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A novel method to obtain a real-time control force strategy using genetic algorithms for dynamic systems subjected to external arbitrary excitations ☆

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ABSTRACT

The paper deals with a discrete differential dynamic programming type of problem. It is an optimal control problem where an external disturbance is controlled over the time horizon by a control force constituted with the well-known convolution approach. The paper presents a simple and novel idea to achieve an optimally controlled response when a linear system is subjected to an arbitrary external disturbance. The proposed approach uses the convolution concept and states that if a control method can be established to restore a unit external disturbance, then the convolution integral can be applied to generate an overall control strategy to control the system when it is subjected to an arbitrary external disturbance. In spite of its simplicity, such a strategy has not been encountered in the literature. The only requirement for this method to be useful is to obtain an optimal control strategy to suppress the vibration of the system when it is subjected to unit response disturbance. To accomplish this, a method from classical optimal control theory such as linear quadratic regulator (LQR) that involves solving the Riccati equation of the associated system can be used. However, genetic algorithm (GA) can be adopted as an alternative way to obtain an optimal control strategy against impulse input. As any arbitrary excitation can be divided into impulses, the convolution concept will constitute the overall optimal control strategy for any arbitrary excitation with simply shifting, scaling and summation (or integration) of the GA-optimized control strategy for each impulse of the arbitrary excitation. The proposed method can be used for real time control applications. Once the control strategy for the impulse disturbance is established, the results can then be used at each time step when online control is performed. Computer simulations were carried out to control the response of a quarter-vehicle active suspension system using the proposed method. The obtained results were compared to those of linear quadratic regulator (LQR) and passive suspension applications. The overall results demonstrated the effectiveness of the proposed method for active suspension systems, especially in suppressing the vehicle body displacement when compared to both the LQR based and passive systems. Furthermore, such a control system proves to be simpler requiring less information to process, which is crucial for real-time applications.

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1. Introduction

Vehicle suspension systems aim to improve the vehicle performance which involves the following criteria [1]:

- *Ride comfort*: Achieving better ride comfort by reducing the vibration transmitted from the road to the vehicle body through the wheels.
- *Handling*: In terms of pitch and roll movements of the vehicle body, which are generated because of the cornering and braking motions.
- *Road holding*: By keeping a good contact force between the wheels and the road all the times.
- *Suspension travel* (*deflection*): This is associated with the limits of suspension working space.
- *Static deflection*: The suspension system should be able to support the payload changes [1].

As the above-mentioned criteria conflict with each other, a good suspension system should achieve a compromise that satisfies different design requirements. Various suspension systems were used to achieve this aim such as passive, semi-active and active suspension systems. The main difference between active systems and the others is that they dissipate energy as well as provide it to the systems they are working with [1]. Research in the area of suspension systems for automotive industry has gained a great interest in recent years from both academia and automobile industry. Many research works have contributed to this field by introducing various methods and techniques to improve the performance and quality of the suspension systems taking the cost, power consumption and the trade offs into consideration. The models suggested in literature varied from one-degree of freedom (1-dof) quarter-vehicle model to multi-degree of freedom full-vehicle model. Feedback and feed-forward control loops involving adaptive and optimal control methods were used by designers to improve the suspension systems' performance developed for different vehicle models.

The effects of adding active components to the passive system in order to achieve better compromise between ride comfort and holding ability was demonstrated in [2,3]. Some practical considerations were also reported such as the operating power consumption which should be carefully taken into account when designing suspension systems. Low energy consumption and good ride comfort were taken as the main design objectives for the active suspension system design presented in [4]. The proposed system has the characteristics of the full bandwidth and the slow active suspension system. Moreover, it utilized two subsystems; the first was an electro-hydraulic system to provide low energy consumption, the second was an electro-hydraulic system with pneumatic actuation mechanism to assure good ride comfort. The design procedure proposed in [5] aimed to find the best parameters of the passive part of the suspension system that would optimize the active control behaviour for minimum power consumption as the main objective. In order to build a pneumatic active suspension system to suppress vibration of a quarter-car model, fuzzy reasoning and disturbance observer were used in [6] in which the active control force was provided by a pneumatic actuator. Deterministic robust control and adaptive control were used in designing active suspension controllers in which the adaptive robust control guaranteed both the transient and steady-state tracking accuracy [7–9]. To investigate the effects of suspension structure on the equivalent parameters, two sets of identified parameters were compared for different suspension systems. The benefit of the parameter identification, which was a crucial process in the design of linear and nonlinear active suspension systems, was demonstrated in [10]. Another work worth to mention was reported by Kucukdemiral et al. where a unified approach referred to as adaptive sliding mode fuzzy control was successfully incorporated an active suspension system [11]. The paper combines the human way of decision-making capabilities of fuzzy logic with the robustness of sliding mode controller and presents prevailing results. The approach surpasses several control techniques including PID and PD type self-tuning fuzzy controllers and proves to overcome the global stability problem. All these methods were performed on a quarter-car model, and the results were presented comparatively.

Active suspension systems are generally considered as two types: fully active and slow active systems [1]. In the fully active suspension systems (also known as high bandwidth systems) the actuator is mounted between the sprung and unsprung masses in parallel with the passive spring [1]. Using such a system would allow the full bandwidth of the suspension to be controlled. However, in the slow active systems (known as low bandwidth systems) the actuator is always in series with the passive system. The lower frequency range is controlled by the actuator and the higher frequency is controlled by the passive system [1]. Fig. 1 shows passive and semi-active suspension systems and Fig. 2 shows the two types of active-suspension systems together, i.e., high-bandwidth (Fig. 2(a)) and low-bandwidth (Fig. 2(b) and (c)) [1,3].

Using active suspension systems would enhance the vehicle performance in terms of ride quality, performance and stability [1]. However, these systems can consume a large amount of power, which is required to operate the control system along with the actuating system (hydraulic and pneumatic). Consequently, necessitating a large amount of power might cause performance degradation [1]. Complexity and high cost have prevented the extensive use of active control for many applications, but the control system stability and superior performance have made it viable for some applications such as suspension systems in automotive industry.

Optimization is one of the most important issues in designing suspension systems; therefore, optimal control theory, which is capable of achieving a desired performance, is commonly employed by researchers to accomplish this purpose. In general, the main aim of optimal control is to generate control signals that minimize (or maximize) a chosen performance

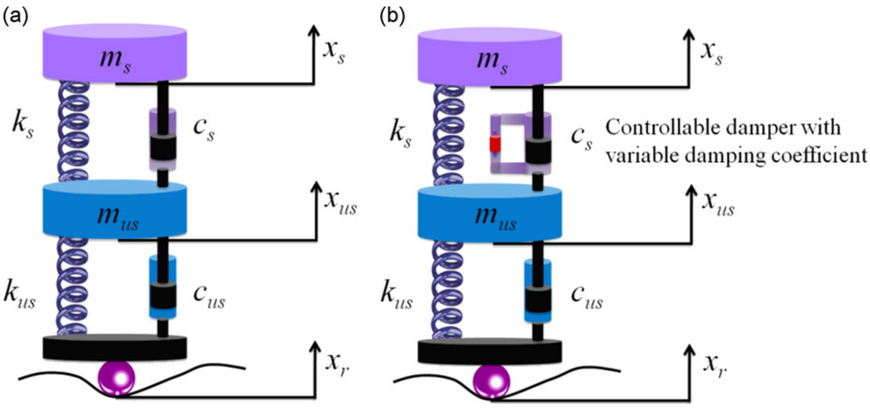

Fig. 1. Automotive suspension systems: (a) passive and (b) semi-active suspensions.

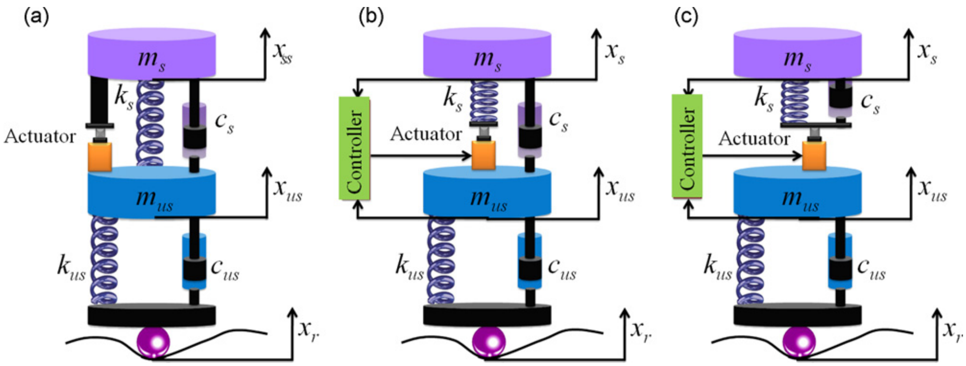

Fig. 2. Active suspension systems: (a) high-bandwidth; (b), (c) low-bandwidth.

index and to make sure that some constraints would not be exceeded [12]. A comprehensive survey on advanced suspension works including the optimal control applications have been presented in [13]. A study on stochastic optimal preview control of vehicle suspensions, which aims to improve the performance of vehicle active suspension system on a random road, was carried out in [14,15], while five different types of suspension systems (fully active, limited active, optimal passive, actively damped and variable damper systems) were designed using stochastic linear quadratic Gaussian (LQG) in [16].

In the present study, a new method was proposed which combines the use of genetic algorithm (GA) and convolution integral concept to find a general control force strategy that minimizes the response of an oscillatory system when it is subjected to external arbitrary excitation. The following sections present the approach in detail and give example applications involving a mathematical model with realistic parameters to demonstrate the effectiveness of the proposed method.

2. Convolution of the control force strategy

The proposed convolution integral based control strategy is aimed to use the convolution concept in designing automotive suspension systems. The study is motivated by the idea of convolution integral that can be used to find the response of the system subjected to an arbitrary excitation if the response of the system to an impulse input is known. An arbitrary excitation is assumed to be constituted of a series of impulses with different amplitudes [17]. Then the total response can be found by simply scaling, shifting and summing (or integrating) the response of the system to these impulses.

Convolution integral is represented in Eq. (1), which consists of the excitation input $F(\tau)$ multiplied by the shifted impulse response function $h(t-\tau)$ [1,17]:

$$x(t)=\int_0^t F(\tau)h(t-\tau)\,\mathrm{d}\tau \tag{1}$$

where $x(t)$ is the system response.

The proposed method follows the similar approach and it was constituted as follows: If the response of a system to an impulse excitation was controlled by a control signal, then it would be possible to obtain a control strategy for the system when it is subjected to an arbitrary excitation by simply scaling, shifting and summing the control forces to the impulse response. The scaling would be performed according to the ratio between the unit impulse and the arbitrary excitation amplitude at each integration interval.

For the proposed method, Eq. (1) is modified to accommodate the arbitrary excitation and the control forces are obtained as

$$G(\tau)=[F(\tau)+\lambda U(t-\tau)] \tag{2}$$

where $F(\cdot)$ is the arbitrary excitation input function, $U(\cdot)$ is the general control force strategy, $h(\cdot)$ is the impulse response function, λ is the scaling factor and will be defined later in Eq. (17).

It is possible to treat $U(\cdot)$ as a part of the external excitation and this would not be in conflict with the idea of the convolution integral. Therefore, the definition of the convolution integral can be applied to constitute a control strategy in order to provide an overall controlled motion

$$x(t)=\int_0^t G(\tau)h(t-\tau)\,\mathrm{d}\tau \tag{3}$$

where $G(\cdot)$ is the actual external input function that results in minimum response of the system.

