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Free transverse vibration analysis of an elastically connected annular and circular double-membrane compound system

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ABSTRACT

In this paper, free transversal vibrations of a systems of two annular and circular membranes connected by a Winkler elastic layer are studied using analytical methods and numerical simulation. At first the motion of each system is described by two homogeneous partial differential equations. The general solutions of the free vibrations are derived by the Bernoulli–Fourier method and the boundary problems are solved. The natural frequencies and natural mode shapes of vibrations of systems under consideration are determined. The investigation of free vibrations prove that the double-membrane systems execute two kinds of vibrations, synchronous and asynchronous. Then for each system two models formulated by using finite element representations are prepared. The FE models are manually tuned to reduce the difference between the natural frequencies of the analytical solutions and the natural frequencies of the FE model calculations, respectively.

1. Introduction

The problem of free transverse vibrations of double-membrane compound systems with elastic constraints is of great theoretical and practical significance because of wide applications in different disciplines of engineering including the pharmaceutical and biotechnological industries, and biomedical devices [1–3]. The membrane systems are modeled by simple or complex two-dimensional continuous systems. The simplest fundamental model of a compound two-dimensional system consists of two parallel membranes which are coupled by a Winkler elastic layer. The fundamental theory of vibration of simple two-dimensional continuous systems (membranes and plates) is elaborated in a number of monographs by, for example, Ziemba [4], Kaliski [5], Rao [6], and others. Transverse vibrations of circular and annular membranes are investigated by many researchers. The problem of free vibrations of non-homogeneous circular and annular membranes is analyzed in [7,8]. In [7] exact solutions for variable density membranes are found by using the dynamic stiffness method. The classical theory of membranes is used in paper [8] to solve the problem of the free vibration of composite membranes with discontinuously varying thickness. In paper [9] free vibrations of the polygonal membrane with a circular core are studied using the vibration theory of membranes. Firstly, the fundamental vibration theory of compound two-dimensional continuous systems are mainly investigated for double-plate systems. Free transverse vibrations of the circular double-plate complex systems are developed in works [1,10–13] using the classical vibration theory of plate. The annular case is investigated in the article [14]. The free vibration problem concerning a mixed rectangular plate–membrane complex system is solved in paper [15] by using the Navier method. These results are utilized

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by the authors of paper [16] to solve the problem of the optimal control of an elastically connected rectangular plate–membrane system. The free and forced vibration problems of the rectangular double-membrane compound continuous system are investigated in works [1,17] by using the vibration theory of membranes. The solution of the same problem can be found in paper [18]. Moreover, in the monograph [1] the theoretical study related to the circular double-membrane complex system is developed. In papers [2,3] practical examples of the use of the membrane complex systems in the pharmaceutical, chemical and biotechnological industries, and biomedical devices are presented. Finite element (FE) representation is a useful technique to solve various dynamic problems connected with engineering structures [19]. Three-dimensional vibrations of circular and annular plates are analyzed in paper [20] by using finite element method. In work [21] the FE technique is utilized to elaborate the algorithm to identify the proper distorted mode shapes of the gear wheel. The solution of the free transverse vibration problem of the elastically connected annular double-membrane compound system by using FE code is presented in work [22]. Usually an FE model needs to be improved by the so-called model updating technique, to predict more accurately the dynamics of a structure. A number of efficient model updating methods are proposed in [23–25]. Paper [26] presents introductory studies that deal with the updating of the FE model of an annular membrane based on the analytical solution data. This paper continues the recent author's investigations concerning the dynamics of structures [27].

The present paper deals with an exact solution of the problem of free transverse vibrations of elastically connected annular and circular double-membrane compound systems. The complete analytical solutions of undamped free vibrations of these systems are derived by using the Bernoulli–Fourier method. Then the analytical solutions are treated as experimental data and they are used to manually tune the FE models of the membrane compound systems. The preliminary studies focused on the preparation of the appropriate FE models of the double-membrane complex systems are provided. Finally, the concluding remarks are given and the adequate mode shapes referring to the appropriate natural frequencies of the systems are illustrated.

2. Formulation of the problem

The objective of this work is the formulation of dynamic models of an elastically connected annular and circular double-membrane compound systems. The mechanical model of the first system consists of two parallel annular membranes connected by a massless, linear, elastic layer of a Winkler type. It is assumed that the membranes are thin, homogeneous and perfectly elastic, and that they have constant thickness [1,17]. The membranes are uniformly tense by adequate constant tensions applied at the edges of the membranes (see Fig. 1). Small vibrations with no damping are considered. The partial differential equations of motion for the free transversal vibrations may be written in the following form [1]:

$$m_1\ddot{w}_1 - S_1\,\Delta w_1 + k(w_1 - w_2) = 0, \quad m_2\ddot{w}_2 - S_2\,\Delta w_2 + k(w_2 - w_1) = 0, \tag{1}$$

where $w_i=w_i(r, \varphi, t)$ is the transverse membrane displacement; r, φ, t are the polar coordinates and the time; r_1, r_2, h_i are the membrane dimensions; ρ_i is the mass density; S_i is the uniform constant tension per unit length; k is the stiffness modulus of a Winkler elastic layer;

$$m_i = \rho_i h_i, \quad \dot{w}_i = \frac{\partial w_i}{\partial t}, \quad \Delta w_i = \frac{\partial^2 w_i}{\partial r^2} + \frac{1}{r}\frac{\partial w_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_i}{\partial \varphi^2}, \quad i = 1, 2. \tag{2}$$

The boundary and periodicity, and initial conditions are

$$\begin{gathered} w_i(r_1, \varphi, t) = w_i(r_2, \varphi, t) = 0, \quad w_i(r, \varphi, t) = w_i(r, \varphi + 2\pi, t), \\ w_i(r, \varphi, 0) = w_{i0}(r, \varphi), \quad \dot{w}_i|_{(r,\varphi,0)} = v_{i0}(r, \varphi), \quad i = 1, 2. \end{gathered} \tag{3}$$

The second system under consideration consists of two parallel circular membranes connected by a massless, linear, Winkler elastic layer. The remaining technical assumptions are the same as in the first case. The governing differential

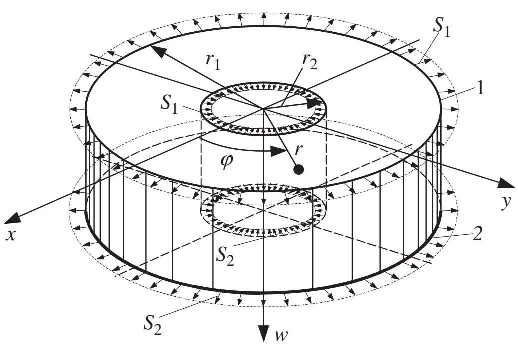

Fig. 1. The physical model of the system.

equations for the free transversal vibrations take the form (1). The boundary, periodicity and initial conditions are different and may be written as

$$w_i(r_1,\varphi,t)=0,\quad w_i(0,\varphi,t)<\infty,\quad w_i(r,\varphi,t)=w_i(r,\varphi+2\pi,t),$$
$$w_i(r,\varphi,0)=w_{i0}(r,\varphi),\quad \dot{w}_i|_{(r,\varphi,0)}=v_{i0}(r,\varphi),\quad i=1,2. \tag{4}$$

3. Free vibration analysis

The objective of this section is the determination of the complete analytical solutions of free vibrations of the analyzed systems. To solve the governing equations (1) the Bernoulli–Fourier method (separation of variables) will be employed.

