Journal of Sound and Vibration 330 (2011) 87–109

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

A quantitative criterion validating coupling power proportionality in statistical energy analysis ☆

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ARTICLE INFO

Article history:
Received 29 September 2006
Received in revised form
6 June 2010
Accepted 2 August 2010
Handling Editor: M.P. Cartmell

ABSTRACT

The response of two general spring-coupled elements is investigated to develop a unifying approach to the weak coupling criterion in Statistical Energy Analysis (SEA). First, the coupled deterministic equations of motion are expressed in the bases given by the uncoupled elements' eigenmodes. Then, an iterative solution is expressed as a succession of exchanges between elements, where uncoupled motion provides the start approximation, converging if the 'coupling eigenvalue' is less than unity, in which case coupling is said to be weak. This definition is related to whether response is 'local' or 'global', encompassing a number of previously defined coupling strength definitions, applying for deterministically described structures. A stochastic ensemble is defined by that its members are equal to the investigated structure but the elements have random frequencies. It is required that the coupling eigenvalue be less than unity for all members of the ensemble. This requirement generates the title subject of the article: 'the modal interaction strength'. It is similar to the previously defined coupling strength criterion characterising the ensemble average energy flow in uni-dimensional waveguides. Finally, SEA models are formulated in terms of the uncoupled elements' modal data.

1. Introduction

1.1. Background

Statistical Energy Analysis (SEA) is a quickly applied approximate method for prediction of stationary high frequency random vibrations in built-up structures [1–3]. It is believed that SEA works if there are many resonances in each element and these are "weakly coupled". However, there is no universally agreed coupling strength criterion assessing the assumption of weak coupling. Therefore, much effort has been put into investigation of the SEA hypothesis that energy flow between two directly coupled elements is proportional to the difference in their modal energies. In what follows,

☆ *Editor's Note*: Apart from improvements in presentation made in response to the present reviewers, the only changes made to this article relative to the version submitted in 2000 are that:

(1) sub headings are introduced to the introduction;
(2) section 1.2 is new (replacing the final four paragraphs of the original introduction;
(3) an update on relevant work since 2000 is included in section 5.3.

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doi:10.1016/j.jsv.2010.08.003

different measures of coupling strength presented in Refs. [4–12] are discussed. It is demonstrated that a unifying approach is possible.

SEA considers the average response of an ensemble of similar structures. Thus in the name SEA: "*Statistical* emphasises that the systems being studied are presumed to be drawn from populations of similar design construction having known distributions of their dynamical parameters" [1]. This ensemble point of view is implicit in the present work. Explicit response statistics, however, are not calculated; being the subject of forthcoming publications. Instead, deterministic calculations establish criteria validating the SEA power balance formulation. It is then required that these criteria be fulfilled for all members of the ensemble (naively) defined by that its members are equal to the investigated structure but the elements have stochastic resonance frequencies. Upon this basis, the coupling strength criterion is evaluated.

There are two approaches to predictive SEA: the modal and the wave approach. The latter relates SEA to numerous asymptotic results in acoustics and high frequency vibration analysis, i.e., results for large elements and for high frequencies. Most of the existing routines for generating SEA energy flow coefficients and for SEA element formulations are based on the wave approach, e.g., Refs. [1,3]. The modal approach is the original one [13,14] and is the approach chosen in the present work.

When applying it, the investigated structure divides into 'elements' and energy conservation equations are formulated with their vibration energies as variables based upon two assumptions. (I) In each element, the response is determined by its 'resonances', so vibrations are described by the subset of the element's eigenmodes having resonance frequencies within the considered frequency band. (II) The energy flow between two connected elements is proportional to the difference of their average modal energy. This is the hypothesis of Coupling Power Proportionality. In the present work, the constant of proportionality is termed "modal energy conductivity" or simply conductivity.

The first assumption, of local resonant vibration, helps define an 'element'. In many cases, it enables response calculations so that, e.g., velocity, sound pressure or stress levels can be determined from SEA results. The first assumption is related to the (perhaps not rigorously defined) concept of 'local' and 'global' modes. Local modes are those proper to an isolated element. In coupled structures, only global modes exist; being solutions to an eigenvalue problem for the entire structure. However, modes may be localised, meaning that all or most vibration energy in a mode may be found within an element. Since it is localised, not much energy is exchanged at the boundary. Hence, it is in principle possible to find conservative boundary conditions for an element such that the local modes' resonance frequencies and their mode shapes are not much different from those of the global modes. In such a situation, the assumption is approximately fulfilled and one is entitled to say, the structure's response is given by its local modes.