It is thought that the GA can be used to setup a control strategy against impulse disturbance so that it can be extended to overcome any arbitrary excitation as described above. The prevailing side of the proposed method is that it can be used for real time control applications. Once the control strategy for the impulse disturbance is established, the results can then be used at each time step when online control is performed. Thus, the system can be controlled in real time irrespective of the nature of the external excitation. Since the control of the system would be quite simple in this way, the implementation of such a system would be rather straight forward. In previous studies, the convolution integral method was applied to the response of the system, whereas in this study to the control force strategy.

3. Mathematical model

The mathematical model developed for this study is for a two-degree of freedom (2-dof) quarter-vehicle model subjected to road disturbances. As shown in Fig. 2, the model comprises (m_s, m_{us}), (k_s, k_{us}), (c_s, c_{us}), which are masses and, stiffness and damping coefficients of the sprung and the unsprung, respectively; u is the control force of the actuator, x_r is the road disturbance, x_s is the sprung mass displacement and x_{us} is the unsprung mass displacement.

The equation of motion for the 2-dof quarter-vehicle model is presented as

$$m_s\ddot{x}_s=c_s(\dot{x}_{us}-\dot{x}_s)+k_s(x_{us}-x_s)+u \tag{4}$$

$$m_{us}\ddot{x}_{us}=c_{us}(\dot{x}_r-\dot{x}_{us})+k_{us}(x_r-x_{us})-c_s(\dot{x}_{us}-\dot{x}_s)-k_s(x_{us}-x_s)-u \tag{5}$$

and in matrix form

$$\begin{bmatrix} m_s & 0 \\ 0 & m_{us}\end{bmatrix}\begin{bmatrix}\ddot{x}_s\\ \ddot{x}_{us}\end{bmatrix}=\begin{bmatrix} -c_s & +c_s \\ +c_s & -(c_s+c_{us})\end{bmatrix}\begin{bmatrix}\dot{x}_s\\ \dot{x}_{us}\end{bmatrix}+\begin{bmatrix} -k_s & +k_s \\ +k_s & -(k_s+k_{us})\end{bmatrix}\begin{bmatrix}x_s\\ x_{us}\end{bmatrix}+\begin{bmatrix}0\\ k_{us}\end{bmatrix}x_r + \begin{bmatrix}0\\ c_{us}\end{bmatrix}\dot{x}_r+\begin{bmatrix}+1\\ -1\end{bmatrix}u \tag{6}$$

Representing the system in state space form as in Eq. (7a) will result in Eq. (7b)

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)+\mathbf{D}\dot{x}_r(t) \tag{7a}$$

$$\begin{bmatrix}\dot{x}_{us}-\dot{x}_r\\ \ddot{x}_{us}\\ \dot{x}_s-\dot{x}_{us}\\ \ddot{x}_s\end{bmatrix}=\begin{bmatrix} 0 & +1 & 0 & 0\\ -\dfrac{k_{us}}{m_{us}} & -\dfrac{c_s+c_{us}}{m_{us}} & \dfrac{k_s}{m_{us}} & \dfrac{c_s}{m_{us}}\\ 0 & -1 & 0 & 1\\ 0 & -\dfrac{c_s}{m_s} & -\dfrac{k_s}{m_s} & -\dfrac{c_s}{m_s}\end{bmatrix}\begin{bmatrix}x_{us}-x_r\\ \dot{x}_{us}\\ x_s-x_{us}\\ \dot{x}_s\end{bmatrix}+\begin{bmatrix}0\\ -\dfrac{1}{m_{us}}\\ 0\\ \dfrac{1}{m_s}\end{bmatrix}u+\begin{bmatrix}-1\\ \dfrac{c_{us}}{m_{us}}\\ 0\\ 0\end{bmatrix}\dot{x}_r \tag{7b}$$

If we assume that,

$$x_1=x_{us}-x_r \tag{8}$$

$$x_2=\dot{x}_{us} \tag{9}$$

$$x_3=x_s-x_{us} \tag{10}$$

$$x_4=\dot{x}_s, \tag{11}$$

then Eqs. (7b) can be expressed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & +1 & 0 & 0 \\ -\dfrac{k_{us}}{m_{us}} & -\dfrac{c_s+c_{us}}{m_{us}} & \dfrac{k_s}{m_{us}} & \dfrac{c_s}{m_{us}} \\ 0 & -1 & 0 & 1 \\ 0 & -\dfrac{c_s}{m_s} & -\dfrac{k_s}{m_s} & -\dfrac{c_s}{m_s} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{1}{m_{us}} \\ 0 \\ \dfrac{1}{m_s} \end{bmatrix} u + \begin{bmatrix} -1 \\ \dfrac{c_{us}}{m_{us}} \\ 0 \\ 0 \end{bmatrix} \dot{x}_r \tag{12}$$

After converting the equation of motion to a state space form, it can be solved in time domain using the Runge–Kutta numerical integration algorithm.

4. The optimization process

An optimization process is used to find the most effective design among many other alternatives, considering the constraints which impose limitations on the controller design considerations. The process effectively utilizes the available resources to reach the desired results [18]. In short, the aim of optimization is to find the optimum solution of a problem subjected to many different constraints, and to improve the system performance to achieve optimal operating conditions [18]. The efficiency and robustness of the optimization method will affect the results that could be obtained as a proper solution to the problem. Genetic algorithm (GA) was chosen as an optimization tool in the current study and it will be explained in the following sections.

4.1. Genetic algorithm

Genetic algorithm (GA) is a global search method based on the principle of natural selection, which are related to the evolution in nature. "Survival of the fittest", Darwin's principle of natural selection, promoted the idea of mimicking the natural selection and using it in the artificial life [19–21]. GA is an optimization method that operates with a population of possible solutions (individuals) of the optimization problem. These solutions are evaluated with respect to their degree of fitness that indicates how well the individual will fit (solve) the optimization problem [19–21].

The terms of the GA could be summarized as follows [19]:

- Population ($\mathbf{P}$): the population consists of individuals $\mathbf{s}_i$ with i: 1, 2, ..., α

$$\mathbf{P} = \{\mathbf{s}_1, \ldots, \mathbf{s}_i, \ldots, \mathbf{s}_\alpha\} \tag{13}$$

- An individual $\mathbf{s}_i$ represents a possible solution of the optimization problem, this individual $\mathbf{s}_i$, which is a μ-dimensional vector, consists of μ variables $\mathbf{s}_j$ with j=1, 2, ..., μ. The variables $\mathbf{s}_{ij}$ are referred to as genes

$$\mathbf{s}_i = [\mathbf{s}_{i1}, \ldots, \mathbf{s}_{ij}, \ldots, \mathbf{s}_{i\mu}] \tag{14}$$

Individuals are usually coded either in binary or real numbers. The selection of the appropriate candidates will be related to a fitness function, which depends on the objective function decided by the designer. A simple form of GA uses three types of operators: selection, crossover and mutation [19,20].

4.1.1. Selection

The selection operator tries to select the fittest individuals from the current generation's population to go on into the next generation [19,20]. Biasing towards the fitter individuals in the population is the main strategy of the selection process, which hopefully, will lead to produce the offspring with higher fitness to be selected to the next generation and so on. One should take into consideration that too-strong selection would cause the highly fit individuals to take over the population reducing the diversity, which is an important factor for change and progress. Too-weak selection will lead to too-slow evolution [19,20].

4.1.2. Crossover

This operator will choose two individuals (parents) to produce two new offspring [20]; consequently, the crossover process produces new assemblage of material by swapping genetic material between the individuals of the population. The new offspring will be better than their parents if they get the best features from their parents [22], but the crossover strategy may lead to the loss of good genetic pattern [19]. Although the crossover process reorganizes patterns, it does not introduce new information to the population, except that some patterns are more effective than the others. In order to ensure that pattern formation process is not biased in any particular way, the crossover process is carried out with the probability that determines how many bits will be passed from a parent to an offspring. The crossover cut points will decide how the genetic material of the new offspring will be made up from the material of the chosen individuals [19].

4.1.3. Mutation

Mutation is the occasional alteration of the value of a randomly selected string; the mutation operator will flip one or more randomly selected bits in the chromosome. This will happen with mutation probability, which is usually very small and has a fixed value assigned before the optimization process is carried out. The importance of mutation comes from the fact that it prevents the evolution process; thus the population from getting stuck in any local minima [19,20,22].

Usually searching methods need loads of auxiliary information to perform properly; some of them need the objective function derivatives to be able to climb the current hill such as the gradient technique. Others need the access to almost all tabular parameters like the greedy technique [21]. The main differences between GAs and the other optimization techniques could be summarized as follows [21]:

- GA does not work with the parameters themselves; actually it works with a code of parameter group.
- GA does not perform search depending on one point; it starts searching based on a population of points.
- GA does not depend on the auxiliary information or the derivatives; it just needs the objective function information.
- GA is not a deterministic search technique, because it uses operators with probabilistic transition rules [21].

Although GA may be seen as a panacea to any optimization problem, a researcher new to GA will find out that unless very lucky, GA does not necessarily work well. There will be a great deal of twigging of various user definable stochastic parameters before getting it produce anything acceptable in terms of performance. Another major problem with GA is that it is not really designed to work in real time optimization applications as mentioned above. So when it comes to using it for control problems, the GA can be very effective if control is a passive control.

However, in the present study a real coded GA, in which each variable is represented by a real number, is used with active application and the aim, as described before, is to find a control force strategy that minimizes the response of the system to an impulse input. The first step of GA is to form initial population of individuals (chromosomes) and each member of the chromosome set is a control force value (U_{ij}) corresponding to a time step. In other words, the time horizon t is divided into a number of steps and at each step a control force value is selected, forming a control force vector. This vector is the chromosome in the GA as depicted in Fig. 3, which represents a control force strategy. Initially a population of these vectors (chromosomes) is generated randomly to start the iteration process of the GA. In order to generate the initial population, the following parameter values should be defined for GA to run:

- The number of variables (no. of variables) represents the number of forces. This is a user definable value.
- The population size that refers to how many chromosomes will be generated during each generation, for example, if the no. of variables is four, the population size will decide how many sets of four variables will be used to evaluate the objective function.
- The search space is needed to be defined by inserting the upper and the lower bounds of each variable.

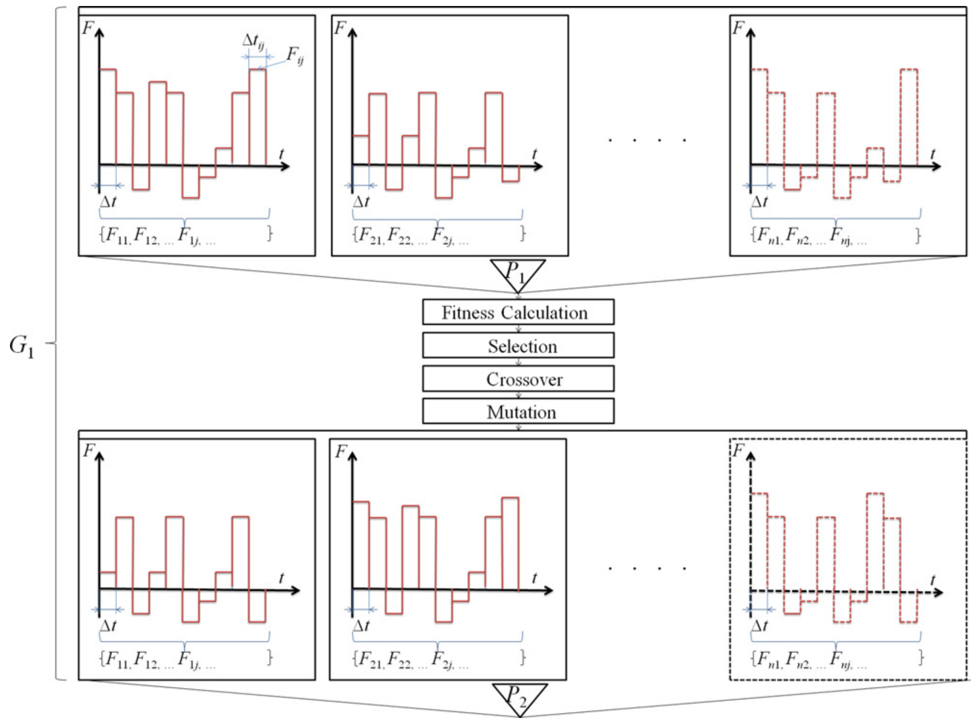

Fig. 3. Genetic operators on the chromosomes for one generation; G: Generation, P: Population. F_{ij}: output variable (here control force, U).