3.1. The annular double-membrane compound system

The general solutions of Eq. (1) can be written in the form [1,4–6]

$$w_i(r,\varphi,t)=W_i(r,\varphi)T(t),\quad i=1,2, \tag{5}$$

$$T(t)=C\sin(\omega t)+D\cos(\omega t), \tag{6}$$

where ω is the natural frequency of the system. Substituting solutions (5) into Eq. (1) gives the following expressions:

$$S_1\,\Delta W_1+(m_1\omega^2-k)W_1+kW_2=0,\quad S_2\,\Delta W_2+(m_2\omega^2-k)W_2+kW_1=0. \tag{7a,b}$$

Now by eliminating the function W_2 the equation system (7a,b) takes the form

$$(\Delta+k_1^2)(\Delta+k_2^2)W_1=0, \tag{8}$$

where

$$k_{1,2}^2=0.5\Big([(m_1\omega^2-k)S_1^{-1}+(m_2\omega^2-k)S_2^{-1}]\Big)$$
$$\pm\Big([(m_1\omega^2-k)S_1^{-1}+(m_2\omega^2-k)S_2^{-1}]^2-4\omega^2[m_1m_2\omega^2-(m_1+m_2)k](S_1S_2)^{-1}\Big)^{1/2}\Big). \tag{9}$$

The coefficients k_1^2 and k_2^2 are both positive when

$$\omega^2>k(m_1+m_2)/(m_1m_2). \tag{10}$$

Condition (10) guarantees the harmonic type of free vibrations [1]. Assuming the solution of Eq. (8) in the form

$$W_1(r,\varphi)=R_1(r)U(\varphi) \tag{11}$$

and introducing it into an equation of type (8)

$$(\Delta+k_i^2)W_1=0,\quad i=1,2 \tag{12}$$

provides the relations

$$r^2R_1''+rR_1'-(n^2-(k_ir)^2)R_1=0,\quad U''+n^2U=0,\quad n=0,1,2,\ldots,i=1,2, \tag{13a,b}$$

where

$$R_1'=\frac{\mathrm{d}R_1}{\mathrm{d}r},\quad U'=\frac{\mathrm{d}U}{\mathrm{d}\varphi}.$$

The general solutions of Eqs. (13a,b) may be written as [1,4–6]

$$R_{1in}(r)=A_{in}J_n(k_ir)+B_{in}Y_n(k_ir),\quad U_n(\varphi)=A_n\sin(n\varphi)+B_n\cos(n\varphi),n=0,1,2,\ldots,\quad i=1,2, \tag{14a,b}$$

where J_n and Y_n are the Bessel functions of the first and second kinds, respectively, A_{in} and B_{in} are the constants which will be determined from the boundary conditions. The mode shape function W_1 can be expressed as

$$W_{1n}(r,\varphi)=R_{1n}(r)U_n(\varphi)=U_n(\varphi)\sum_{i=1}^{2}R_{1in}(r)=[A_n\sin(n\varphi)+B_n\cos(n\varphi)]\sum_{i=1}^{2}[A_{in}J_n(k_ir)+B_{in}Y_n(k_ir)]. \tag{15}$$

In order to obtain the solution for the function W_2 it will be assumed to be in the form

$$W_2(r,\varphi)=R_2(r)U(\varphi) \tag{16}$$

and using the equation system (7a,b) one can determine the mode shape function W_2 in the form

$$W_{2n}(r,\varphi)=R_{2n}(r)U_n(\varphi)=U_n(\varphi)\sum_{i=1}^{2}R_{2in}(r)=[A_n\sin(n\varphi)+B_n\cos(n\varphi)]\sum_{i=1}^{2}d_i[A_{in}J_n(k_ir)+B_{in}Y_n(k_ir)], \tag{17}$$

where

$$d_i=(S_1k_i^2+k-m_1\omega^2)k^{-1}=k(S_2k_i^2+k-m_2\omega^2)^{-1},\quad i=1,2. \tag{18}$$

The boundary conditions (3) take the form

$$R_i(r_1)=R_i(r_2)=0, \quad U(\varphi)=U(\varphi+2\pi), \quad i=1,2. \tag{19}$$

Substituting Eqs. (15) and (17) into the boundary conditions (19) gives

$$\begin{aligned}
&A_{1n}J_n(k_1r_1)+B_{1n}Y_n(k_1r_1)+A_{2n}J_n(k_2r_1)+B_{2n}Y_n(k_2r_1)=0,\\
&A_{1n}J_n(k_1r_2)+B_{1n}Y_n(k_1r_2)+A_{2n}J_n(k_2r_2)+B_{2n}Y_n(k_2r_2)=0,\\
&d_1(A_{1n}J_n(k_1r_1)+B_{1n}Y_n(k_1r_1))+d_2(A_{2n}J_n(k_2r_1)+B_{2n}Y_n(k_2r_1))=0,\\
&d_1(A_{1n}J_n(k_1r_2)+B_{1n}Y_n(k_1r_2))+d_2(A_{2n}J_n(k_2r_2)+B_{2n}Y_n(k_2r_2))=0.
\end{aligned} \tag{20}$$

The existence of a nontrivial solution of Eqs. (20) yields the characteristic determinant

$$(d_1-d_2)^2\begin{vmatrix} J_n(k_1r_1) & Y_n(k_1r_1)\\ J_n(k_1r_2) & Y_n(k_1r_2)\end{vmatrix}\cdot\begin{vmatrix} J_n(k_2r_1) & Y_n(k_2r_1)\\ J_n(k_2r_2) & Y_n(k_2r_2)\end{vmatrix}=0. \tag{21}$$

From relation (21) it is shown that

$$k_1=k_2=k_{mn}, \quad m=1,2,3,\ldots, \quad n=0,1,2,\ldots . \tag{22}$$

The proper values of the coefficient k_{mn} satisfying the relation (21) may be found from the secular equation [26]

$$J_n(k_{mn}r_1)Y_n(k_{mn}r_2)-J_n(k_{mn}r_2)Y_n(k_{mn}r_1)=0. \tag{23}$$

Then taking into account Eq. (9) the frequency equation can be expressed as [1,17]

$$\omega^4-((m_1S_2+m_2S_1)k_{mn}^2+(m_1+m_2)k)(m_1m_1)^{-1}\omega^2+k_{mn}^2((S_1+S_2)k+k_{mn}^2S_1S_2)(m_1m_2)^{-1}=0. \tag{24}$$

The natural frequencies of the double-membrane system are determined from the relation

$$\begin{aligned}\omega_{1,2mn}^2=0.5\Big(&[(S_2k_{mn}^2+k)m_1+(S_1k_{mn}^2+k)m_2](m_1m_2)^{-1}\mp\Big([(S_2k_{mn}^2+k)m_1+(S_1k_{mn}^2+k)m_2]^2(m_1m_2)^{-2}\\
&-4k_{mn}^2[k_{mn}^2S_1S_2+(S_1+S_2)k](m_1m_2)^{-1}\Big)^{1/2}\Big), \quad \omega_{1mn}<\omega_{2mn}.\end{aligned} \tag{25}$$

The time functions (6) and the mode shapes of free vibrations (15) and (17) are related to the natural frequencies ω_{imn} by

$$T_{imn}(t)=C_{imn}\sin(\omega_{imn}t)+D_{imn}\cos(\omega_{imn}t), \tag{26}$$

$$\begin{aligned}
W_{1imn}(r,\varphi)&=R_{1imn}(r)U_n(\varphi)=B_{imn}(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))U_n(\varphi),\\
W_{2imn}(r,\varphi)&=R_{2imn}(r)U_n(\varphi)=d_{imn}B_{imn}(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))U_n(\varphi),
\end{aligned} \tag{27}$$

where

$$d_{imn}=(S_1k_{mn}^2+k-m_1\omega_{imn}^2)k^{-1}=k(S_2k_{mn}^2+k-m_2\omega_{imn}^2)^{-1}, \quad d_{1mn}>0, \quad d_{2mn}<0, \quad B_{imn}=1, \quad i=1,2, \tag{28}$$

$$e_{mn}=-Y_n(k_{mn}r_1)/J_n(k_{mn}r_1). \tag{29}$$

Finally the general solution of the free vibrations of the system under consideration can be written in the following form:

$$\begin{aligned}
w_1(r,\varphi,t)&=\sum_{i=1}^{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}W_{1imn}(r,\varphi)T_{imn}(t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\Big(W_{1mn}(r,\varphi)\sum_{i=1}^{2}\Big(K_{imn}^{(1)}\sin(\omega_{imn}t)+L_{imn}^{(1)}\cos(\omega_{imn}t)\Big)\\
&\quad+W_{2mn}(r,\varphi)\sum_{i=1}^{2}(K_{imn}^{(2)}\sin(\omega_{imn}t)+L_{imn}^{(2)}\cos(\omega_{imn}t))\Big),\\
w_2(r,\varphi,t)&=\sum_{i=1}^{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}W_{2imn}(r,\varphi)T_{imn}(t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\Big(W_{1mn}(r,\varphi)\sum_{i=1}^{2}d_{imn}\Big(K_{imn}^{(1)}\sin(\omega_{imn}t)+L_{imn}^{(1)}\cos(\omega_{imn}t)\Big)\\
&\quad+W_{2mn}(r,\varphi)\sum_{i=1}^{2}d_{imn}(K_{imn}^{(2)}\sin(\omega_{imn}t)+L_{imn}^{(2)}\cos(\omega_{imn}t))\Big),
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
W_{1mn}(r,\varphi)&=(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))\sin(n\varphi),\\
W_{2mn}(r,\varphi)&=(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))\cos(n\varphi)
\end{aligned} \tag{31}$$

are the eigenfunctions. The constants $K_{imn}^{(1)}, K_{imn}^{(2)}, L_{imn}^{(1)}, L_{imn}^{(2)}, (i=1,2)$ are determined from the initial conditions. From Eqs. (31) it can be shown that the nodal lines of the membrane face deflections may be described in the form

$$\sin(n\varphi)=0, \quad \cos(n\varphi)=0, \quad e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r)=0. \tag{32}$$

The free vibrations of the double-membrane system are realized by synchronous ($d_{1mn}>0$, ω_{1mn}) and asynchronous ($d_{2mn}<0$, ω_{2mn}) displacements. It simply means that in modes related to frequencies ω_{1mn}, the corresponding points on the two membranes vibrate in phase with each other, whereas in modes related to frequencies ω_{2mn}, they are in antiphase.