The second assumption, of coupling power proportionality, is based on the result derived by Scharton and Lyon who show it to be exactly true for the frequency-averaged energy flow between two conservatively coupled oscillators [14]. This result is extended, on a mode-to-mode basis, to the coupling between sets of oscillators (modes in connected structures), see Ref. [1, Section 3.2]. The second assumption is fundamental to the formulation of SEA equations, being also the basis for the evaluation of their parameters. Moreover, coupling power proportionality simplifies these evaluations, since it implies that the energy flow between two directly coupled elements could be considered, while neglecting the rest of the structure.

Woodhouse demonstrates that for three coupled oscillators, unless they are 'weakly coupled', the hypothesis of coupling power proportionality is not valid [15]. Similarly, modal response cannot conceivably be local unless coupling is weak. Consequently, the assumptions of local resonant modes and coupling power proportionality are valid only if the elements are 'weakly coupled'.

Despite the significance of the weak coupling assumption, there is no universally agreed measure for coupling strength proving the assumption. However, different observations can be found in the literature. Some of these are:

(1) *Measures of dynamic coupling strength*: For e.g., the stiffness or mass impedance of a connecting structure in relation to characteristic impedances in the connected elements, e.g., Refs. [4,15]; the number of straps connecting two plates [12]; etc. Dynamic coupling strength is proportional, or at least related, to SEA conductivity but its relation to the weak coupling assumption is not always evident.

(2) *Vibrational behaviour*: Coupling is weak if: (a) the modal behaviour of the uncoupled substructure is not greatly affected when the subsystems are coupled [6]; (b) the Green's function of a directly excited substructure is approximately equal to that of the uncoupled substructure [7]; (c) the non dimensional rise time C_s of an indirectly impulse excited element is not much less than unity [12]; (d) the maximum coupling eigenvalue is less than unity [11]. Here, the uncoupled response provides the start approximation for an iterative solution of the coupled equations of motion. The coupling appears as a succession of exchanges between the elements. This process converges when exchanges are decreasing with increasing order in which case coupling is said to be weak [11].

(3) *The Smith criterion*: Coupling is weak if the ratio of coupling loss factor to dissipation loss factor is small for both connected elements [5]. This coupling strength measure characterises the solutions to SEA equations [5]. Whether it is also useful for assessing the validity of these equations is debatable [9,10]. (3b) Coupling is weak if the ratio of modal energies in an indirectly and a directly excited substructure is less than or of the order of ε^n where ε is a small number and n is the number of connections separating the elements [16,17]. This order of 'smallness', however, does not guarantee that indirect couplings are negligible [10,16].

(4) *The Gamma criterion*: Coupling is weak if gamma is smaller than unity, where the non-dimensional parameter gamma (but for a constant factor) is the SEA conductivity divided by the product of the modal overlaps in two connected elements [8–10]. (The modal overlap is the ratio of resonance 3-dB bandwidth to the average separation between resonances.) Studies of uni-axial waveguides have shown that if coupling is weak in this manner, travelling wave estimates of SEA conductivities are accurate for predicting ensemble averages of energies. Finnveden shows that only for small 'gamma' coupling power proportionality holds in a three-element structure [10]. Mace demonstrates that for small 'gamma', ensemble averaged coupling power is determined by those members of the ensemble having equal uncoupled frequencies, while for strong coupling, many of the members of the ensemble contribute to the average energy exchange at the 'global' resonances of the system [9,18]. Thus, gamma signifies from an ensemble point of view whether response in one-dimensional systems is local or global.

1.2. Article content

The analyses in Refs. [8–10,18] are based on the wave approach to SEA. In the following, SEA coupling strength is studied using the modal approach. Section 2 starts with a review of the standard SEA of spring-coupled structures, which ends by a fundamental and original result: the "standard SEA" formulation for two spring-coupled oscillators. In Section 2, the derivation is formulated as a mere consequence of standard SEA procedures, while it is demonstrated later in Section 4.

Section 3 considers the energy balance for deterministic spring-coupled structures based on Langley's general derivation of the SEA equations [7]. As in Refs. [4,16], the formulation is simplified by employing a series expansion; the original contribution being a quantitative criterion for convergence that is expressed in the system parameters. This criterion was in another context derived by Bessac, Gagliardini and Guyader [11], and the measure used is the coupling eigenvalue. Upon this basis, it is deduced that the qualitative weak coupling criteria 2a and 2b can be quantified by the criterion 2d.

Section 4 considers ensembles of spring-coupled structures. The ensemble members are identical to the nominal structure except for having random natural frequencies. First two coupled oscillators are considered. The coupling power and the input power are formulated on a non-dimensional form consisting of three parameters only: the first two describes the difference between an uncoupled oscillator's natural frequency and the frequency of excitation, while the third parameter describes the coupling; it is termed the modal interaction strength. The new non-dimensional form allows an original discussion of the energy flow characteristics in ensembles of two-oscillator structures. It turns out the modal interaction strength is the largest coupling eigenvalue for any member of the ensemble and any frequency.