The chromosomes (individuals) of the initial population will undergo the evaluation process in which GA uses an objective function (performance index) to evaluate and select the best individuals. The objective function (*Obj*) in this study can be defined as minimization of the sum of absolute values of all the selected variables over a given time horizon. The selected variables in this case are the sprung mass displacement responses over the whole integration time:

$$Obj = \sum_{i=1}^{n} \sum_{j=1}^{v} \left| x_s(t) \right| \tag{15}$$

where n is the population size, v is the number of variables and $t \in [0,T]$.

One can notice that there are no explicit control force terms in the objective function (*Obj*). However, x_s is produced as a result of the external disturbance and the applied control force. Achieving the objective, that is minimization of overall response of the system, depends on selection of correct control force values at each time step. This involves selection of control forces on a time horizon to constitute the control strategy. The evaluated individuals will assign a fitness value that depends on how well they will solve the problem.

The individuals will go through the selection process, in which each population will be “stacked” based on the ratio $R(j)$ as follows:

$$R(j) = \frac{\sum_{i=1}^{j} Obj(i)}{\sum_{i=1}^{n} Obj(i)} \tag{16}$$

where j is the index of the current population and n is the population size, $R(n)=1$.

The ratio $R(j)$ will generate a wheel, which is divided into regions; the area of these regions will be in proportion with the fitness of the associated variables. GA will generate a random number between 0 and 1, and try to find the region in which this number is placed. This selection process is expected to find the large regions that represent the fitter individuals. After that the crossover operator starts, which involves the exchange of a variable between two of the selected individuals. The variable to be selected is randomly determined as follows: a random integer is generated between one and the maximum number of variables in each chromosome, and this indicates the variable number to be exchanged.

However, there could sometimes be a problem with GA depending on the fitness distribution among the population; that is the best individual may not get selected to the next generation. To ensure that the best individuals will not be lost in the processes, the “elitism” technique is utilized in the selection process. Elitism ensures that GA retains some of the best individuals at each generation. In this work, it was ensured that the best individual of previous generation would populate at least 10% of the next generation. GA was iterated until an acceptable convergence was reached and satisfactory solutions were obtained.

The GA based optimization process introduced was used to find the best control force strategy in order to control the system when it is subjected to an impulse input. After this stage, one can proceed to the next step of the proposed method, which involves use of convolution integral as described above to find the general control force strategy against arbitrary excitation. To do so, the control force strategy against each impulse of the arbitrary excitation should be found. Nevertheless, this does not require to re-run the whole optimization process to get the control force strategy against each of the impulses of the arbitrary excitation. Instead, the ratio (λ_i) between the unit impulse and the arbitrary excitation impulses should be calculated. This ratio equals to

$$\lambda_i = \frac{x_{ri}}{x_r} \tag{17}$$

where x_r is the amplitude of the unit impulse, x_{ri} is the amplitude at the ith iteration, treated as an impulse in the short time interval due to arbitrary excitation. The control force strategy for each of these impulses equals to the optimized control force strategy of the unit impulse multiplied by the ratio λ_i. These control force values are shifted with time and summed up constituting the general control force strategy, which results in the minimum response of the sprung mass against the arbitrary excitation.

5. Linear quadratic regulator

Optimal control theory has been widely applied to the design of vehicle suspension system as reported in literature. In order to compare the performance of proposed method to that of a widely accepted optimal control technique, the application of linear quadratic regulator (LQR) to control a quarter-vehicle model is introduced in this section.

Let us take an example linear dynamical system to be controlled, which is represented in the standard state space form as in Eq. (18) with $x(t_0) = x_0$ as in [12]

$$\dot{x}(t) = \mathbf{B}x(t) + \mathbf{C}u(t) \tag{18}$$

where as usual $\mathbf{B}$ is the $n \times n$ state matrix, $\mathbf{C}$ is the $n \times r$ input matrix.

The optimal linear regulator problem of the system at hand is to determine the optimal control $u(t)$, $t \in [t_0,T]$, which will minimize the quadratic form of the cost function J (performance index) given as

$$J = \int_0^T (\mathbf{x}'\mathbf{R}\mathbf{x} + \mathbf{u}'\mathbf{Q}\mathbf{u})\,\mathrm{d}t \tag{19}$$

where J is the quadratic performance index to be minimized. The superscript ($'$) denotes the matrix transposition. $\mathbf{R}$ is a real symmetric $n \times n$ positive semi-definite matrix, $\mathbf{Q}$ is a real symmetric $r \times r$ positive definite matrix. T is the terminal time and $T > t_0$, t_0=0 [23]. The selection of the trajectory weighting matrix $\mathbf{R}$ and the control weighting matrix $\mathbf{Q}$ is based on the control system designer's experience and is usually determined by experimentation. These weighting matrices execute the main rule for deciding the relative importance of suppressing the response of specific system state or bounding the control attempt [1,23].

One of the most common techniques in literature to find the control gain matrix that minimizes the cost function J introduced in Eq. (19) is dynamic programming, in which it is stated that if the control over some time period (t_0 to t_f) is optimal; then, over all sub-intervals (t_m, t_f) the control would also be optimal [1]. Consequently, based on this method the sequence of the optimal control is computed using the relationships of backward recursion, which starts with the best possible final point [1]. It was proven in [24] that the control signal (u) that would minimize the aforementioned performance index is found as

$$u = -\mathbf{Q}^{-1}\mathbf{C}'\mathbf{K}\,\mathbf{x} = -\mathbf{D}\,\mathbf{x} \tag{20}$$

$$\mathbf{D} = \mathbf{Q}^{-1}\mathbf{C}'\mathbf{K} \tag{21}$$

where $\mathbf{K}$ is a ($m \times m$) symmetric matrix and can be calculated from the following equation:

$$\dot{\mathbf{K}} = \mathbf{KCQ}^{-1}\mathbf{C}'\mathbf{K} - \mathbf{R} - \mathbf{B}'\mathbf{K} - \mathbf{KB} \tag{22}$$

Eq. (22) is the generalized form of Riccati differential equation. The matrix $\mathbf{D}$ is referred to as the optimal control gain matrix.

6. Numerical results

In order to perform numerical tests, a computer program was developed using Visual Basic 6. The integration, which uses the Runge–Kutta numerical integration method, was carried out for t=4 s with the time interval of Δt=0.02 s.

6.1. Response to a shock input with active suspension (i.e. with control forces)

An optimization process is carried out to find the optimal control force strategy, which minimizes the response of the sprung mass to the shock input. The shock input and the generated GA based control force against this input are plotted in Fig. 4. In this investigation the ranges of several parameters were studied. In particular, actuator forces were carefully determined after experimenting with a number of possible actuator ranges. Another important parameter studied was force reversals. Initially, the actuator was allowed to generate as many values as the number of integration steps. This was, as explained before, equivalent to the number of variables in the control strategy. Initially, it was assumed that allowing the maximum number of variables would enable GA to iterate and find the most effective number of variables so that the control strategy becomes optimum. It was expected that GA would eliminate "unnecessary" variables (of forces) by zeroing their magnitudes. As each control force represents a reversal and as it is known from literature that the total number of reversals is related to the degree of freedom of the system under investigation, one would expect for this system that the number of reversal would be four according to the Pontryagin principle. After a lengthy investigation and many

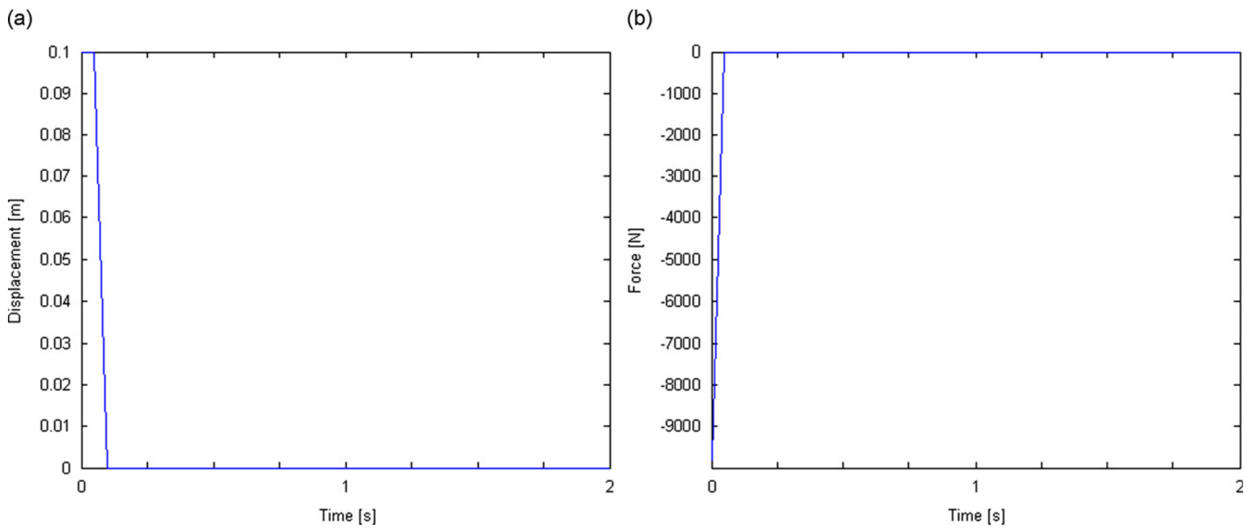

Fig. 4. (a) The shock input and (b) the generated GA based control force against it.

experiments, it was found that GA was not able to reach such a number. It was then decided that the number four would be the total number of forces as imposed by the GA scheme. The population size, which enables GA to search for the range and density of sampling points at which fitness was assessed, was initially chosen to be equal to the number of the integration steps. Also, during the early part of the experimentation, the total number of generations was chosen to be equal to the number of the integration steps. Although there is no direct association between the number of generations and the number of genes in population (in the current case the number of forces was initially taken to be the same as the number of integration steps), it was accepted that they were related and more genes in chromosome would slow down the convergence. After many runs with different numbers of chromosomes and generations, it was found that a good convergence could be achieved by using the numbers presented in Table 2 and these numbers reduced the amount of total time spent during the GA iterations.

Fig. 5 shows the response of the sprung mass when it is subjected to a shock disturbance and controlled by the optimized actuator forces. The magnitude of the optimized control forces suppressing the deflections to the shock disturbance is depicted in Fig. 4(b). The exact values of the forces are presented in Table 3.

It is known that genetic algorithm (GA) may fail for some applications; however, in the present study GA was able to converge to specific force values after a certain number of generations. The number of generations was chosen after experimenting with quite a few different numbers of generations in such a way to ensure that the convergence would occur within the selected range. This can be seen for the optimized control force values in Figs. 6 and 7, as GA was able to converge to the force values presented in Table 3 within the range.