In both cases two mode shapes W_{1mn} and W_{2mn} related to the one coefficient k_{mn} exist [1,17,22]. To solve the initial value problem the complete knowledge of the orthogonality condition of normal modes of vibration is needed. Then the orthogonality condition of mode shape functions takes the classical form

$$\int_{r_2}^{r_1}\int_0^{2\pi} W_{1mn}(r,\varphi)W_{1sl}(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_{r_2}^{r_1}\int_0^{2\pi} W_{2mn}(r,\varphi)W_{2sl}(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\sin(n\varphi)\sin(l\varphi)\,\mathrm{d}\varphi$$
$$\times \int_{r_2}^{r_1}(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))(e_{sl}J_l(k_{sl}r)+Y_l(k_{sl}r))r\,\mathrm{d}r = \int_0^{2\pi}\cos(n\varphi)\cos(l\varphi)\,\mathrm{d}\varphi$$
$$\times \int_{r_2}^{r_1}(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))(e_{sl}J_l(k_{sl}r)+Y_l(k_{sl}r))r\,\mathrm{d}r = \begin{cases} 0, & m\neq s \vee n\neq l, \\ a_{mn}^2, & m=s\wedge n=l, \end{cases} \tag{33}$$

where

$$a_{mn}^2 = \int_{r_2}^{r_1}\int_0^{2\pi} W_{1mn}^2(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_{r_2}^{r_1}\int_0^{2\pi} W_{2mn}^2(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\sin^2(n\varphi)\,\mathrm{d}\varphi\int_{r_2}^{r_1}(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))^2 r\,\mathrm{d}r$$
$$= \int_0^{2\pi}\cos^2(n\varphi)\,\mathrm{d}\varphi\int_{r_2}^{r_1}(e_{mn}J_n(k_{mn}r)+Y_n(k_{mn}r))^2 r\,\mathrm{d}r = \pi e_{mn}^2\int_{r_2}^{r_1}J_n^2(k_{mn}r)r\,\mathrm{d}r + \pi\int_{r_2}^{r_1}Y_n^2(k_{mn}r)r\,\mathrm{d}r, \quad m,\ n=1,2,3,\ldots \tag{34}$$

and

$$a_{m0}^2 = \int_0^{2\pi}\int_{r_2}^{r_1} W_{2m0}^2(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi} d\varphi\int_{r_2}^{r_1}(e_{m0}J_0(k_{m0}r)+Y_0(k_{m0}r))^2 r\,\mathrm{d}r$$
$$= 2\pi e_{m0}^2\int_{r_2}^{r_1}J_0^2(k_{m0}r)r\,\mathrm{d}r + 2\pi\int_{r_2}^{r_1}Y_0^2(k_{m0}r)r\,\mathrm{d}r, W_{1m0}(r,\varphi)=0,$$
$$W_{2m0}(r,\varphi) = e_{m0}J_0(k_{m0}r)+Y_0(k_{m0}r). \tag{35}$$

Substituting the general solutions (30) into the initial conditions (3) yields

$$w_{10} = \sum_{m=1}^{\infty} W_{2m0}(r,\varphi)\sum_{i=1}^{2}L_{im0}^{(2)} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(W_{1mn}(r,\varphi)\sum_{i=1}^{2}L_{imn}^{(1)} + W_{2mn}(r,\varphi)\sum_{i=1}^{2}L_{imn}^{(2)}\right),$$
$$w_{20} = \sum_{m=1}^{\infty} W_{2m0}(r,\varphi)\sum_{i=1}^{2}d_{im0}L_{im0}^{(2)} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(W_{1mn}(r,\varphi)\sum_{i=1}^{2}d_{imn}L_{imn}^{(1)} + W_{2mn}(r,\varphi)\sum_{i=1}^{2}d_{imn}L_{imn}^{(2)}\right),$$
$$v_{10} = \sum_{m=1}^{\infty} W_{2m0}(r,\varphi)\sum_{i=1}^{2}\omega_{im0}K_{im0}^{(2)} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(W_{1mn}(r,\varphi)\sum_{i=1}^{2}\omega_{imn}K_{imn}^{(1)} + W_{2mn}(r,\varphi)\sum_{i=1}^{2}\omega_{imn}K_{imn}^{(2)}\right),$$
$$v_{20} = \sum_{m=1}^{\infty} W_{2m0}(r,\varphi)\sum_{i=1}^{2}d_{im0}\omega_{im0}K_{im0}^{(2)} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(W_{1mn}(r,\varphi)\sum_{i=1}^{2}d_{imn}\omega_{imn}K_{imn}^{(1)} + W_{2mn}(r,\varphi)\sum_{i=1}^{2}d_{imn}\omega_{imn}K_{imn}^{(2)}\right). \tag{36}$$

To find the unknown constants the following operations are made. At first the relations (36) are multiplied by the eigenfunctions W_{1sl} or W_{2sl}. Then the results are integrated over the membrane surface and the orthogonality conditions (34) and (35) are used. Then it is possible to obtain the formulas from which the unknown constants are calculated:

$$L_{1mn}^{(1)} = (b_{mn})^{-1}\int_{r_2}^{r_1}\int_0^{2\pi} W_{1mn}(r,\varphi)(w_{20}-d_{2mn}w_{10})r\,\mathrm{d}r\,\mathrm{d}\varphi,$$
$$L_{2mn}^{(1)} = (-b_{mn})^{-1}\int_{r_2}^{r_1}\int_0^{2\pi} W_{1mn}(r,\varphi)(w_{20}-d_{1mn}w_{10})r\,\mathrm{d}r\,\mathrm{d}\varphi, \tag{37}$$

$$L_{1mn}^{(2)} = (b_{mn})^{-1}\int_{r_2}^{r_1}\int_0^{2\pi} W_{2mn}(r,\varphi)(w_{20}-d_{2mn}w_{10})r\,\mathrm{d}r\,\mathrm{d}\varphi,$$
$$L_{2mn}^{(2)} = (-b_{mn})^{-1}\int_{r_2}^{r_1}\int_0^{2\pi} W_{2mn}(r,\varphi)(w_{20}-d_{1mn}w_{10})r\,\mathrm{d}r\,\mathrm{d}\varphi, \tag{38}$$

$$K_{1mn}^{(1)} = (\omega_{1mn}b_{mn})^{-1}\int_{r_2}^{r_1}\int_0^{2\pi} W_{1mn}(r,\varphi)(v_{20}-d_{2mn}v_{10})r\,\mathrm{d}r\,\mathrm{d}\varphi,$$
$$K_{2mn}^{(1)} = (-\omega_{2mn}b_{mn})^{-1}\int_{r_2}^{r_1}\int_0^{2\pi} W_{1mn}(r,\varphi)(v_{20}-d_{1mn}v_{10})r\,\mathrm{d}r\,\mathrm{d}\varphi, \tag{39}$$

$$K_{1mn}^{(2)} = (\omega_{1mn}b_{mn})^{-1}\int_{r_2}^{r_1}\int_0^{2\pi} W_{2mn}(r,\varphi)(v_{20}-d_{2mn}v_{10})r\,\mathrm{d}r\,\mathrm{d}\varphi,$$

$$K_{2mn}^{(2)} = (-\omega_{2mn} b_{mn})^{-1} \int_{r_2}^{r_1} \int_0^{2\pi} W_{2mn}(r,\varphi)(v_{20} - d_{1mn} v_{10}) r \, \mathrm{d}r \, \mathrm{d}\varphi, \tag{40}$$

$$L_{1m0}^{(2)} = (b_{m0})^{-1} \int_{r_2}^{r_1} \int_0^{2\pi} W_{2m0}(r,\varphi)(w_{20} - d_{2m0} w_{10}) r \, \mathrm{d}r \, \mathrm{d}\varphi,$$
$$L_{2m0}^{(2)} = (-b_{m0})^{-1} \int_{r_2}^{r_1} \int_0^{2\pi} W_{2m0}(r,\varphi)(w_{20} - d_{1m0} w_{10}) r \, \mathrm{d}r \, \mathrm{d}\varphi, \tag{41}$$

$$K_{1m0}^{(2)} = (\omega_{1m0} b_{m0})^{-1} \int_{r_2}^{r_1} \int_0^{2\pi} W_{2m0}(r,\varphi)(v_{20} - d_{2m0} v_{10}) r \, \mathrm{d}r \, \mathrm{d}\varphi,$$
$$K_{2m0}^{(2)} = (-\omega_{2m0} b_{m0})^{-1} \int_{r_2}^{r_1} \int_0^{2\pi} W_{2m0}(r,\varphi)(v_{20} - d_{1m0} v_{10}) r \, \mathrm{d}r \, \mathrm{d}\varphi, \tag{42}$$

where

$$b_{mn} = (d_{1mn} - d_{2mn}) a_{mn}^2, \quad b_{m0} = (d_{1m0} - d_{2m0}) a_{m0}^2. \tag{43}$$