Sets of coupled modes, having random, uniformly probable, natural frequencies, are then analysed and resonant response within a frequency band is considered. In this case, the modal interaction strength is of the order of the largest coupling eigenvalue for any member of the ensemble and any frequency. Upon this basis, the standard SEA formulation for spring coupled structures is validated. The standard modal approach in SEA is based upon the assumption that the energy flow between sets of modes is given on a mode-to-mode basis [1, Fig. 3.4]. It is demonstrated originally here that this assumption is valid if the modal interaction is small.

In short: for spring coupled structures, it is originally demonstrated that, the article's title subject, the modal interaction strength is an ensemble property; can be expressed in standard SEA parameters; validates the weak coupling criteria 2a, 2b, 2d and 3 for all members of the ensemble; and is similar to the criterion 4, which was derived for one dimensional systems that support one kind of wave motion. For small modal interaction strengths, the standard SEA approach is valid and, most spectacularly, so is also the "standard SEA" formulation for two spring-coupled oscillators, which is presented in Section 2 and demonstrated in Section 4.

2. Statistical energy analysis of spring coupled structures

As background for the analysis of coupling strength, which follows in Sections 3 and 4, the SEA of two spring-coupled elements is reviewed.

2.1. Standard SEA of two spring coupled elements

In SEA, the variables of interest are the modal energies times analysis bandwidth, also referred to as modal power potentials [1]

$$E_{m\cdot i} = E_i/n_i, \quad n_i = N_i/\Omega, \tag{2.1}$$

where E_i is the total vibration energy in element i within the frequency band Ω, N_i is the number of resonances within the frequency band and n_i is the modal density. The coupling power hypothesis states that energy flow between two elements is proportional to the difference in their modal energy:

$$P^{1,2}_{\text{coup}} = C(E_{m\cdot 1} - E_{m\cdot 2}), \tag{2.2}$$

where the coefficient C is defined by this equation and $P_{\mathrm{coup}}^{1,2}$ is the coupling power. It is proposed that the non-dimensional parameter C be termed the 'modal energy conductivity', or, in any given context, simply the 'conductivity'.

The governing equation in SEA of energy conservation is

$$\begin{bmatrix} M_1+C & -C \\ -C & M_2+C \end{bmatrix} \begin{bmatrix} E_{m\cdot 1} \\ E_{m\cdot 2} \end{bmatrix} = \begin{bmatrix} P_{\mathrm{in}\cdot 1} \\ P_{\mathrm{in}\cdot 2} \end{bmatrix}, \tag{2.3}$$

where P_{in} is the input power and M is the modal overlap based on the 3-dB bandwidth, which is given by

$$M = \eta\omega n, \tag{2.4}$$

where η is the loss factor. It is assumed in the formulation of Eq. (2.3) that the power dissipated in an element is given by

$$P_{\mathrm{diss}}^{i} = \eta_i \omega E_i = M_i E_{m\cdot i}. \tag{2.5}$$

2.1.1. Input power

The frequency averaged input power from a white force excitation of bandwidth Ω and amplitude f to a weakly damped single-degree-of-freedom oscillator (one mode in a substructure) is approximately given by (see Appendix A and B).

$$P_{\mathrm{in}} = \frac{1}{\Omega}\int_{\Omega} |f|^2 \mathrm{Re}(Y)\,\mathrm{d}\omega \approx \begin{cases} |f|^2 \dfrac{\pi}{2m\Omega}, & \omega_1 \in \Omega, \\ 0, & \omega_1 \notin \Omega, \end{cases} \tag{2.6}$$

where

$$\omega_1 = \sqrt{k/m}, \quad Y = -\mathrm{i}\omega/(k - \mathrm{i}\omega c - \omega^2 m), \tag{2.7}$$

where a time dependence $\mathrm{e}^{-\mathrm{i}\,\omega\, t}$ is implicitly assumed and where f is the rms complex amplitude of the force, Y is the input mobility, k is the spring stiffness, c is the viscous damping coefficient and m is the mass of the oscillator. Eq. (2.6) is asymptotically exact in the limit of zero damping. It is very accurate for small damping in which case it is also valid for mass- or stiffness-proportional damping, i.e., when the input mobility is $Y = -\mathrm{i}\omega/(k(1-\mathrm{i}\eta_e) - \mathrm{i}\omega c - \omega^2 m(1+\mathrm{i}\eta_m))$, where η_e and η_m are loss factors.

Neglecting the influence of connected elements, the one-oscillator result is generalised to apply for an element, described by its local modes. Thus, the input power is given by

$$P_{\mathrm{in}} = \langle f^2 \rangle Y_c, \tag{2.8}$$

$$Y_c = \frac{\pi}{2}\frac{N}{m\Omega}, \tag{2.9}$$

where $\langle f^2 \rangle$ is the mean square force on the element and m is the mass. Y_c is for large elements asymptotically equal to the characteristic mobility, the mobility for an infinite element. Eq. (2.8) is derived for rain-on-the-roof excitation or as the expected value of the input power for a point force at a random location within the element. (Derivations are found, e.g., in Ref. [20, Section IV.4] and in Ref. [21].)