Figs. 8–10 show the controlled acceleration, suspension deflection and tyre deflection responses to the shock excitation, respectively. The results obtained by applying the proposed method to control the quarter-vehicle response to a shock excitation were compared to the response of the passive suspension system under the same shock excitation. The passive and controlled sprung mass displacement responses are presented in Fig. 11, while responses of passive and active accelerations, suspension deflections and tyre deflections are shown in Figs. 12–14, respectively, for comparison purpose.

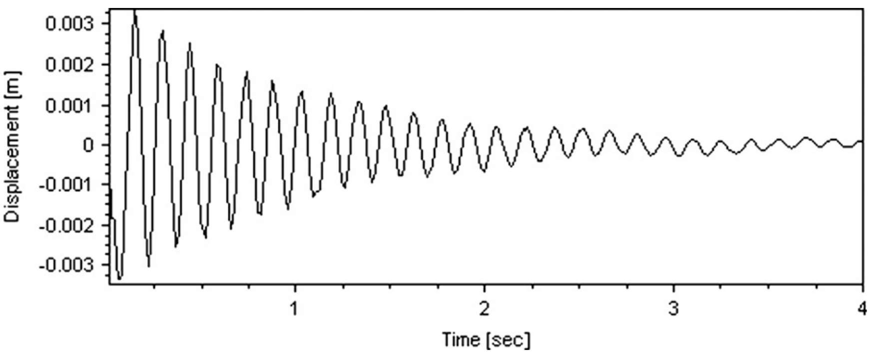

Fig. 5. The controlled displacement response of the sprung mass m_2 to the shock input.

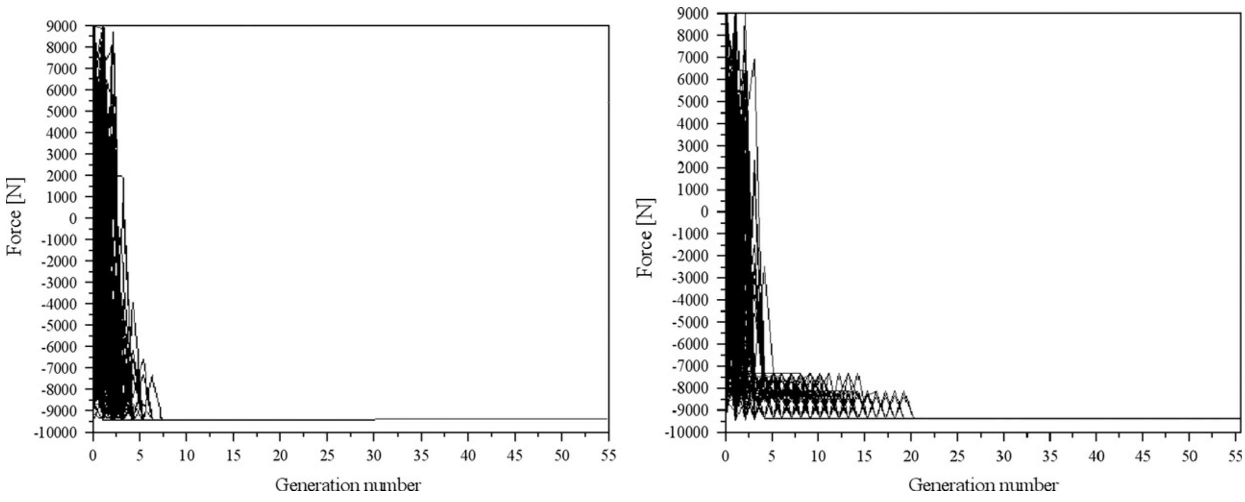

Fig. 6. Genetic algorithm convergence to the values of U_1 and U_2, respectively.

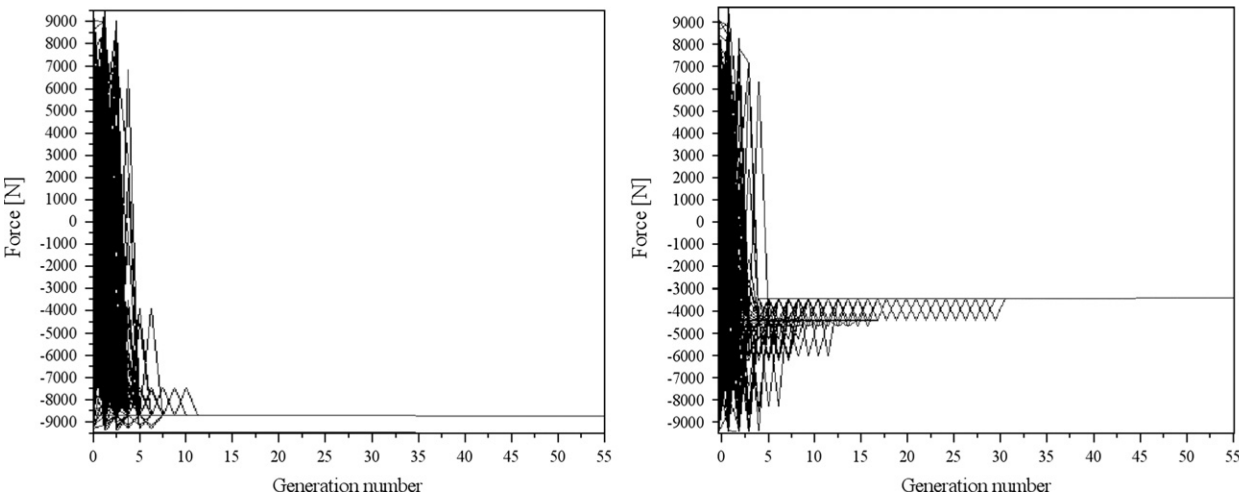

Fig. 7. Genetic algorithm convergence to the values of U_3 and U_4, respectively.

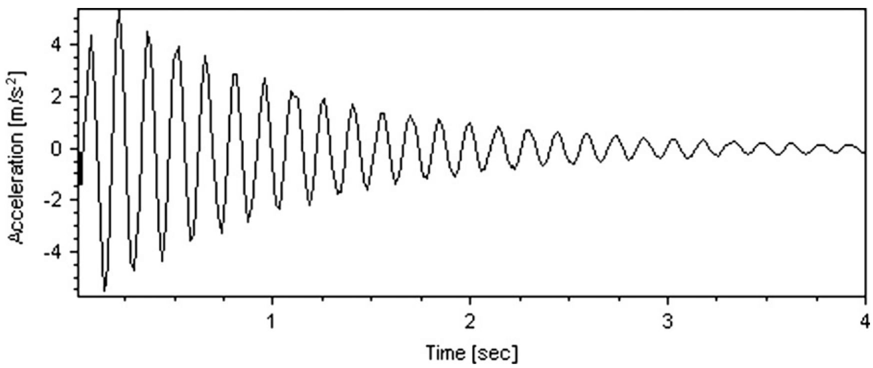

Fig. 8. The controlled acceleration response of the sprung mass m_2 to the shock input.

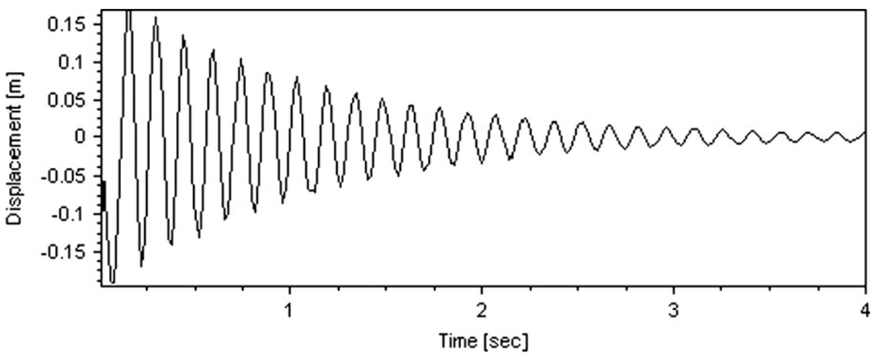

Fig. 9. The controlled suspension deflection response.

6.2. Application of the proposed method to achieve active suspension control against arbitrary excitation

As presented, the aim is to control the sprung mass response to an arbitrary excitation by means of the control force strategy established through the proposed method. To do this, first a control force strategy optimized by GA was set up. Then an arbitrary excitation input was generated as shown in Fig. 15 and it was divided into impulses. The ratio λ was calculated for each of these impulses using Eq. (17). Next, the control rule established as described above was shifted and scaled by λ for each impulse of the arbitrary excitation and by summing up the generated control forces against the arbitrary excitation were obtained as depicted in Fig. 16. This control force strategy was applied to control the sprung mass response when it was subjected to arbitrary excitation; the controlled response against this arbitrary excitation is shown in

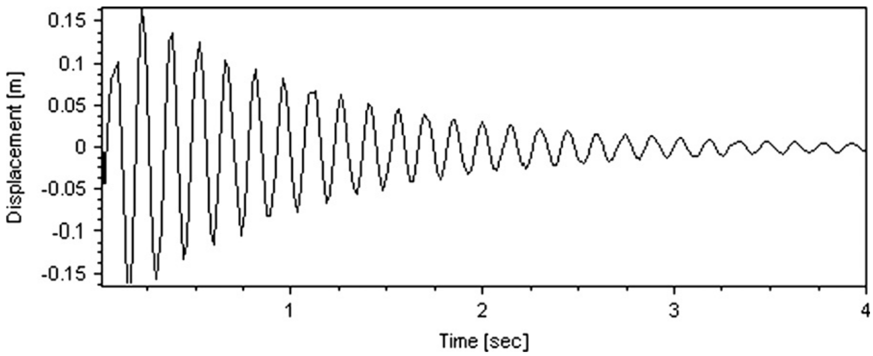

Fig. 10. The controlled tyre deflection response.

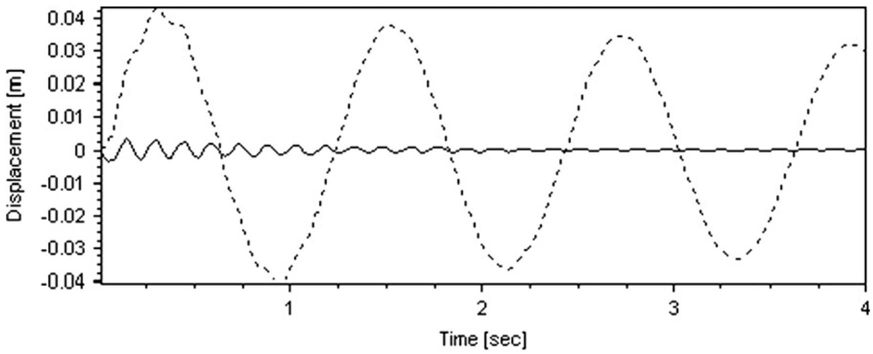

Fig. 11. The sprung mass m_2 displacement response to the shock excitation passive (dashed line), active (solid line).

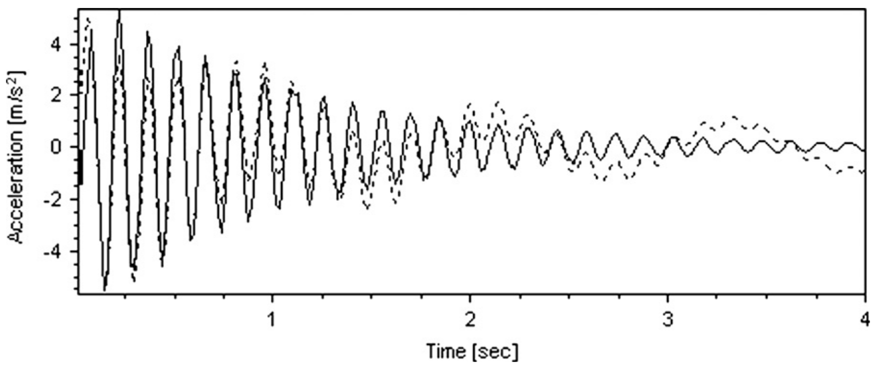

Fig. 12. The sprung mass m_2 acceleration response to the shock excitation passive (dashed line) and active (solid line).