In case of initial conditions taken as circular symmetry functions, i.e.

$$w_i(r,\varphi,0) = w_{i0}(r), \quad \dot{w}_i|_{(r,\varphi,0)} = v_{i0}(r), \quad i = 1,2, \tag{44}$$

the vibrations of the considered system are realized in the nature of circular symmetry and the corresponding integral constants are given as

$$L_{1mn}^{(1)} = L_{2mn}^{(1)} = L_{1mn}^{(1)} = L_{2mn}^{(2)} = 0, \quad K_{1mn}^{(1)} = K_{2mn}^{(1)} = K_{1mn}^{(2)} = K_{2mn}^{(2)} = 0 \tag{45}$$

and

$$L_{1m0}^{(2)} = 2\pi(b_{m0})^{-1} \int_{r_2}^{r_1} (e_{m0} J_0(k_{m0} r) + Y_0(k_{m0} r))(w_{20} - d_{2m0} w_{10}) r \, \mathrm{d}r,$$
$$L_{2m0}^{(2)} = -2\pi(b_{m0})^{-1} \int_{r_2}^{r_1} (e_{m0} J_0(k_{m0} r) + Y_0(k_{m0} r))(w_{20} - d_{1m0} w_{10}) r \, \mathrm{d}r, \tag{46}$$

$$K_{1m0}^{(2)} = 2\pi(\omega_{1m0} b_{m0})^{-1} \int_{r_2}^{r_1} (e_{m0} J_0(k_{m0} r) + Y_0(k_{m0} r))(v_{20} - d_{2m0} v_{10}) r \, \mathrm{d}r,$$
$$K_{2m0}^{(2)} = -2\pi(\omega_{2m0} b_{m0})^{-1} \int_{r_2}^{r_1} (e_{m0} J_0(k_{m0} r) + Y_0(k_{m0} r))(v_{20} - d_{1m0} v_{10}) r \, \mathrm{d}r. \tag{47}$$

The general solution for the circular symmetry free vibration case of the system can be expressed as

$$w_1(r,\varphi,t) = \sum_{m=1}^{\infty} W_{2m0}(r,\varphi) \sum_{i=1}^{2} (K_{im0}^{(2)} \sin(\omega_{im0} t) + L_{im0}^{(2)} \cos(\omega_{im0} t)),$$
$$w_2(r,\varphi,t) = \sum_{m=1}^{\infty} W_{2m0}(r,\varphi) \sum_{i=1}^{2} d_{im0}(K_{im0}^{(2)} \sin(\omega_{im0} t) + L_{im0}^{(2)} \cos(\omega_{im0} t), \tag{48}$$

where the integral constants $K_{im0}^{(2)}$ and $L_{im0}^{(2)}$ may be calculated from relations (46) and (47), respectively.

3.2. *The circular double-membrane compound system*

Like in the previous case the general solutions take the form (5) and (6). For this case it is assumed that r_2=0. Then the boundary conditions (19) become

$$R_i(r_1) = 0, \quad R_i(0) < \infty, \quad U(\varphi) = U(\varphi + 2\pi), \quad i = 1,2 \tag{49}$$

and the system equations (20) can be written as [1]

$$A_{1n} J_n(k_1 r_1) + A_{2n} J_n(k_2 r_1) = 0,$$
$$d_1 A_{1n} J_n(k_1 r_1) + d_2 A_{2n} J_n(k_2 r_1) = 0, \tag{50}$$

where as mentioned earlier J_n is the Bessel function of the first kind. The existence of nontrivial solution of Eq. (50) referring to coefficients k_1 and k_2 yields the secular equation [1]

$$(d_2 - d_1) J_n(k_1 r_1) J_n(k_2 r_1) = 0, \tag{51}$$

from which it is shown that k_1=k_2=k_{mn}, and eigenvalue k_{mn} may be found from the relation

$$J_n(k_{mn} r_1) = 0. \tag{52}$$

Taking into account the proper values of the coefficient k_{mn} which satisfy Eq. (52), the natural frequencies of the circular double-membrane system may be determined from Eq. (25). The time functions (6) are related to the natural frequencies

ω_{imn} by Eq. (26). Then the mode shapes of free vibrations of the circular membrane system case are determined from the relations [1]

$$W_{1imn}(r,\varphi)=R_{1imn}(r)U_n(\varphi)=A_{imn}J_n(k_{mn}r)U_n(\varphi),$$
$$W_{2imn}(r,\varphi)=R_{2imn}(r)U_n(\varphi)=d_{imn}A_{imn}J_n(k_{mn}r)U_n(\varphi), \tag{53}$$

where

$$W_{1mn}(r,\varphi)=J_n(k_{mn}r)\sin(n\varphi),\quad W_{2mn}(r,\varphi)=J_n(k_{mn}r)\cos(n\varphi) \tag{54}$$

are the eigenfunctions and d_{imn} can be calculated from Eq. (28). Then the nodal lines of the membrane face deflections may be described as follows [1,5]:

$$\sin(n\varphi)=0,\quad \cos(n\varphi)=0,\quad J_n(k_{mn}r)=0. \tag{55}$$

As in the previous case the free vibrations of the circular double-membrane system are realized by synchronous (in-phase) and asynchronous (out-of-phase) displacements. The general solution of the free vibrations of the system can be obtained by substituting Eqs. (54) into relations (30). The orthogonality condition of the mode shape functions takes the form [1]

$$\int_0^{r_1}\int_0^{2\pi}W_{1mn}(r,\varphi)W_{1sl}(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi=\int_0^{r_1}\int_0^{2\pi}W_{2mn}(r,\varphi)W_{2sl}(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi=\int_0^{2\pi}\sin(n\varphi)\sin(l\varphi)\,\mathrm{d}\varphi$$
$$\times\int_0^{r_1}J_n(k_{mn}r)J_l(k_{sl}r)r\,\mathrm{d}r=\int_0^{2\pi}\cos(n\varphi)\cos(l\varphi)\,\mathrm{d}\varphi\int_0^{r_1}J_n(k_{mn}r)J_l(k_{sl}r)r\,\mathrm{d}r=\begin{cases}0, & m\neq s\vee n\neq l,\\ a_{mn}^2, & m=s\wedge n=l,\end{cases} \tag{56}$$

where a_{mn}^2 for this case are given by

$$a_{mn}^2=\int_0^{r_1}\int_0^{2\pi}W_{1mn}^2(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi=\int_0^{r_1}\int_0^{2\pi}W_{2mn}^2(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi=\int_0^{2\pi}\sin^2(n\varphi)\,\mathrm{d}\varphi\int_0^{r_1}J_n^2(k_{mn}r)r\,\mathrm{d}r$$
$$=\int_0^{2\pi}\cos^2(n\varphi)\,\mathrm{d}\varphi\int_0^{r_1}J_n^2(k_{mn}r)r\,\mathrm{d}r=\pi\int_0^{r_1}J_n^2(k_{mn}r)r\,\mathrm{d}r,\quad m,n=1,2,3,\ldots \tag{57}$$

and

$$a_{m0}^2=\int_0^{2\pi}\int_0^{r_1}W_{2m0}^2(r,\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{r_1}J_0^2(k_{m0}r)r\,\mathrm{d}r=2\pi\int_0^{r_1}J_0^2(k_{m0}r)r\,\mathrm{d}r, W_{1m0}(r,\varphi)=0,\quad W_{2m0}(r,\varphi)=J_0(k_{m0}r). \tag{58}$$

Taking into account the initial conditions (4) and eigenfunctions (54), the integral constants $L_{imn}^{(1)}$, $L_{imn}^{(2)}$, $K_{imn}^{(1)}$ and $K_{imn}^{(2)}$ may be calculated from Eqs. (37), (38), (39), (40), (41) and (42), respectively. For the circular symmetry free vibration case of the system the general solution takes the form (48). The integral constants $K_{im0}^{(2)}$ and $L_{im0}^{(2)}$ can be calculated from the relations [1]

$$L_{1m0}^{(2)}=2\pi(b_{m0})^{-1}\int_0^{r_1}J_0(k_{m0}r)(w_{20}-d_{2m0}w_{10})r\,\mathrm{d}r,$$
$$L_{2m0}^{(2)}=-2\pi(b_{m0})^{-1}\int_0^{r_1}J_0(k_{m0}r)(w_{20}-d_{1m0}w_{10})r\,\mathrm{d}r, \tag{59}$$

$$K_{1m0}^{(2)}=2\pi(\omega_{1m0}b_{m0})^{-1}\int_0^{r_1}J_0(k_{m0}r)(v_{20}-d_{2m0}v_{10})r\,\mathrm{d}r,$$
$$K_{2m0}^{(2)}=-2\pi(\omega_{2m0}b_{m0})^{-1}\int_0^{r_1}J_0(k_{m0}r)(v_{20}-d_{1m0}v_{10})r\,\mathrm{d}r, \tag{60}$$

where

$$w_i(r,\varphi,0)=w_{i0}(r),\quad \dot{w}_i|_{(r,\varphi,0)}=v_{i0}(r),\quad i=1,2 \tag{61}$$

are the circular symmetry initial conditions of the circular double-membrane system.