Eq. (2.9) applies for frequency band averages. Skudrzyk [22] reveals that for reverberant vibrations at higher frequencies (above the first few resonances) the space averaged point mobility for a particular frequency is limited by its value at resonance

$$Y \le Y_c/(\pi M/2). \tag{2.10}$$

If the modal overlap is greater than unity, this expression is not valid, whereas in this case the average mobility is to a good approximation equal to the characteristic mobility. It should be noted, the limit in Eq. (2.10) is not exact but depends on the position in the element, however, for uniform elements (beams, plates, air-volumes, etc.) it is accurate on average.

2.1.2. SEA conductivity

For two elements, coupled at a point via a spring, the conductivity is according to Manning [21, Eq. (23a)]

$$C = \frac{2}{\pi}\frac{\langle \mathrm{Re}(Y_1)\rangle \langle \mathrm{Re}(Y_2)\rangle}{|\langle Y_1\rangle + \langle -\mathrm{i}\omega/k_c + Y_2\rangle|^2} = \frac{2}{\pi}\frac{Y_{c1}Y_{c2}}{|\langle Y_1\rangle + \langle -\mathrm{i}\omega/k_c + Y_2\rangle|^2}, \tag{2.11}$$

where k_c is the coupling spring stiffness, Y_i is the point mobility in element i at the spring connection and $\langle\ \rangle$ denotes that appropriate averaging (frequency, space, ensemble, etc.) should be performed. It can be argued whether in Eq. (2.11) it is correct to average each term individually.

Exact ensemble averages of vibration energies in two rods, end-coupled via a spring of stiffness k_c, are calculated in Refs. [8–10]. These ensemble averages are described by the SEA Eq. (2.3) where in this case the conductivity is given by [10]

$$C = \frac{C_\infty}{Q - C_\infty/M_1 - C_\infty/M_2}; \tag{2.12}$$

$$Q=\left[1+\frac{2\pi C_\infty}{\tanh(\pi M_1)\tanh(\pi M_2)}+\left(\frac{\pi C_\infty}{\tanh(\pi M_1)}\right)^2+\left(\frac{\pi C_\infty}{\tanh(\pi M_2)}\right)^2-(\pi C_\infty)^2\right]^{1/2} \tag{2.13}$$

where the 'travelling wave' estimate of the conductivity, for arbitrary dynamic coupling strength, is given by

$$C_\infty=\frac{2}{\pi}\frac{Y_{c1}Y_{c2}}{Y_{c1}^2+Y_{c2}^2+(\omega/k_c)^2}. \tag{2.14}$$

In Refs. [8–10] it is found that if the factor γ is small,

$$\gamma=2C_\infty/(\pi M_1M_2), \tag{2.15}$$

that is if

$$\gamma<1, \tag{2.16}$$

then the conductivity C of Eq. (2.12) is to a fair approximation equal to the travelling wave estimate (2.14). If γ is larger than unity, the conductivity decreases and, if the modal overlaps M_1 and M_2 are of the same order of magnitude, this decrease is directly proportional to the square root of their product. (This conclusion has been confirmed for plate structures by Yap and Woodhouse [23] and Mace and Rosenberg [24].)

If γ is smaller than unity and if the dynamic coupling strength is weak, i.e., if the mobility of the spring is much higher than the characteristic mobilities of the connected elements, the conductivity in Eq. (2.12) simplifies to

$$C=\frac{2k_c^2}{\pi\omega^2}Y_{c1}Y_{c2}. \tag{2.17}$$

The same result is found from Eq. (2.11), if it is assumed that the dynamic coupling strength is weak in the sense that the mobility of the spring is much higher than the mobilities of the connected elements, even at resonance. From Eq. (2.10), for reverberant elements this requirement is fulfilled provided that

$$k_c/\omega>>Y_{c1}/(\pi M_1/2) \quad \text{and} \quad k_c/\omega>>Y_{c2}/(\pi M_2/2). \tag{2.18}$$

2.1.3. Many randomly positioned springs

For general spring coupling there is no simple expression for the conductivity since it depends on the precise spring positions and of the elements' mode shapes. Simplifications are provided, however, if the springs are randomly positioned or equally if the vibration fields in the elements are random and the springs are separated by distances that are of the order of the wavelength or longer. In either of these cases, the coupling powers through the springs are uncorrelated and the total conductivity is given by the sum of the conductivities for the individual springs. Thus, for weak coupling, in analogy with Eq. (2.17), the conductivity is given by

$$C=\sum_p\frac{2k_{c\cdot p}^2}{\pi\omega^2}Y_{c1}Y_{c2}, \tag{2.19}$$

where $k_{c\cdot p}$ is the strength of spring p.