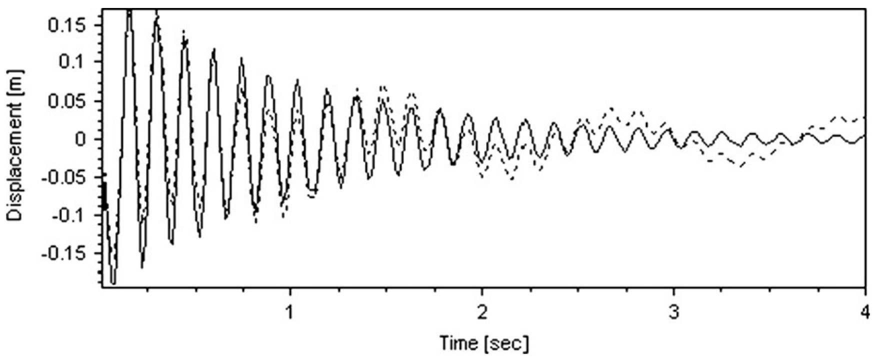

Fig. 13. Suspension deflection response to the shock input passive (dashed line) and active (solid line).

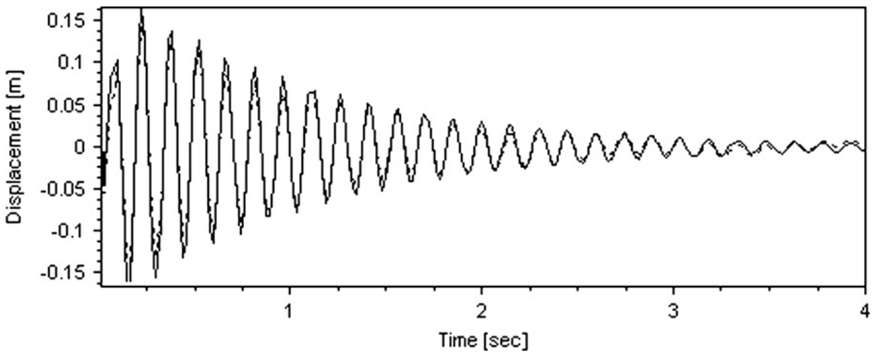

Fig. 14. Tyre deflection response to the shock input passive (dashed line) and active (solid line).

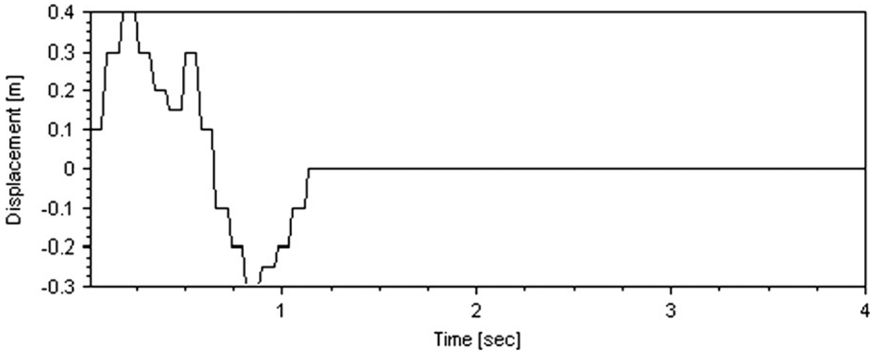

Fig. 15. An arbitrary excitation input.

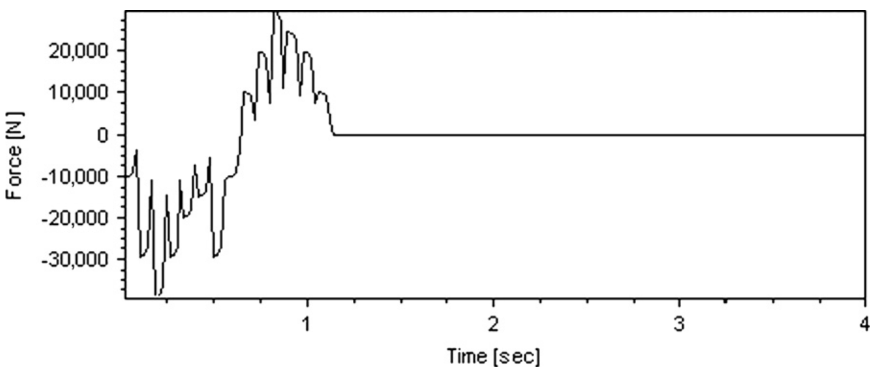

Fig. 16. The generated control force against the arbitrary input given in Fig. 15.

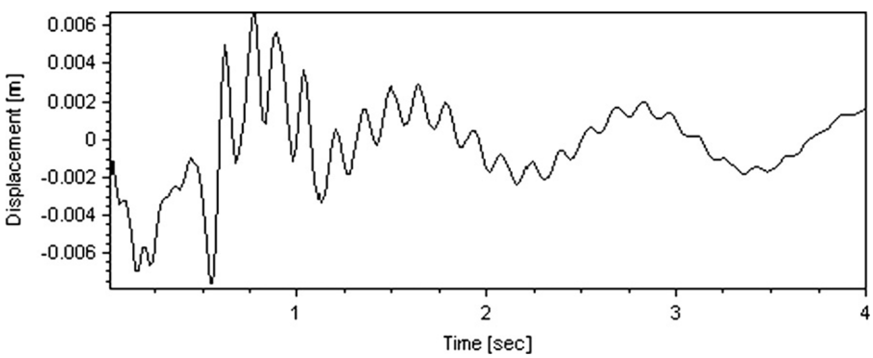

Fig. 17. The controlled m_2 displacement response to the arbitrary excitation.

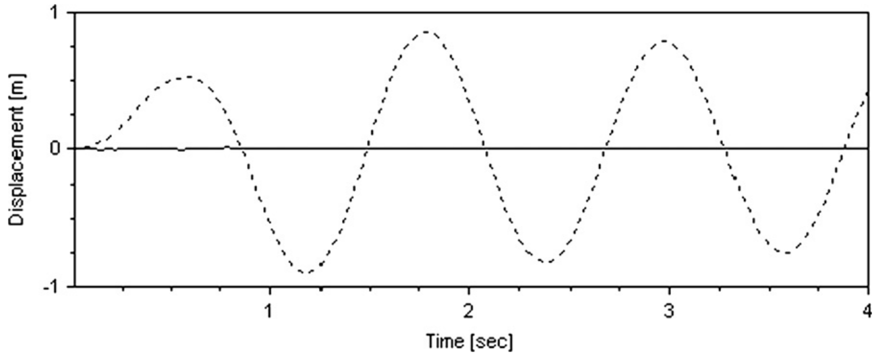

Fig. 18. The sprung mass m_2 displacement response to the arbitrary excitation: passive (dashed line) and active (solid line).

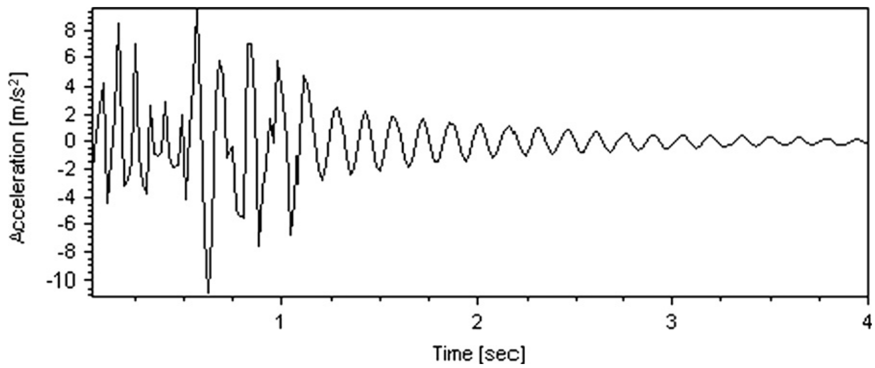

Fig. 19. The controlled m_2 acceleration response to the arbitrary excitation.

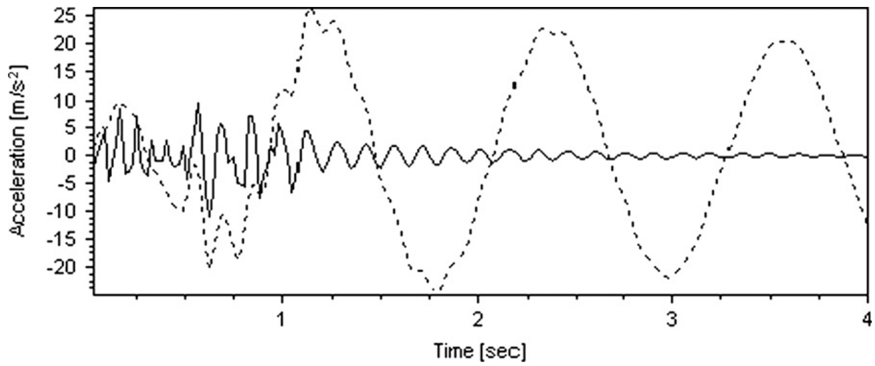

Fig. 20. The sprung mass m_2 acceleration response to the arbitrary excitation passive (dashed line) and active (solid line).

Fig. 17. Displacements generated by passive and active suspensions against the arbitrary excitation are presented in Fig. 18 for comparison purpose. The controlled acceleration response is shown in Fig. 19 and may be compared to the passive response plotted together in Fig. 20. Similarly, the actively controlled suspension deflection and tyre deflection responses are shown in Figs. 21 and 23, and they can be compared to the passive responses plotted in Figs. 22 and 24, respectively.

6.3. The optimal control strategy using linear quadratic regulator (LQR)

In order to solve the Riccati equation given in matrix form in Eq. (22) to minimize the performance index presented in Eq. (19), the gain matrix **D** given in Eq. (21) should be determined so that it would be possible to find an optimal control rule as a function of the state variables of the system. To achieve this, one should first set the quadratic performance index as follows:

$$J = \int_0^T (x'_2 \mathbf{R} x_2 + u' \mathbf{Q} u)\,\mathrm{d}t \tag{23}$$

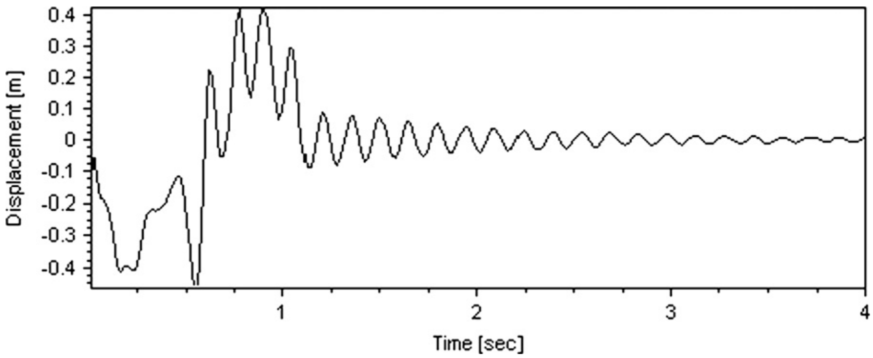

Fig. 21. The controlled suspension deflection response to the arbitrary input.