4. The finite element representations of the systems

In this section for both systems the finite element (FE) models are formulated to discretize the continuous models given by the system equations (1). The equations of motion are first transformed into a set of independent or decoupled differential equations cast in modal generalized coordinates through the use of the mode shapes of the structure. The response of the system is then received by superimposing the solutions of the decoupled modal equations [19]. To find the eigenpairs (eigenvalue, eigenvector) connected with the natural frequencies and natural mode shapes of the system, the block Lanczos method is employed [19].

As mentioned earlier, the FE models are treated as approximations of the exact systems. The quality of the approximation model depends on the type and density of the mesh and the manner of the application of the tensile forces per unit length of the membranes. In this work the impact of the manner of the membranes tensile forces application in the FE models on the quality of the accurate model approximation is analyzed. In order to compare the continuous systems analysis results with the FE models solutions, two finite element models for each system are prepared and discussed by using the ANSYS FE code.

4.1. The annular membrane system

As mentioned earlier the mechanical model of the system under consideration consists of two parallel annular membranes and a massless, linear, elastic layer of the Winkler type which connects membranes.

The first FE model is realized as follows. The layer is modeled by a finite number of parallel massless springs. The stiffness modulus k_S of each spring can be obtained from the relation [22]

$$k_S = \frac{kp_0}{b}, \tag{62}$$

where p_0 is the area of the membrane large face and b is the number of the springs. The spring-damper element (combin14) defined by two nodes with the option "*3-D longitudinal*" is used to realize the elastic layer. The damping capability of the element are omitted. The four-node quadrilateral element (shell63) with six degrees of freedom in each node and with the element stiffness option "*Membrane only*" is used to realize each membrane. The uniform constant tension is applied to the outer edges of each membrane by using the FE code system standard procedure. The boundary conditions are realized as follows. All nodes lying on the outer edges of the membranes are simply supported with a possibility to slide freely in the radial direction, and all nodes lying on the inner edges of the membranes are pinned. The prepared model shown in Fig. 2 consists of 19,080 shell elements, and 9324 combin elements, respectively.

The second FE model is the same as the first, but the application of the tensile forces is different. To each node lying on the outer edge is imposed a concentrated tensile force S_{0j} in the radial direction. The proper value of the force is selected experimentally by numerical simulation. The outer edge of each membrane includes 324 nodes.

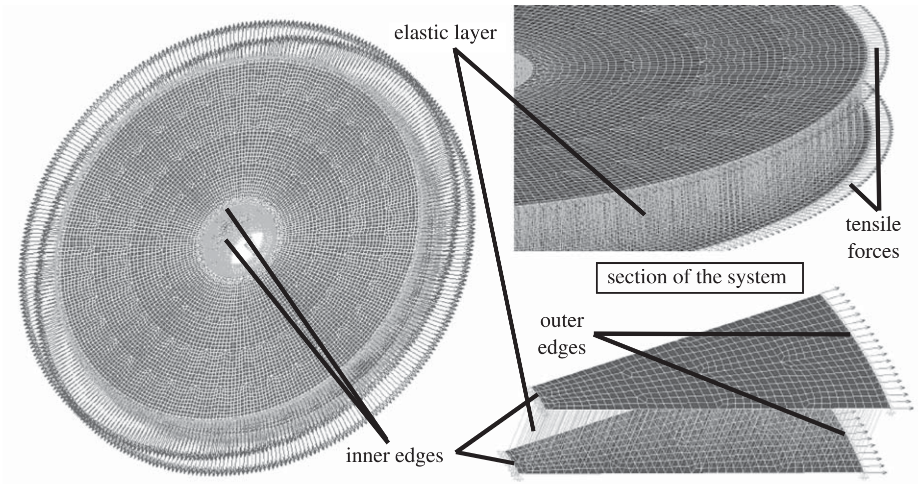

Fig. 2. Finite element model of the annular membrane system.

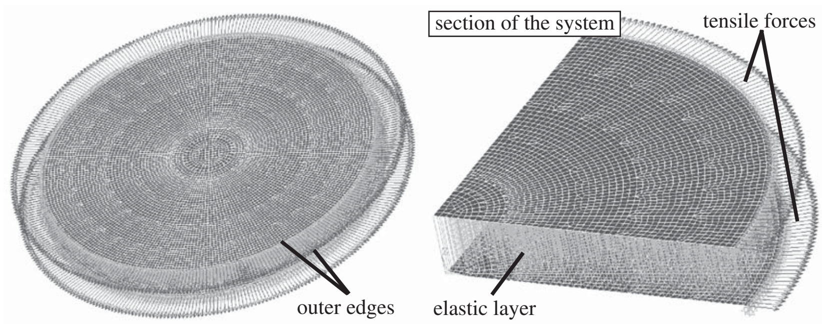

Fig. 3. Finite element model of the circular membrane system.

4.2. The circular membrane system

In this case the Winkler layer is realized as for the previous system. The circular membranes are modeled using orthotropic shell63 elements with the element stiffness option "*Membrane only*" and assuming isotropic properties. As in the previous subsection the first FE model is realized by assuming that constant tension is applied to the outer edges of the membranes using the FE code system standard procedure. The boundary conditions are realized by pinning all nodes lying on the outer edges of the membranes but with a possibility to slide freely in the radial direction. The developed model is displayed in Fig. 3 and it consists of 19,856 shell elements, and 9729 combin elements, respectively.

As in the previous subsection the second FE model is almost the same as the first with the exception of the application of the tensile force. In this case the uniform constant tension is realized by concentrated tensile forces S^*_{0j} imposed in the radial direction to all nodes lying on the outer edges of the membranes. The proper value of the force is selected experimentally by numerical simulation. The outer edge of each membrane includes 324 nodes.

5. Numerical analysis

Numerical analysis results of the annular and circular double-membrane compound systems free vibration are obtained using the models suggested earlier. For each approach, only the first ten natural frequencies and mode shapes are discussed and compared for these kinds of vibrations. The special case where both membranes are identical and have the same values of the uniform constant tension is analyzed. The parameters characterizing the systems used in calculations are shown in Table 1. In Table 1, E and ν are Young's modulus of elasticity and Poisson's ratio, respectively. In this paper the continuous models given by the analytical solutions (see Section 3) are considered as exact, compared to the finite element models, which are treated as approximations of the precise models. The difference between the accurate and the FE models is defined by [23]

$$\varepsilon_{imn} = \frac{\omega^f_{imn} - \omega^c_{imn}}{\omega^c_{imn}} 100\%, \tag{63}$$

where ω^f_{imn} and ω^c_{imn} are the natural frequencies of the FE and exact models, respectively. Eq. (63) is the so-called frequency error [23–25]. Because of manual tuning the modal assurance criterion (MAC) is not used.