2.2. Weakly coupled oscillators

The SEA formulation above is derived using a mobility approach [21] and a wave approach [9,10]. The expressions are equally expressed for sets of spring-coupled oscillators (modes in connected substructures). Thus, if the sets are 'weakly coupled', the SEA equations are as in Eq. (2.3) with

$$P_{\text{in}}=\langle f^2\rangle Y_{ci};\quad Y_{ci}=\frac{\pi}{2}\frac{N_i}{m_i\Omega};\quad M_i=\eta_i\omega\frac{N_i}{\Omega}; \tag{2.20}$$

$$C=\frac{2}{\pi}\frac{k_c^2}{\omega^2}Y_{c1}Y_{c2}=\frac{\pi}{2}\frac{\chi^2}{\omega^2}\frac{N_1N_2}{\Omega^2}, \tag{2.21}$$

where

$$\chi^2=k_c^2/m_1m_2. \tag{2.22}$$

From this it follows that for coupled oscillators γ, Eq. (2.15), is given by

$$\gamma=\frac{2C_\infty}{\pi M_1M_2}=\frac{\chi^2}{\eta_1\eta_2\omega^4}. \tag{2.23}$$

In the following sections the interaction of spring coupled structures are investigated to gain understanding about when Eqs. (2.3), (2.8) and (2.17) are valid. It is demonstrated that γ is the parameter characterising modal interaction and coupling power from an ensemble point of view. It is proposed that γ be termed 'modal interaction strength'.

2.2.1. Two weakly coupled oscillators

The equations above are applied for two weakly spring-coupled oscillators having their resonance frequencies within the frequency band Ω. In this case $N_1=N_2=1$. The expression in Eq. (2.23) for the coupling strength measure γ is independent of the mode count and hence it applies to the two-oscillator problem. Notably, γ is independent of bandwidth, that is, of the probability that there is resonant coupling.

From this it is concluded that if: (A) two oscillators are weakly coupled according to the criterion $\gamma \ll 1$, (B) the dynamic coupling strength is weak, and (C) their uncoupled resonance frequencies are uniformly probable within the frequency band Ω, then the energy balance between them is written as in Eq. (2.3), with

$$M_i = \eta_i \omega/\Omega, \quad C = \frac{2}{\pi}\frac{k_c^2}{\omega^2} Y_{c1} Y_{c2} = \frac{\pi}{2}\frac{\gamma^2}{\omega^2 \Omega^2}. \tag{2.24}$$

The generic problem of two statistically defined, weakly coupled, oscillators is the foundation for much of our understanding of the modal approach to SEA. Curiously enough, to the best of the author's knowledge, the standard SEA formulation for this problem appeared for the first time in Ref. [19]. Albeit, Eq. (2.24) follows directly upon the discussion in Eq. [1, Section 3.2]. Also, the formulation follows from Newland's analysis in Ref. [4].

3. Deterministic analysis of spring coupled structures

In the previous section SEA models for structures having motion described by their uncoupled resonant modes were formulated upon the assumption of 'weak coupling'. In this section, deterministic models of two spring-coupled elements are investigated to further our understanding of the implications of this assumption and the criteria for its validity. Also, the evaluation of SEA parameters from modal data is demonstrated.

3.1. Equations of motion

The element motions are described by their uncoupled modes $\{\psi_{n.i}\}$. These orthogonal modes are scaled so that

$$\int_{V_i} \psi_{n\cdot i}(\mathbf{x})\psi_{m\cdot i}(\mathbf{x})\mathrm{d}\mathbf{x} = V_i \delta_{nm}, \tag{3.1}$$

where V_i is the volume of element i. Harmonic motion of the form $\mathrm{e}^{-\mathrm{i}\omega t}$ is assumed.

The displacement $w_{i\cdot b}$ of the uncoupled element i to a harmonic point force excitation with complex amplitude f_i is given by Green's function $G_{ii\cdot b}$

$$w_{i\cdot b}(\mathbf{x}) = \int_{V_i} G_{ii\cdot b}(\mathbf{x}|\mathbf{y}) f_i(\mathbf{y})\,\mathrm{d}\mathbf{y}; \tag{3.2}$$

$$G_{ii\cdot b}(\mathbf{x}|\mathbf{y}) = \sum_n \psi_{n\cdot i}(\mathbf{x})\psi_{n\cdot i}(\mathbf{y})/\alpha_{n\cdot i}, \tag{3.3}$$

$$\alpha_{n\cdot i} = m_i(\omega_{n\cdot i}^2 - \omega^2(1+\mathrm{i}\eta_i)), \tag{3.4}$$

where m_i is the mass, $\omega_{n\cdot i}$ is the eigenfrequency associated with the eigenfunction $\psi_{n\cdot i}$ and η_i is the loss factor; for viscous damping $\eta = c/\omega$, where c is the specific viscous damping constant. $1/\alpha_{n\cdot i}$ is the modal receptance.