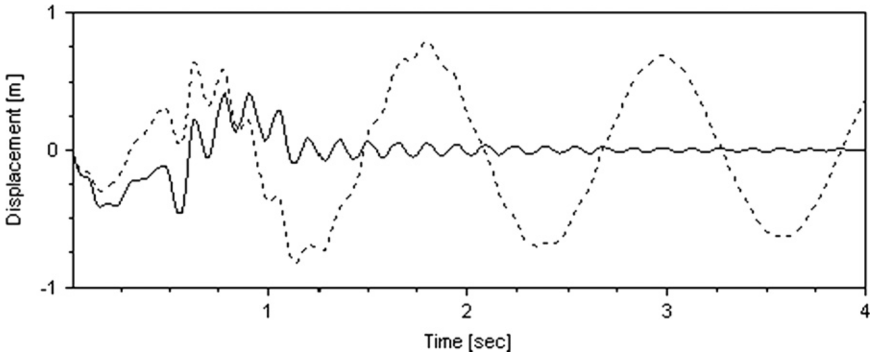

Fig. 22. Suspension deflection response to the arbitrary input passive (dashed line) and active (solid line).

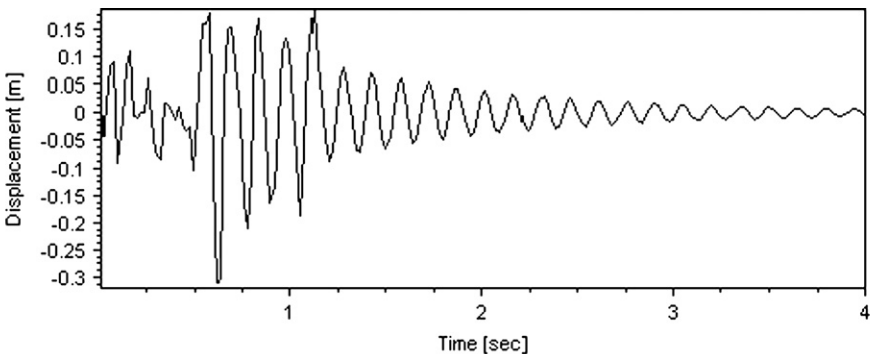

Fig. 23. The controlled tyre deflection response to the arbitrary input.

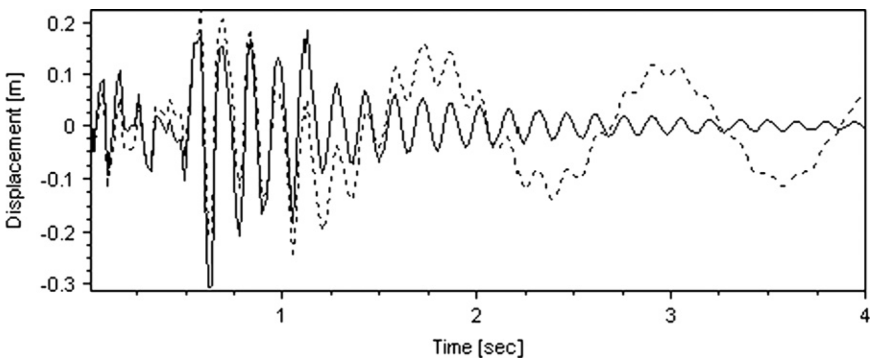

Fig. 24. Tyre deflection response to the arbitrary input passive (dashed line), active (solid line).

where x_2 is the sprung mass displacement. The matrices **R** and **Q** were chosen after many tests in such a way to make the results obtained by the LQR method comparable to the results obtained by the proposed method. The state and control weighting matrices **R** and **Q**, respectively, are given as follows:

$$\mathbf{R}=\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \mathbf{Q}=\left[4.5\times10^{-13}\right] \tag{24}$$

Then the performance index will be

$$J=\int_0^T\left(x_2^2+\left[4.5\times10^{-13}\right]u^2\right)\mathrm{d}t \tag{25}$$

The system matrices **B** and **C** will become

$$\mathbf{B}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-\dfrac{k_{\mathrm{us}}+k_s}{m_{us}}&\dfrac{k_s}{m_{\mathrm{us}}}&-\dfrac{c_s+c_{\mathrm{us}}}{m_{\mathrm{us}}}&\dfrac{c_s}{m_{\mathrm{us}}}\\\dfrac{k_s}{m_s}&-\dfrac{k_s}{m_s}&\dfrac{c_s}{m_s}&-\dfrac{c_s}{m_s}\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}0\\0\\-\dfrac{1}{m_{\mathrm{us}}}\\\dfrac{1}{m_s}\end{bmatrix} \tag{26}$$

The values of the model variables are given in Table 1; some of these variables are taken from [25]. In order to find the 4×4 matrix **K**, the Riccati equation in Eq. (22) should be solved. However, for numerical integration of the differential equations it would be more appropriate to start with the initial values, so the time is reversed as follows:

$$\tau=T-t \tag{27}$$

and Eq. (22) will take the form:

$$\frac{\mathrm{d}K}{\mathrm{d}\tau}=\mathbf{R}+\mathbf{B'K}+\mathbf{KB}-\mathbf{KCQ}^{-1}\mathbf{C'K} \tag{28}$$

with the zero initial values [24].

Table 1
The values of the quarter-vehicle variables.

Model variables	Values	Unit
m_s	2500	kg
m_{us}	320	kg
k_s	80,000	N m^{-1}
k_{us}	500,000	N m^{-1}
c_s	500	N s m^{-1}
c_{us}	20	N s m^{-1}
The upper bound of (u_i)	+10,000	N
The lower bound of (u_i)	−10,000	N

Table 2
The parameters values required to implement the optimization process.

Parameter	Value
No. of variables (control forces)	4
Population size	300
No. of generations	100

Table 3
Control force values obtained using genetic algorithm against shock input.

Force	U_1	U_2	U_3	U_4
Value (N)	−9921.36	−9769.85	−9147.76	−3661.89

Runge–Kutta numerical integration algorithm was used to solve Eq. (28). The convergence of the elements of matrix **K** is almost completed after t=1.0 s. Then the matrix **K** is found to be

$$\mathbf{K}=\begin{bmatrix} 0.00008723468070 & 0.00065297820393 & 0.00000014905327 & 0.00000876347192 \\ 0.00065297820393 & 0.06042025294314 & 0.00002964812207 & 0.00182109008197 \\ 0.00000014905327 & 0.00002964812207 & 0.00000008722491 & 0.00000230420118 \\ 0.00000876347192 & 0.00182109008197 & 0.00000230420118 & 0.00011865145798 \end{bmatrix} \tag{29}$$

In order to find the optimal control gain matrix **D**, the matrices **Q**, **C**, **K** are substituted into Eq. (21):

$$\mathbf{D}=\begin{bmatrix} d_1 & d_2 & d_3 & d_4 \end{bmatrix} \tag{30}$$

where $d_1=-6754.66066500$, $d_2=-1412857.00293167$, $d_3=-1442.45028500$ and $d_4=-89{,}466.56556556$.

As a result the optimal control forces that minimize the sprung mass response to any excitation as a function of the system state variables can be found by substituting **D** into Eq. (20):

$$u=-\begin{bmatrix} d_1 & d_2 & d_3 & d_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=-d_1x_1-d_2x_2-d_3x_3-d_4x_4 \tag{31}$$

In the following paragraphs, the results obtained by means of the LQR based optimal control method will be illustrated and they compared with the results obtained via the proposed method, i.e. convolution of the control force strategy, to control the response of the quarter-vehicle model when it is subjected to the shock and arbitrary excitation.

Fig. 25 shows the optimal control forces generated by LQR to control the quarter-vehicle response to a shock excitation, while in Fig. 26 these forces are compared to the control strategy optimized by genetic algorithm against the same shock excitation. Likewise, the optimal control forces generated by LQR to control the quarter-vehicle response to the arbitrary excitation are shown in Fig. 27 and compared to the control strategy established by the proposed method in Fig. 28.

The sprung mass controlled displacement response by means of LQR against shock excitation is shown in Fig. 29 and is compared to the controlled displacement response using the proposed method in Fig. 30. The actively controlled displacement response to the arbitrary excitation using LQR method is plotted in Fig. 31 and it could be compared to the

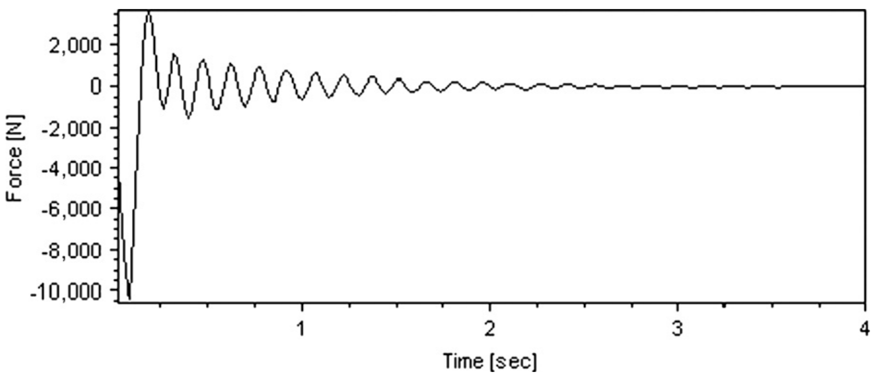

Fig. 25. The generated control force strategy by LQR against the shock input.

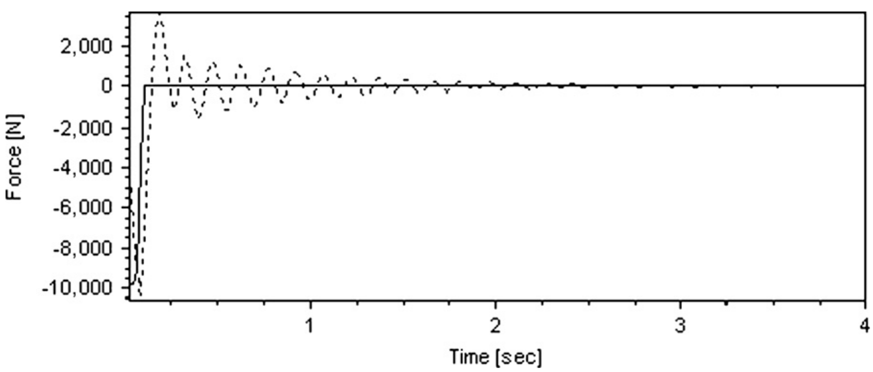

Fig. 26. The generated control force strategy against shock input LQR (dashed line), GA(solid line).

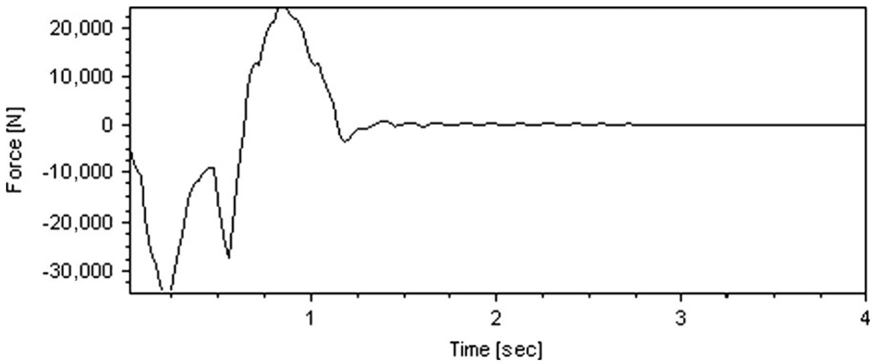

Fig. 27. The generated control force strategy by LQR against arbitrary input.