5.1. The continuous models

For both systems the natural frequencies are determined as follows. At first the proper values of the coefficient k_{mn} are evaluated by numerical calculation from Eqs. (23) (the annular case) and (52) (the circular case), respectively. Then taking into account the proper values of the factor k_{mn}, the natural frequencies for both systems are determined from the numerical solution of Eq. (25). The mode shapes of vibrations corresponding to the presented pairs of the natural frequencies are presented in the appendix. The natural mode shapes of vibration are connected with the eigenfunctions W_{1mn} and W_{2mn} which are described by Eq. (31) for the annular membrane system, and by Eq. (54) for the circular membrane system. Taking into account the eigenfunction W_{1mn} the mode shapes of vibration are described by the expressions [1]

$$W_{1imn} = W_{1mn}, \quad W_{2imn} = d_{imn} W_{1mn}, \quad d_{1mn} = -d_{2mn} = 1, \quad i = 1,2 \tag{64}$$

and the mode shapes connected with the function W_{2mn} can be determined from the relation

$$W_{1imn} = W_{2mn}, \quad W_{2imn} = d_{imn} W_{2mn}. \tag{65}$$

Both systems under consideration perform two kinds of vibrations: in-phase vibrations ($d_{1mn} > 0$) with lower frequencies ω_{1mn} and out-of-phase vibrations ($d_{2mn} < 0$) with higher frequencies ω_{2mn} ($\omega_{1mn} < \omega_{2mn}$) [1,17]. The deflection shape of the membrane large face is identical for any pair of natural frequencies ω_{imn}. The vibrations are executed by both membranes with equal absolute values of the amplitudes (d_{1mn}=1, d_{2mn}=−1). In the synchronous vibrations case the elastic layer is not deformed on the transverse direction.

In Tables 2 and 3 the results of the calculation for the annular membrane system are shown. The values of the natural frequencies corresponding to the in-phase vibrations of the system are presented in Table 2. The results of the calculation for the out-of-phase vibrations of the annular membrane system are shown in Table 3. Worth pointing out is the fact that the appearance sequence of the natural frequencies connected with the adequate mode shapes for both kinds of vibrations is the same.

Table 1
Parameters characterizing the annular and circular membrane systems.

r_1 (m)	r_2 (m)	h_1 (m)	h_2 (m)	ρ_1 (kg m^{-3})	ρ_2 (kg m^{-3})	E (Pa)	ν	S_1 (N m^{-1})	S_2 (N m^{-1})	k (N m^{-3})
0.5	0.1	0.001	0.001	7.85×10^3	7.85×10^3	2.05×10^{11}	0.29	10^3	10^3	2.9×10^4

Table 2
Natural frequencies of the annular membrane system ω_{1mn} (Hz).

n (m)	0	1	2	3	4	5	6
1	13.697	15.224	18.727	22.948	27.259	31.481	35.702
2	27.978	28.966	31.66				

Table 3
Natural frequencies of the annular membrane system ω_{2mn} (Hz).

n (m)	0	1	2	3	4	5	6
1	19.359	20.468	23.191	26.716	30.5	34.325	38.233
2	31.143	32.034	34.49				

Table 4
Natural frequencies of the circular membrane system ω_{1mn} (Hz).

n (m)	0	1	2	3	4	5
1	8.667	13.787	18.457	22.948	27.259	31.481
2	19.805	25.194	30.223			
3	31.121					

Table 5
Natural frequencies of the circular membrane system ω_{2mn} (Hz).

n (m)	0	1	2	3	4	5
1	16.195	19.422	22.974	26.716	30.5	34.325
2	24.07	28.668	33.175			
3	33.996					

Table 6
Natural frequencies of the annular membrane system ω_{1mn} (Hz) (the first FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	14.331	15.563	18.637	22.542	26.678	30.825	34.928
2	29.138	29.882	32.006				

In Tables 4 and 5 the values of the natural frequencies corresponding to the circular membrane system are presented. The results of the calculation for the in-phase vibrations of the system are displayed in Table 4. The results of the calculation for the out-of-phase vibrations of the system are shown in Table 5. As in a previous case the appearance sequence of the natural frequencies connected with the adequate mode shapes for both kinds of vibrations is the same (see Tables 4 and 5).

Not all the first ten mode shapes of the annular membrane system have their counterparts in the first ten mode shapes of the circular membrane system. The set of mode shapes related to the annular case includes the mode shapes connected with the frequencies ω_{116} and ω_{216}. The set of mode shapes related to the circular case includes the mode shapes coupled with the frequencies ω_{130} and ω_{230}. The appearance sequence of the adequate natural frequencies related to the annular case is different compared with the frequencies related to the circular case (see Tables 2–5).

5.2. The finite element models

In this subsection, numerical solution results are presented using prepared FE models. Tables 6–13 demonstrate the results achieved for the annular membrane system. The natural frequencies and the frequency errors (see Eq. (63)) obtained by using the first FE model of the annular system are presented in Tables 6–9. In the second FE model case the proper value of the concentrated tensile force is selected experimentally to minimize the relation defined by Eq. (63). The results presented in Tables 10–13 are achieved for S_{0j}=9.64 [N].

Table 7
Natural frequencies of the annular membrane system ω_{2mn} (Hz) (the first FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	19.531	20.45	22.875	26.158	29.805	33.577	37.389
2	32.129	32.798	34.727				

Table 8
Frequency error ε_{1mn} (%) (the first FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	4.6288	2.2267	−0.4806	−1.7692	−2.1314	−2.0838	−2.168
2	4.1461	3.1623	1.0929				

Table 9
Frequency error ε_{2mn} (%) (the first FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	0.8885	−0.0879	−1.3626	−2.0886	−2.2787	−2.1792	−2.2075
2	3.166	2.385	0.6872				

Table 10
Natural frequencies of the annular membrane system ω_{1mn} (Hz) (the second FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	14.289	15.518	18.584	22.477	26.602	30.736	34.828
2	29.055	29.796	31.914				

Table 11
Natural frequencies of the annular membrane system ω_{2mn} (Hz) (the second FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	19.501	20.416	22.832	26.102	29.736	33.496	37.295
2	32.053	32.72	34.642				

Table 12
Frequency error ε_{1mn} (%) (the second FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	4.3221	1.9311	−0.7636	−2.0525	−2.4102	−2.3665	−2.448
2	3.8495	2.8654	0.8023				

Table 13
Frequency error ε_{2mn} (%) (the second FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5	6
1	0.7335	−0.2541	−1.548	−2.2983	−2.5049	−2.4151	−2.4534
2	2.922	2.1415	0.4407				

In the in-phase vibrations case the biggest difference between the analytical results and the first FE solution can be visible for the frequencies ω_{110}, ω_{120} and ω_{121}, respectively. The best compatibility is obtained for frequency ω_{112}. In the out-of-phase vibrations case the biggest distinction between the analytical results and the first FE solution may be observable for the frequency ω_{220}. The best result is achieved for frequency ω_{221}.

In the second FE model case the calculation results are slightly different compared with the previous case. In the synchronous vibrations case the biggest difference between the analytical and FE model is observed for the frequencies ω_{110} and ω_{120}. The best compatibility is observed for the frequency ω_{122}. In the asynchronous vibrations case the biggest distinction between the models is noticed for frequency ω_{120}.

Worth pointing out is the fact that in the second FE model case the value of the frequency error for the two frequencies is above 3 percent in the synchronous vibration case (in the first FE model case it is for three frequencies) and in the asynchronous vibration case the value of the frequency error for all frequencies is below 3 percent (in the first FE model case for one frequency the frequency error is above 3 percent).

Tables 14–21 show the results obtained for the circular membrane system. The numerical calculations achieved by using the first FE model of the circular system are displayed in Tables 14–17. The solution results received by using the second FE model of the circular system are presented in Tables 18–21. These results are achieved for $S^*_{0j} = 9.68\,[N]$ (Figs. 4–9).

In the in-phase vibrations case the value of the frequency error for all natural frequencies of the first FE model is below 1 percent which is a more-than-satisfactory result. In the out-of-phase vibrations case the value of the frequency error above 1 percent is observable for the frequencies ω_{210}, ω_{214} and ω_{221}.