The elements are spring coupled and the uncoupled modes for one element are defined by the condition that the motion of the other element is blocked (as in Ref. [14]). The additional coupling force for coupled motion is then given solely by the other elements motion at the coupling point. Thus, if the elements are connected by one spring, the displacements are given by

$$\begin{aligned} w_1(\mathbf{x}) &= w_{1\cdot b}(\mathbf{x}) + k_c G_{11\cdot b}(\mathbf{x}|\mathbf{x_{1c}}) w_2(\mathbf{x_{2c}}), \\ w_2(\mathbf{x}) &= w_{2\cdot b}(\mathbf{x}) + k_c G_{22\cdot b}(\mathbf{x}|\mathbf{x_{2c}}) w_1(\mathbf{x_{1c}}), \end{aligned} \tag{3.5}$$

where w_i is the displacement in element i, k_c is the strength of the spring and $\mathbf{x_{ic}}$ is the coupling point coordinate for element $\mathbf{i}$.

The uncoupled modes form complete bases and are used to express the solutions to the coupled equations of motion. Thus, it is assumed

$$w_i(\mathbf{x}) = \sum_n \mathbf{w_i}[n]\psi_{n\cdot i}(\mathbf{x}); \quad w_{i\cdot b}(\mathbf{x}) = \sum_n \mathbf{w_{i\cdot b}}[n]\psi_{n\cdot i}(\mathbf{x}), \tag{3.6}$$

where the vectors $\mathbf{w_i}$ and $\mathbf{w_{i\cdot b}}$ contain the modal amplitudes of elements $\mathbf{i}$ for coupled and uncoupled motion. Using the orthogonality relation (3.1), Eq. (3.5) are conveniently expressed in matrix form as

$$[\mathbf{I}-\mathbf{K}]\begin{bmatrix}\mathbf{w_1}\\ \mathbf{w_2}\end{bmatrix} = \begin{bmatrix}\mathbf{w_{1b}}\\ \mathbf{w_{2b}}\end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix}\mathbf{0} & \mathbf{K_{12}}\\ \mathbf{K_{21}} & \mathbf{0}\end{bmatrix}, \tag{3.7}$$

where $\mathbf{I}$ is an identity matrix and the sub matrices $\mathbf{K_{12}}$ and $\mathbf{K_{21}}$ of the non-dimensional coupling matrix $\mathbf{K}$ are given by

$$\mathbf{K_{12}}[n,m] = k_c \psi_{n\cdot 1}(\mathbf{x_{1c}})\psi_{m\cdot 2}(\mathbf{x_{2c}})/\alpha_{n\cdot 1},$$
$$\mathbf{K_{21}}[n,m] = k_c \psi_{n\cdot 2}(\mathbf{x_{2c}})\psi_{m\cdot 1}(\mathbf{x_{1c}})/\alpha_{n\cdot 2}. \tag{3.8}$$

Eq. (3.7) is the basis for the analysis of the motion of coupled sets of modes. It is here derived for spring-coupled sub structures only but equations on the same form apply for the gyroscopic and mass coupling conditions that were considered by Lyon and Maidanik [13] and by Scharton and Lyon [14]. Similar equations apply for fluid–structure coupling and also for quite general coupling of structural systems, if the modal sub-structuring technique developed by Maxit and Guyader is used [28,29].

3.1.1. Iterative solution for weak coupling

When convergence is verified, Eq. (3.7) can be solved by a series expansion of $[\mathbf{I}-\mathbf{K}]^{-1}$. The equation then becomes [11]

$$\begin{bmatrix}\mathbf{w_1}\\ \mathbf{w_2}\end{bmatrix} = [\mathbf{I}+\mathbf{K}+\mathbf{K^2}+\cdots]\begin{bmatrix}\mathbf{w_{1b}}\\ \mathbf{w_{2b}}\end{bmatrix}. \tag{3.9}$$

This also can be written as

$$\mathbf{w_1} = [\mathbf{I}+\mathbf{K_{12}K_{21}}+(\mathbf{K_{12}K_{21}})^2+\cdots][\mathbf{w_{1b}}+\mathbf{K_{12}w_{2b}}];$$
$$\mathbf{w_2} = [\mathbf{I}+\mathbf{K_{21}K_{12}}+(\mathbf{K_{21}K_{12}})^2+\cdots][\mathbf{w_{2b}}+\mathbf{K_{21}w_{1b}}]. \tag{3.10}$$