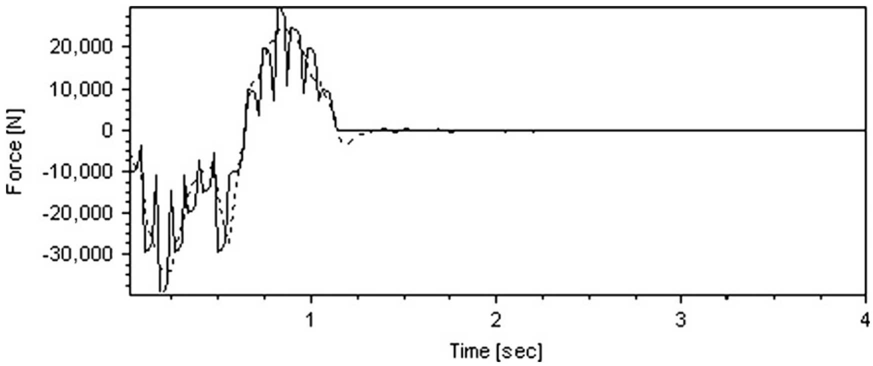

Fig. 28. The generated control force strategy against arbitrary input by LQR (dashed line) and by the proposed method (solid line).

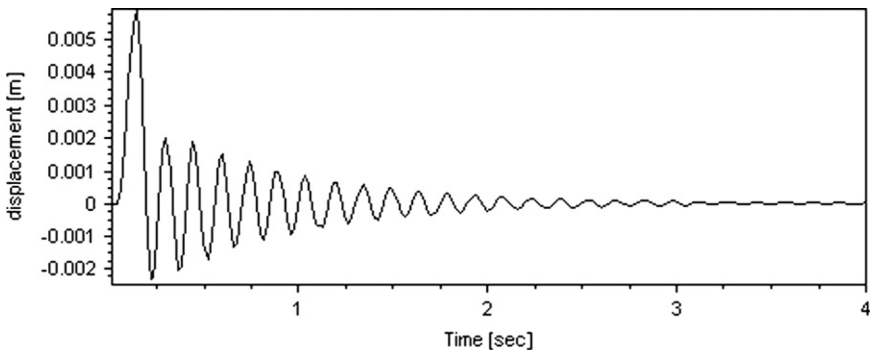

Fig. 29. The sprung mass m_2 displacement response to the shock excitation by LQR.

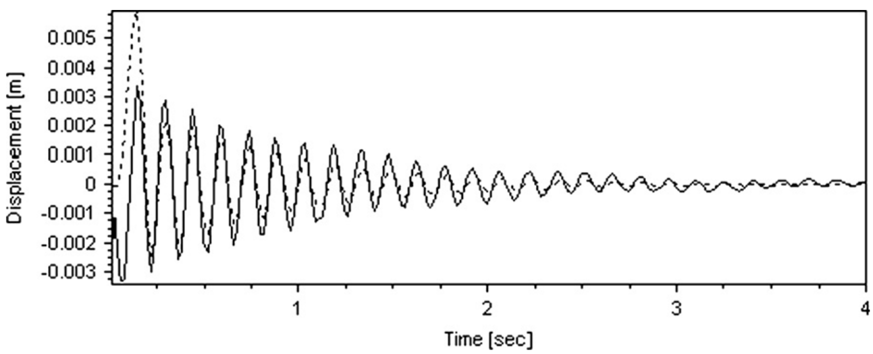

Fig. 30. The sprung mass m_2 displacement response to the shock excitation LQR (dashed line), the propose method (solid line).

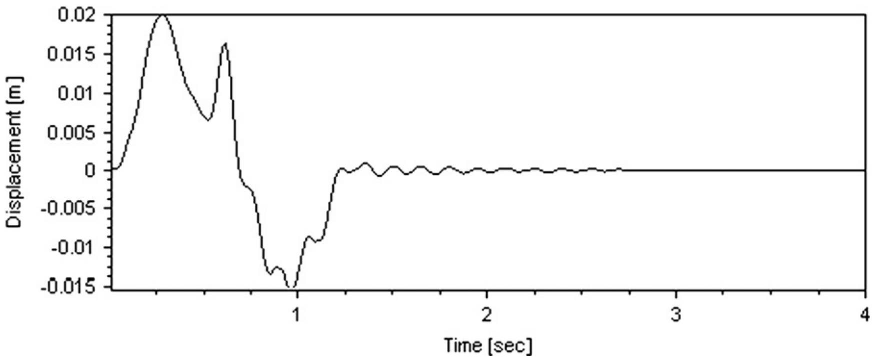

Fig. 31. m_2 displacement response to the arbitrary excitation using LQR.

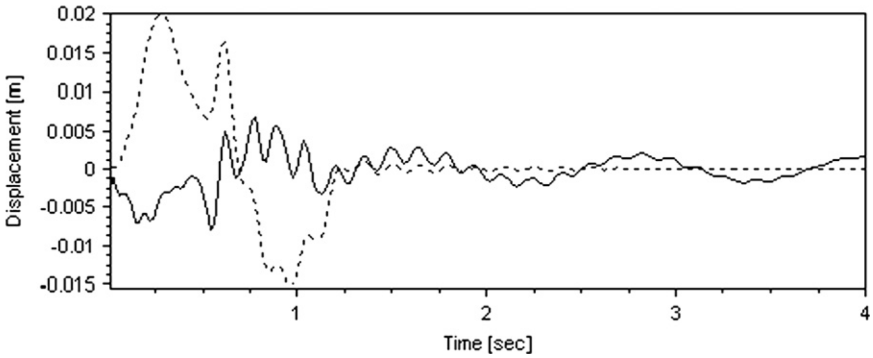

Fig. 32. The sprung mass m_2 displacement response to the arbitrary excitation LQR (dashed line), the proposed method (solid line).

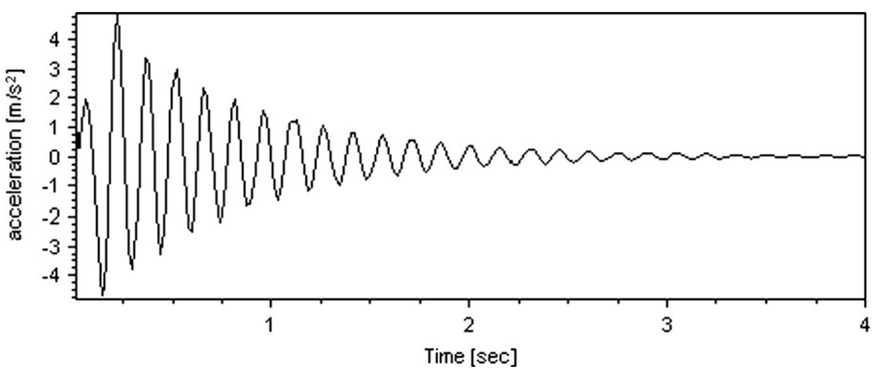

Fig. 33. The sprung mass m_2 acceleration response to the shock excitation using LQR.

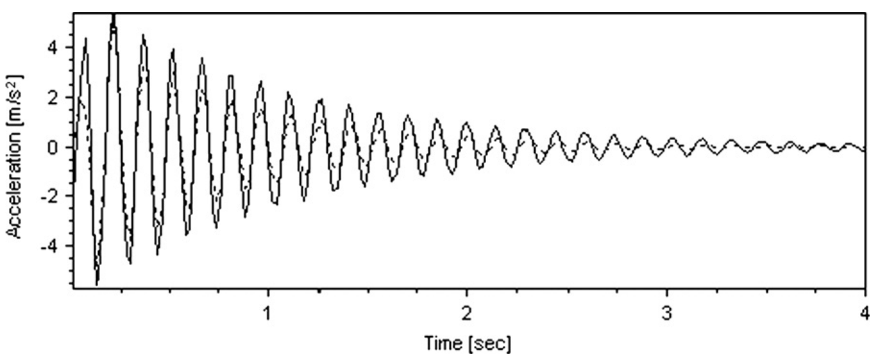

Fig. 34. The sprung mass m_2 acceleration response to the shock excitation LQR (dashed line), using the proposed method (solid line).

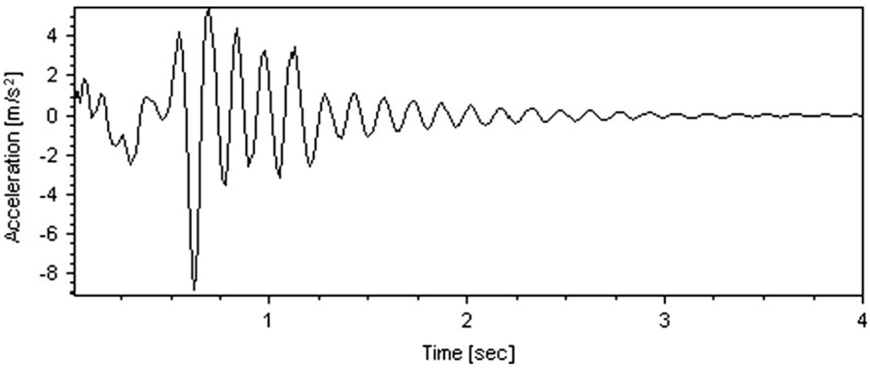

Fig. 35. The sprung mass m_2 acceleration response to the arbitrary excitation by LQR.

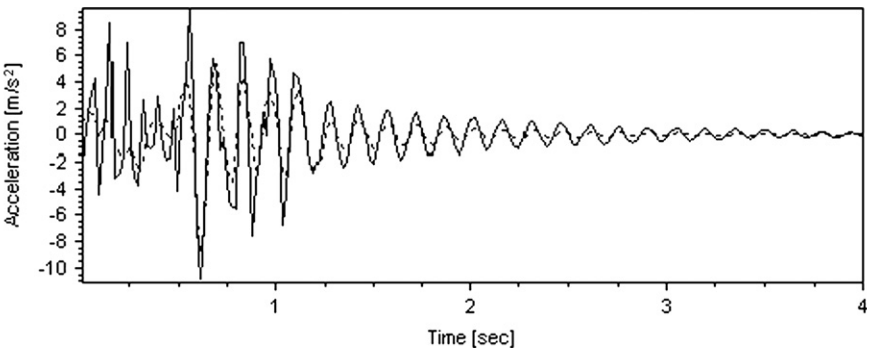

Fig. 36. The sprung mass m_2 acceleration response to the arbitrary excitation using LQR (dashed line) and the proposed method (solid line).

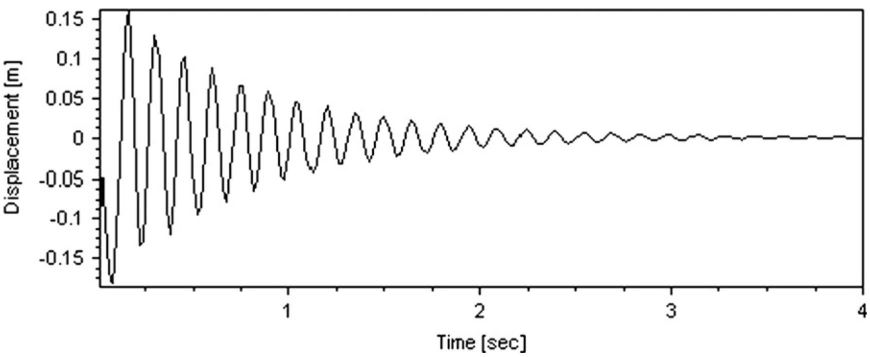

Fig. 37. Suspension deflection response to the shock input using LQR.

response obtained using the proposed method in Fig. 32. The actively controlled sprung mass acceleration response using the LQR method to the shock excitation is shown in Fig. 33 and the response to the arbitrary excitation is shown in Fig. 35. Both are compared to the acceleration response obtained by means of the proposed method in Figs. 34 and 36. The suspension deflection and tyre deflection responses with LQR are shown in Figs. 37 and 41 for shock excitation and, in Figs. 39 and 43 for arbitrary excitation. These responses are compared to the ones obtained by the proposed method in Figs. 38 and 42 for shock excitation and, in Figs. 40 and 44 for arbitrary excitation, respectively.