Table 14
Natural frequencies of the circular membrane system ω_{1mn} (Hz) (the first FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5
1	8.64	13.768	18.455	22.93	27.277	31.536
2	19.84	25.219	30.263			
3	31.124					

Table 15
Natural frequencies of the circular membrane system ω_{2mn} (Hz) (the first FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5
1	16.444	19.313	22.776	26.515	30.36	34.248
2	24.292	28.996	33.297			
3	33.871					

Table 16
Frequency error ε_{1mn} (%) (the first FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5
1	−0.3115	−0.1378	−0.0108	−0.0784	0.066	0.1747
2	0.1767	0.0992	0.1324			
3	0.0096					

Table 17
Frequency error ε_{2mn} (%) (the first FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5
1	1.5375	−0.5612	−0.8618	−0.7524	1.0316	−0.2243
2	0.9223	1.1441	0.3677			
3	−0.3677					

Table 18
Natural frequencies of the circular membrane system ω_{1mn} (Hz) (the second FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5
1	8.633	13.757	18.44	22.912	27.255	31.51
2	19.824	25.199	30.239			
3	31.099					

Table 19
Natural frequencies of the circular membrane system ω_{2mn} (Hz) (the second FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5
1	16.44	19.305	22.764	26.499	30.34	34.224
2	24.279	28.978	33.275			
3	33.847					

Table 20
Frequency error ε_{1mn} (%) (the second FE model, in-phase vibrations).

n (m)	0	1	2	3	4	5
1	−0.3923	−0.2176	−0.0921	−0.1569	−0.0147	0.0921
2	0.0959	0.0198	0.0529			
3	−0.0707					

Table 21
Frequency error ε_{2mn} (%) (the second FE model, out-of-phase vibrations).

n (m)	0	1	2	3	4	5
1	1.5128	−0.6024	−0.9141	−0.8123	0.9651	−0.2943
2	0.8683	1.0813	0.3014			
3	−0.4383					

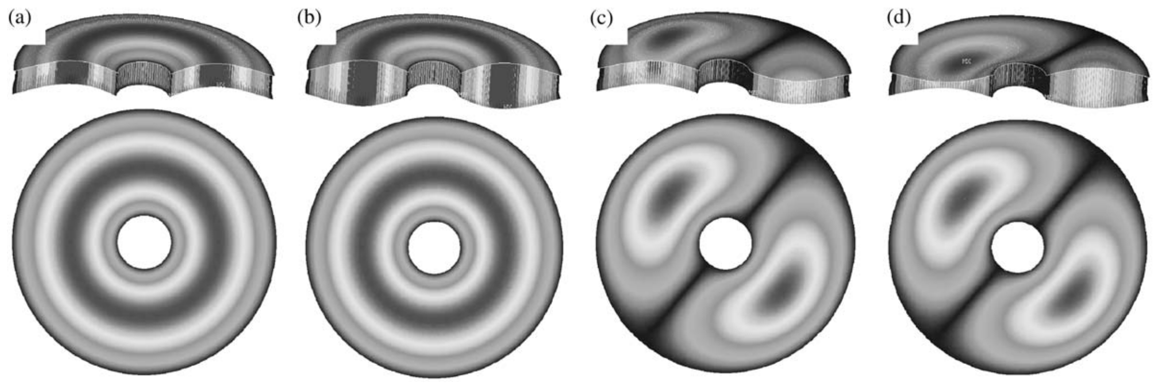

Fig. 4. The mode shapes corresponding to the frequencies: (a) ω_{110}, (b) ω_{210}, (c) ω_{111} and (d) ω_{211}.

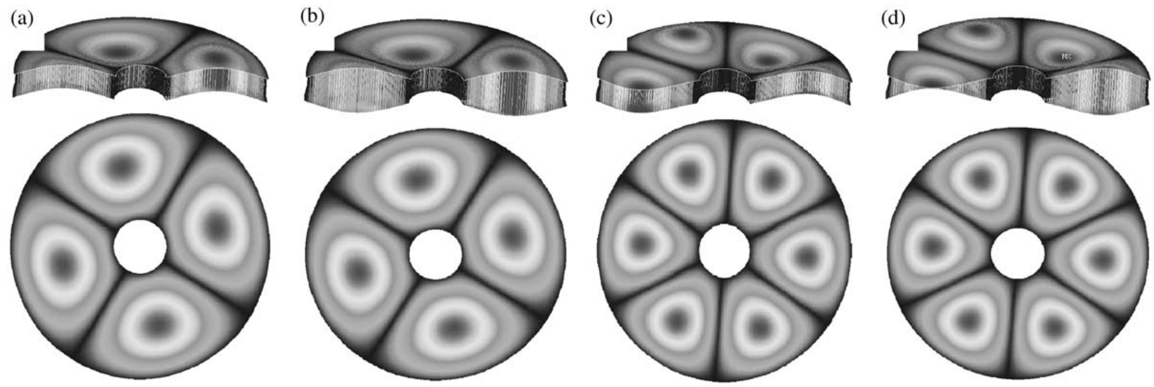

Fig. 5. The mode shapes corresponding to the frequencies: (a) ω_{112}, (b) ω_{212}, (c) ω_{113} and (d) ω_{213}.

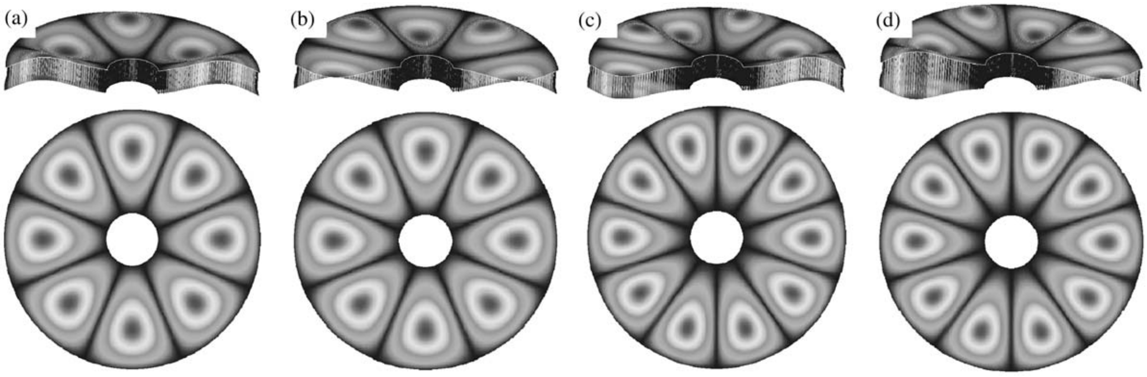

Fig. 6. The mode shapes corresponding to the frequencies: (a) ω_{114}, (b) ω_{214}, (c) ω_{115} and (d) ω_{215}.

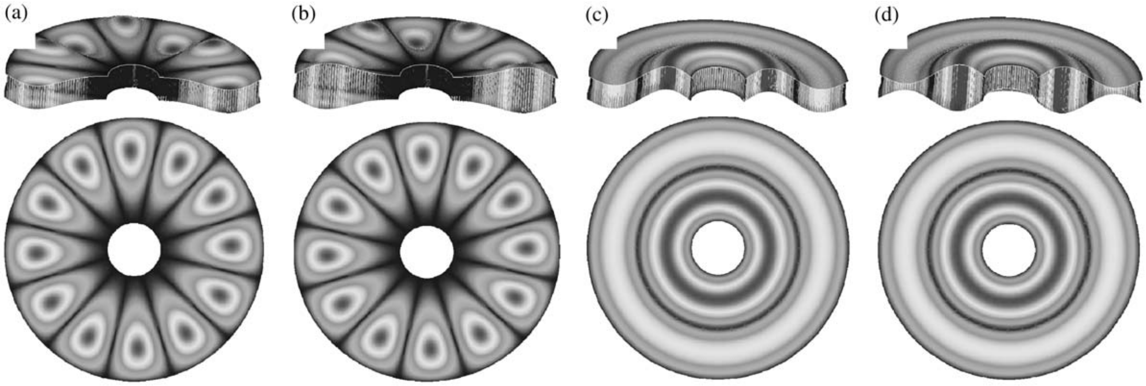

Fig. 7. The mode shapes corresponding to the frequencies: (a) ω_{116}, (b) ω_{216}, (c) ω_{120} and (d) ω_{220}.

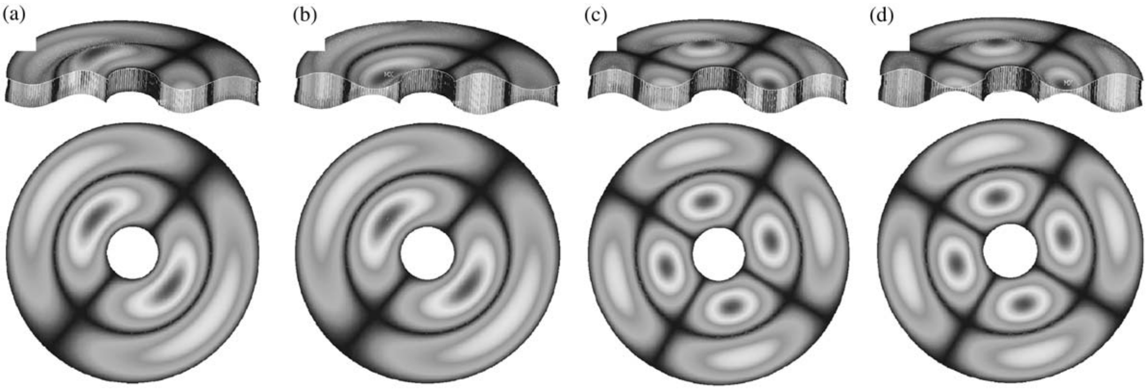

Fig. 8. The mode shapes corresponding to the frequencies: (a) ω_{121}, (b) ω_{221}, (c) ω_{122} and (d) ω_{222}.