The rate of convergence can be estimated by considering the nth iterate of $\mathbf{w_1}$ which is denoted $\mathbf{w_1}^{(n)}$. The norm of the difference of the computed value of $\mathbf{w_1}$ in the nth iteration is

$$|\mathbf{w_1}^{(n)}-\mathbf{w_1}^{(n-1)}| = |(\mathbf{K_{12}K_{21}})^n\mathbf{w_{1b}}+(\mathbf{K_{12}K_{21}})^n\mathbf{K_{12}w_{2b}}| \le |(\mathbf{K_{12}K_{21}})^n|\,|\mathbf{w_{1b}}+\mathbf{K_{12}w_{2b}}| = |\lambda_{\max}^2|^n|\mathbf{w_1}^{(0)}| \tag{3.11}$$

where $\lambda_{\max}^2$ is the largest eigenvalue of $\mathbf{K_{12}K_{21}}$. (The maximum norm is used, see, e.g., Ref. [26, Section 5.5.2]). $\lambda_{\max}$ is denoted the 'coupling eigenvalue'.

Eq. (3.11) demonstrates that the rate of convergence of Eq. (3.9) is determined by the coupling eigenvalue and that the iteration converges if $|\lambda_{\max}|<1$.

The series expansion (3.9) is similar to the one employed by Newland [4,25]. The contribution by Bessac et al. is the criterion for convergence [11]. This criterion is determined by the coupling eigenvalue, which is calculated in Section 3.3.

3.2. Coupling via many springs

If elements are coupled by many springs, their displacements are given by

$$w_1(\mathbf{x}) = w_{1\cdot b}(\mathbf{x})+\sum_p k_{c\cdot p}G_{11\cdot b}(\mathbf{x}|\mathbf{x_{1c\cdot p}})w_2(\mathbf{x_{2c\cdot p}})$$
$$w_2(\mathbf{x}) = w_{2\cdot b}(\mathbf{x})+\sum_p k_{c\cdot p}G_{22\cdot b}(\mathbf{x}|\mathbf{x_{2c\cdot p}})w_1(\mathbf{x_{1c\cdot p}}) \tag{3.12}$$

where $k_{c\cdot p}$ is the strength of spring p, connected to the two elements at $\mathbf{x}_{1c\cdot\mathbf{p}}$ and $\mathbf{x}_{2c\cdot\mathbf{p}}$, respectively. Similarly, the coupling matrix $\mathbf{K}$ in Eq. (3.7) is given by

$$\mathbf{K_{12}}[n,m] = \sum_p k_{c\cdot p}\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{m\cdot 2}(\mathbf{x_{2c\cdot p}})/\alpha_{n\cdot 1};$$
$$\mathbf{K_{21}}[n,m] = \sum_p k_{c\cdot p}\psi_{n\cdot 2}(\mathbf{x_{2c\cdot p}})\psi_{m\cdot 1}(\mathbf{x_{1c\cdot p}})/\alpha_{n\cdot 2}. \tag{3.13}$$

Besides the substitutions of Eq. (3.12) for Eq. (3.5) and Eq. (3.13) for Eq. (3.8), the analysis in Section 3.1 applies equally when the elements are coupled by many springs.

3.3. Properties of the coupling matrix

3.3.1. Coupling by one spring

Work by Bessac et al. [11], is repeated here for completeness.

First, they show that if λ is an eigenvalue to $\mathbf{K}$ then λ^2 is an eigenvalue of $\mathbf{K_{12}K_{21}}$ or equally of $\mathbf{K_{21}K_{12}}$ [11, Eqs. (10a–d)]. The elements of these matrices are given by

$$\mathbf{K_{12}K_{21}}[n,m] = -\frac{k_c^2}{\omega^2}\frac{-i\omega\psi_{n\cdot 1}(\mathbf{x_{1c}})\psi_{m\cdot 1}(\mathbf{x_{1c}})}{\alpha_{n\cdot 1}}Y_2(\mathbf{x_{2c}});$$
$$\mathbf{K_{21}K_{12}}[n,m] = -\frac{k_c^2}{\omega^2}\frac{-i\omega\psi_{n\cdot 2}(\mathbf{x_{2c}})\psi_{m\cdot 2}(\mathbf{x_{2c}})}{\alpha_{n\cdot 2}}Y_1(\mathbf{x_{1c}}), \tag{3.14}$$

where the input mobilities at the coupling points $Y_1(\mathbf{x_{1c}})$ and $Y_2(\mathbf{x_{2c}})$ are given by

$$Y_1(\mathbf{x_{1c}}) = \sum_r \frac{-i\omega(\psi_{r\cdot 1}(\mathbf{x_{1c}}))^2}{\alpha_{r\cdot 1}};$$
$$Y_2(\mathbf{x_{2c}}) = \sum_r \frac{-i\omega(\psi_{r\cdot 2}(\mathbf{x_{2c}}))^2}{\alpha_{r\cdot 2}}. \tag{3.15}$$

Please note, the uncoupled system is here defined for the blocked condition, where the coupling spring is included in the element but the other element is kept motion-less. The input mobilities $Y_1(\mathbf{x_{1c}})$ and $Y_2(\mathbf{x_{2c}})$ are calculated for this condition.