7. Discussion

Based on the numerical results obtained and plotted in Figs. 11–18, one can easily observe the superiority of the proposed method in reducing the quarter-vehicle displacement response for both shock and arbitrary excitations over the passive suspension response. The controlled displacement response of the vehicle body to the shock excitation using the proposed method was very similar with the displacement response using the LQR method but with some reduction in

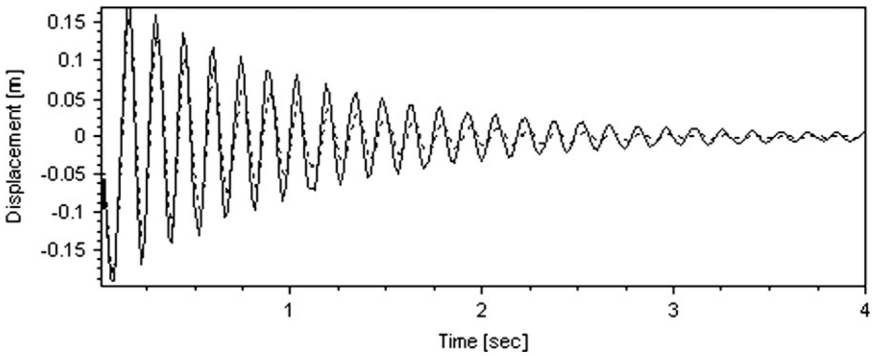

Fig. 38. Suspension deflection response to the shock input LQR (dashed line), using the proposed method (solid line).

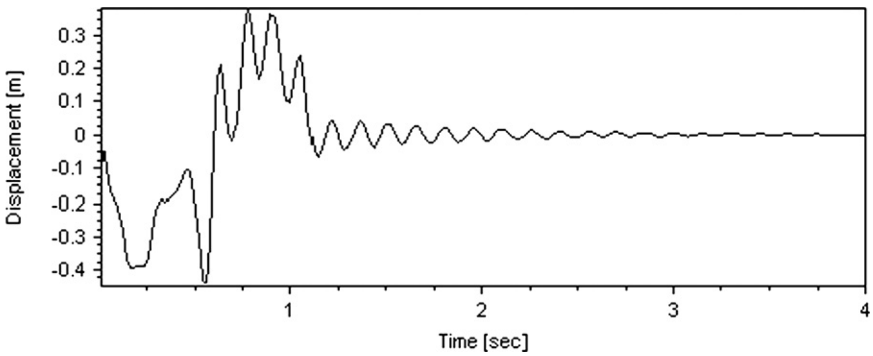

Fig. 39. Suspension deflection response to the arbitrary input using LQR.

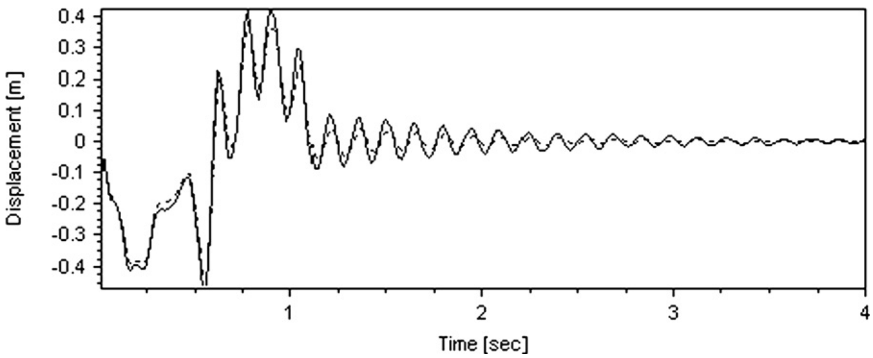

Fig. 40. Suspension deflection response to the arbitrary input LQR (dashed line), the proposed method (solid line).

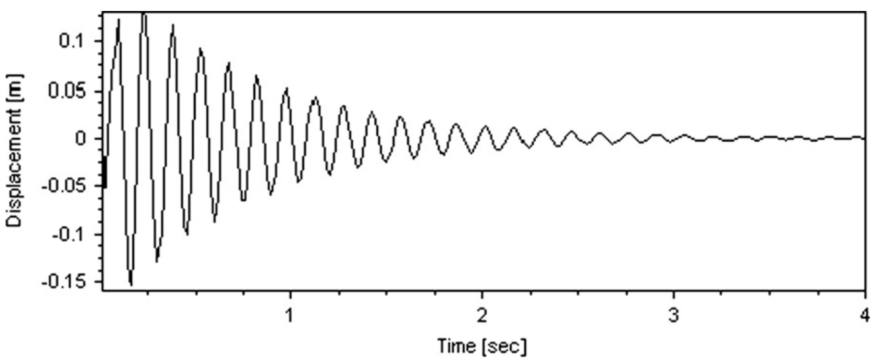

Fig. 41. Tyre deflection response to the shock input using LQR.

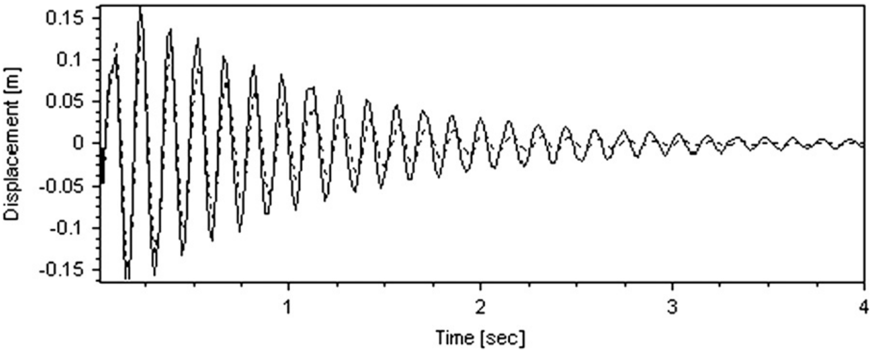

Fig. 42. Tyre deflection response to the shock input LQR (dashed line), using the proposed method (solid line).

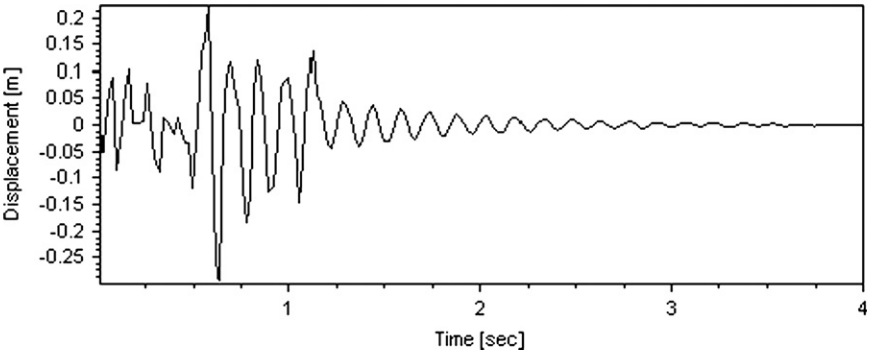

Fig. 43. Tyre deflection response to the arbitrary input using LQR.

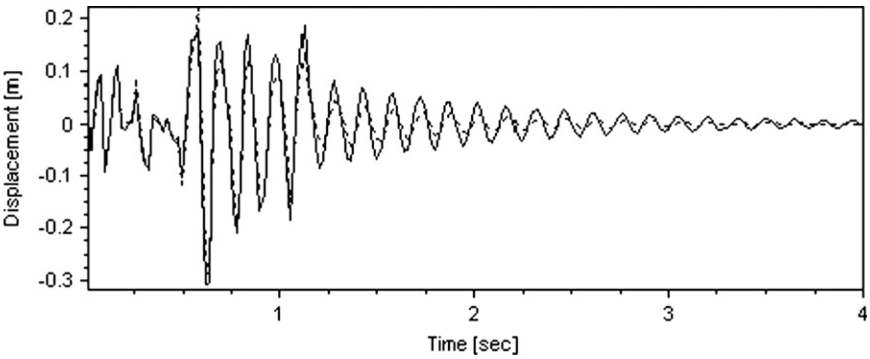

Fig. 44. Tyre deflection response to the arbitrary input. LQR (dashed line), using the proposed method (solid line).

the peak value, as shown in Fig. 30. A noteworthy achievement was the improvement in the sprung mass displacement response to the arbitrary excitation using the proposed method when compared to the response with LQR as seen in Fig. 32. As expected there was a slight increase in the sprung mass acceleration response due to the shock excitation and a marginally higher acceleration response in the case of the arbitrary excitation using the proposed method when compared to the responses using the LQR method as shown in Figs. 34 and 36. The increase in the sprung mass acceleration response was something expected due to the fact that genetic algorithm attempted primarily to establish a control strategy that could minimize the displacement response of the system as clearly stated in the objective function formula. The sprung mass acceleration response using the proposed method is found to be much more controlled than that of passive system as Fig. 20 reveals.

In order to see the effect of the proposed method on the other suspension criteria, the suspension deflection and tyre deflection responses were also checked. The controlled suspension deflection response to the arbitrary excitation obtained with the proposed method was almost identical to that of achieved with the LQR method as shown in Fig. 40. Both exhibited a significant improvement over the one generated by the passive suspension as shown in Fig. 22. Also, the tyre deflection response to the arbitrary excitation was very similar to that obtained with the LQR method as shown in Fig. 44, and was much improved in comparison to the tyre deflection recorded with the passive system as seen in Fig. 24. Therefore, it can be concluded that in comparison to the performance of the LQR method, the proposed method significantly improved the displacement response of the vehicle body with a slight increase in the acceleration response. This was foreseen and found acceptable since the objective function designed for the genetic algorithm established a control strategy to minimize the displacement response of the system. There was also no degradation in the suspension deflection or the tyre deflection for both shock and arbitrary excitations, and a superior performance for all the suspension criteria in comparison to the passive suspension was achieved.

It should be noted that the construction of the control strategy using the proposed method is not based on the system states like conventional methods such as the LQR method, which needs to access all the system states to create the required control forces. Therefore, this is another great advantage, which will lead to a considerable reduction in the number of sensors needed for measurements as the only information required is the amplitude of the disturbance input. This consequently decreases the complexity and cost of the control system.

8. Conclusion

A new approach to the control of the response of oscillatory systems subjected to arbitrary excitation is proposed in this paper. The proposed method combines the use of genetic algorithm and the concept of convolution integral to obtain a general control force strategy against arbitrary excitation. While convolution integral is usually applied to the response of the system; in the present study, it is applied to the control force strategy, which is considered as novelty. Another contribution of the proposed method is that it is suitable for real-time control of the system subjected to an arbitrary excitation. As a matter of fact, an experimental quarter-vehicle active suspension setup has been constructed in the laboratory and an advanced PLC based real-time implementation works of the proposed method have been already initiated. The initial results have been found promising since they confirm the simulation results. The real-time control application of the system involving the optimization studies of the code, the test rig and the experiments is considered as the main topic of a forthcoming research. This second major part of the work is planned as a subsequent publication, which will address the significant issues in real-time applications like computation time and memory usage. In conclusion, the proposed method, which was developed to control the response of a quarter-vehicle active suspension system subjected to an arbitrary external disturbance yielded noteworthy improvements especially on the vehicle body displacement response in comparison to the results obtained using both the LQR method and as expected a passive suspension system. Another great contribution is that the proposed control method has proven that these superior results could be achieved by means of a reduced number of sensors yielding a less complicated control system.

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