

Fig. 4. Nonlinear double harmonic modal amplitudes.

double harmonic generation, i.e. it represents the generated modes, the primary generating mode being at half the frequency.

It can be seen that the L(0,1) to L(0,1) conversion shows increased efficiency at two frequencies (110, 500 kHz). The dispersion curve (Fig. 3) shows that the phase velocity of the L(0,1) mode is relatively flat in these ranges. More specifically, the phase velocity (ω_2/κ_2) of the L(0,1) mode at these frequencies is almost equal to the phase velocity (ω_1/κ_1) of the same mode at half frequencies (55, 250 kHz). This results in phase matching where the denominator in Eq. (8) tends to zero and Eq. (9) starts becoming applicable. At these frequencies L(0,1) becomes a synchronous mode which grows with distance. The same phenomenon can be see for the F(1,1) to L(0,1) conversion. At high frequencies a phase matching occurs between the primary F(1,1) mode and the secondary L(0,1) mode and this is captured in Fig. 4 by a continuous increase in the associated A_m value. The plot also shows that the flexural mode (F(1,1)) is almost nonexistent at the double harmonic.

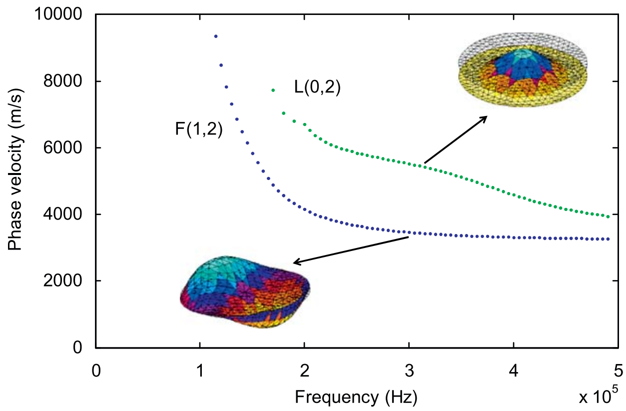

Fig. 5. Phase velocity dispersion curve for F(1,2), L(0,2) modes in the steel rod.

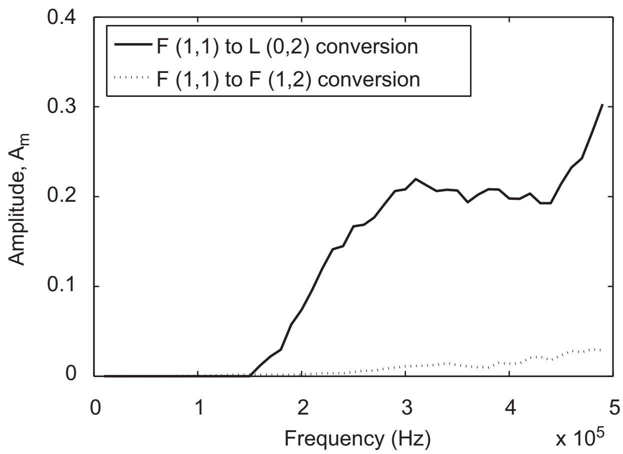

Fig. 6. Nonlinear double harmonic modal amplitudes for F(1,1) to L(0,2) and F(1,1) to F(1,2) conversions.

Although conditions for phase matching are present even for F(1,1) to F(1,1) conversion at high frequencies, this mode is absent because there is no power transfer to the first-order flexure mode at the double harmonic (Eqs. (11) and (12)). This is in accordance with theoretical results.

The same effect is confirmed in higher order modes. Fig. 5 shows the phase velocity dispersion curve for the same rod but with two particular higher order modes extracted. The figure also shows the modeshapes of the two modes at a frequency of 300 kHz. The F(1,2) mode has the same sinusoidal variation in the θ direction as the F(1,1) mode. According to the theoretical considerations, this mode would not be generated at the double harmonic. L(0,2), on the other hand, has a symmetric angular variation and theory suggests that it is likely to be generated at the nonlinear double harmonic.

Fig. 6 shows the variation of A_m for F(1,1) to F(1,2) and F(1,1) to L(0,2) conversions as a function of frequency. The frequency vector is that of the double harmonic generation. In line with theoretical predictions, SAFE predicts an almost zero generation of the F(1,2) mode at the double harmonic. Small deviations of A_m from a value of zero at higher frequency are due to the insufficiency of the mesh discretization at these frequencies. L(0,2) mode, on the other hand, shows a nonzero generation at all frequencies after its cutoff.

6. Conclusions

It can be seen from the above analysis that a primary generating mode in a rod with an angular order p will generate a rod mode with an angular order l at the nth higher harmonic if and only if $l = kp$ for some values of k where:

1. k spans all odd numbers from 1 to n when n is odd (odd harmonics).
2. k spans all even numbers from 0 to n when n is even (even harmonics).

Therefore, the conclusions applicable to odd harmonics are:

1. A longitudinal primary generating mode will not produce any flexural modes. A longitudinal primary generating mode can only produce longitudinal modes.
2. Only selected modes can be generated by flexural primary generating modes. For example, a first-order flexural mode ($p=1$) at the triple harmonic ($n=3$) can only generate the first-order ($l=1$) and third-order ($l=3$) flexural modes. In general, a p th order flexural mode can only generate at the n th harmonic flexural modes of orders equal to odd multiples of p, up to np.

The conclusions applicable to even harmonics are:

1. Longitudinal modes can be generated irrespective of whether the primary generating mode is longitudinal or flexural. Moreover, a longitudinal primary generating mode does not produce flexural modes.
2. As in the case of odd harmonics, only selected modes can be generated by flexural primary generating modes. For example, a first-order flexural mode ($p=1$) at the double harmonic can only generate the longitudinal ($l=0$) and second-order ($l=2$) flexural modes. In general, a p th order flexural mode can only generate at the n th harmonic the longitudinal mode and flexural modes of orders equal to even multiples of p, up to np.

If we constrain our analysis to longitudinal modes and flexural modes of the first order then similarities between rod wave nonlinearity and Lamb wave nonlinearity become clear. The results are, in fact, the rod analogue of the nonlinear bulk wave results of Gol'dberg and the nonlinear plate wave results of Srivastava and Lanza di Scalea. In the constrained domain, it can be concluded that while both longitudinal modes and flexural modes of the first order are present at odd harmonics, only longitudinal modes can be present at even harmonics. This, when compared with Gol'dberg's result of the absence of transverse modes at the double harmonic and Srivastava and Lanza di Scalea's result of the absence of antisymmetric plate modes at even harmonics, shows that the analysis is in line with the existing theory and extends it to the domain of a rod.

Semi-analytical finite element simulations further corroborated a part of the theoretical results.

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