In the second FE model case the results are almost the same as in the first FE model case. In the synchronous vibration case the value of the frequency error for all frequencies is below 1 percent. In the asynchronous vibrations case the value of the frequency error above 1 percent is observable for the frequencies ω_{210} and ω_{221}, respectively.

Obviously, all the modes attained from FE solution have their counterparts in the corresponding analytical models. It is surprising that in the annular membranes system case there is less compatibility between the analytical results and the FE solution results compared with the circular case. Taking into account this phase of search it seems that the second FE model would be better to simulate the compound membrane systems under investigation. For all FE models with the exception of the asynchronous vibrations case of the annular membrane systems, it is visible that the biggest differences

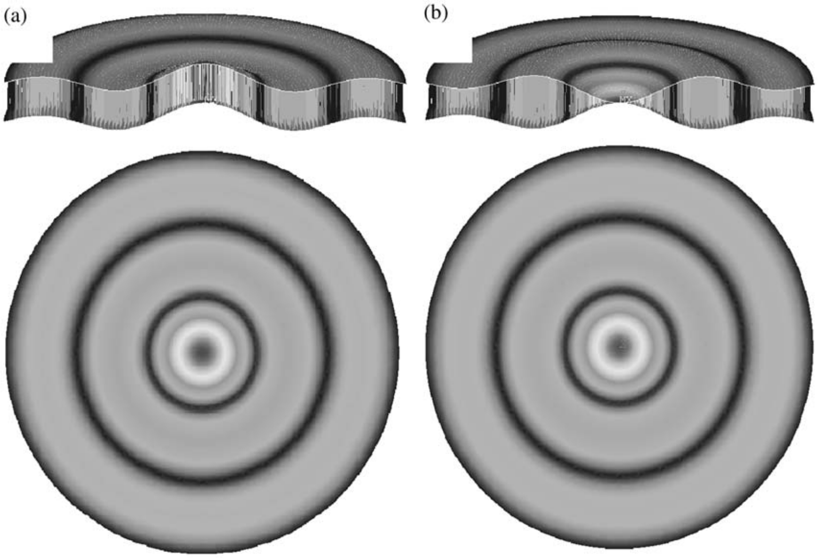

Fig. 9. The mode shapes corresponding to the frequencies: (a) ω_{130} and (b) ω_{230}.

appear in the natural frequencies ω_{i10} (i=1, 2). The main advantage of using the second FE models is the knowledge regarding the value of the concentrated tensile force applied to all the nodes lying on the outer edges.

6. Conclusions

This paper deals with the free transverse vibrations of an elastically connected annular and circular double-membrane systems. The exact solution of the free vibrations of the systems under consideration is found by using the separation of variables method. It is visible that a more complicated solution is found to the annular membrane system case. Numerical simulations are executed for the special case where both membranes are identical and have the same values of the uniform constant tension. It is displayed that the systems execute both synchronous and asynchronous motions. Then the exact solutions of the membrane systems are treated as a testing data and they are used to manually tune the FE models of the membrane systems. Two FE models of each complex system are investigated. In this paper the impact of the manner of the membranes tensile forces application in the FE models on the quality of the accurate model approximation is analyzed. At this stage of search it seems that the second FE model would be better to simulate the analyzed complex membrane system. The principal profit of using the second FE models is the knowledge related to the value of the concentrated tensile force applied to all nodes lying on the outer edges of the membranes. It is particularly helpful for the design engineers of similar structures. The results presented show that in the annular membrane system case there is less compatibility between the analytical results and the FE solution results compared with the circular system case. To achieve better compatibility between the tuned FE models and the continuous systems, further research concerning the type and density of the mesh is needed. Additionally, in the annular membrane system case the research focused on the realization of the boundary conditions of the FE models is necessitated. It is worth considering investigations focused on the modeling of the elastic layer connecting membranes while taking into account the mass of the linking layer. The presented and intended future research in this paper will provide key information for the execution of automatic updating of FE models of the systems under investigation on the basis of the experimental data.

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References

[1] Z. Oniszczuk, *Vibration Analysis of the Compound Continuous Systems with Elastic Constraints*, Publishing House of Rzeszów University of Technology, Rzeszów, 1997 (in Polish).
[2] M.Y. Jaffrin, Dynamic shear-enhanced membrane filtration: a review of rotating disks, rotating membranes and vibrating systems, *Journal of Membrane Science* 324 (2008) 7–25.

[3] J.W. Tack, G.J. Verkerke, E.B. van der Houwen, H.F. Mahieu, H.K. Schutte, Development of a double-membrane sound generator for application in a voice-producing element for laryngectomized patients, *Annals of Biomedical Engineering* 34 (2006) 1896–1907.
[4] S. Ziemba, *Vibration Analysis*, Vol. II, PWN, Warsaw, 1959 (in Polish).
[5] S. Kaliski, *Vibrations and Waves in Solids*, IPPT PAN, Warsaw, 1966 (in Polish).
[6] S.S. Rao, *Vibration of Continuous Systems*, Wiley, Hoboken, 2007.
[7] M. Jabareen, M. Eisenberger, Free vibrations of non-homogeneous circular and annular membranes, *Journal of Sound and Vibration* 240 (2001) 409–429.
[8] P.A.A. Laura, C.A. Rossit, S. La Malfa, Transverse vibrations of composite, circular annular membranes: exact solution, *Journal of Sound and Vibration* 216 (1998) 190–193.
[9] C.Y. Wang, On the polygonal membrane with a circular core, *Journal of Sound and Vibration* 215 (1998) 195–199.
[10] V.X. Kunukkasseril, A.S.J. Swamidas, Normal modes of elastically connected circular plates, *Journal of Sound and Vibration* 30 (1973) 99–108.
[11] S. Chonan, The free vibrations of elastically connected circular plate systems with elastically restrained edges and radial tensions, *Journal of Sound and Vibration* 49 (1976) 129–136.
[12] A.S.J. Swamidas, V.X. Kunukkasseril, Vibration of circular double-plate systems, *The Journal of the Acoustical Society of America* 63 (1978) 1832–1840.
[13] M. Szewczyk, S. Kukla, I. Zamojska, Free vibration of a system of simply supported circular plates connected by elastic elements, *Modelling of Engineering* 33 (2007) 153–158 (in Polish).
[14] S. Chonan, Resonance frequencies and mode shapes of elastically restrained, prestressed annular plates attached together by flexible cores, *Journal of Sound and Vibration* 67 (1979) 487–500.
[15] Z. Oniszczuk, Free transverse vibrations of an elastically connected rectangular plate–membrane complex system, *Journal of Sound and Vibration* 264 (2003) 37–47.
[16] I. Kucuk, I. Sadek, Optimal control of an elastically connected rectangular plate–membrane system, *Journal of Computational and Applied Mathematics* 180 (2005) 345–363.
[17] Z. Oniszczuk, Transverse vibrations of elastically connected rectangular double-membrane compound system, *Journal of Sound and Vibration* 221 (1999) 235–250.
[18] I. Kucuk, Optimality conditions for the control of a double-membrane complex system by a modal decomposition technique, *Applied Mathematical Modelling* 32 (2008) 562–574.
[19] C. de Silva, *Vibration and Shock Handbook*, Taylor & Francis, Boca Raton, 2005.
[20] C.F. Liu, Y. Lee, Finite element analysis of three-dimensional vibrations of thick circular and annular plates, *Journal of Sound and Vibration* 233 (2000) 63–80.
[21] S. Noga, Analysis of toothed gear transverse vibrations, *Cracow University of Technology Publishers, Technical Journal, Mechanics* 9-M (2008) 85–94 (in Polish).
[22] S. Noga, Numerical analysis of a free transverse vibration of an elastically connected annular double-membrane compound system, *Acta Mechanica Slovaca* 3-A (2008) 307–312.
[23] M. Friswell, J. Mottershead, *Finite Element Model Updating in Structural Dynamics*, Kluwer Academic Publishers, Dordrecht, 1995.
[24] S.V. Modak, T.K. Kundra, B.C. Nakra, Comparative study of model updating methods using simulated experimental data, *Computer and Structures* 80 (2002) 437–447.
[25] G.H. Kim, Y.S. Park, An improved updating parameter selection method and finite element model update using multiobjective optimisation technique, *Mechanical Systems and Signal Processing* 18 (2004) 59–78.
[26] S. Noga, Free transverse vibration analysis of an annular membrane, *Vibrations in Physical Systems* XXIII (2008) 283–288.
[27] S. Noga, Dynamical analysis of the low-power electrical engine rotor, *Proceedings of Xth European Mechanics of Materials Conference "Multi-Phase and Multi-Components Materials Under Dynamic Loading"*, Kazimierz Dolny, 2007, pp. 457–465.