It is seen in Eq. (3.14) that the matrix $\mathbf{K_{12}K_{21}}$, apart from a constant factor, is equal to the outer product of two vectors with entries $\{\psi_{n\cdot 1}(\mathbf{x_{1c}})/\alpha_{n\cdot 1}\}$ and $\{\psi_{m\cdot 1}(\mathbf{x_{1c}})\}$, respectively. Thus, the matrix has unity rank and therefore only one non-zero eigenvalue. The trace of a matrix is equal to the sum of its eigenvalues, so, as the matrix has only one non-zero eigenvalue $\lambda^2_{\max}$, it is found that

$$\lambda^2_{\max} = \sum_n \mathbf{K_{12}K_{21}}[n,n] = \sum_n -\frac{k_c^2}{\omega^2}\frac{-i\omega\psi_{n\cdot 1}(\mathbf{x_{1c}})\psi_{n\cdot 1}(\mathbf{x_{1c}})}{\alpha_{n\cdot 1}} Y_2(\mathbf{x_{2c}}) = -\frac{k_c^2}{\omega^2} Y_1(\mathbf{x_{1c}}) Y_2(\mathbf{x_{2c}}). \tag{3.16}$$

The same result applies to $\mathbf{K_{12}K_{21}}$, so Eq. (3.11) estimates equally the rate of convergence in both connected elements.

3.3.2. Coupling by many springs

The proof in Ref. [11] that λ^2 is an eigenvalue of $\mathbf{K_{12}K_{21}}$ (and equally of $\mathbf{K_{21}K_{12}}$) if λ is an eigenvalue of $\mathbf{K}$ is valid also for coupling by many springs. The entries of $\mathbf{K_{12}K_{21}}$ are in this case given by

$$\mathbf{K_{12}K_{21}}[n,m] = \sum_r \sum_p \left[k_{c\cdot p}\frac{\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{r\cdot 2}(\mathbf{x_{2c\cdot p}})}{\alpha_{n\cdot 1}}\right] \sum_q \left[k_{c\cdot q}\frac{\psi_{r\cdot 2}(\mathbf{x_{2c\cdot q}})\psi_{m\cdot 1}(\mathbf{x_{1c\cdot q}})}{\alpha_{r\cdot 2}}\right]. \tag{3.17}$$

The magnitude of the largest eigenvalue of this matrix is limited by the sum of the magnitudes of its diagonal elements:

$$\lambda^2_{\max} \le \sum_n \left|\mathbf{K_{12}K_{21}}[n,n]\right| = \sum_n \left|\sum_r \sum_p \sum_q \left[k_{c\cdot p}\frac{\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{r\cdot 2}(\mathbf{x_{2c\cdot p}})}{\alpha_{n\cdot 1}}\right]\left[k_{c\cdot q}\frac{\psi_{r\cdot 2}(\mathbf{x_{2c\cdot q}})\psi_{n\cdot 1}(\mathbf{x_{1c\cdot q}})}{\alpha_{r\cdot 2}}\right]\right| \tag{3.18}$$

Generally, this expression must be evaluated numerically.

Simplification, however, is provided for random spring positions. Thus, for random location of the spring connections in the second element, the expected values of the entries of $\mathbf{K_{12}K_{21}}$ are given by the terms in the summations over p and q for which $p=q$, since the other terms do on average sum to zero. Thus, the expected values of the entries of $\mathbf{K_{12}K_{21}}$ are for random spring positions given by

$$\mathbf{K_{12}K_{21}}[n,m] = \sum_p -\frac{k_{c\cdot p}^2}{\omega^2}\frac{-i\omega\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{m\cdot 1}(\mathbf{x_{1c\cdot p}})}{\alpha_{n\cdot 1}} Y_2(\mathbf{x_{2c\cdot p}}), \tag{3.19}$$

where the input mobility Y_2 is defined in Eq. (3.15). Using the triangle inequality, an upper bound to the norm of $\mathbf{K_{12}K_{21}}$ is given by

$$|\mathbf{K_{12}K_{21}}| \le \sum_p |(\mathbf{K_{12}K_{21}})_p|, \tag{3.20}$$

where the entries of the matrix $(\mathbf{K_{12}K_{21}})_p$ equals those of $\mathbf{K_{12}K_{21}}$ for the case when only coupling spring p is present. Upon this basis, the results in Section 3.3.1 are used to find an estimate of the maximum eigenvalue of $\mathbf{K_{12}K_{21}}$

$$|\lambda^2_{\max}| = \sum_p \frac{k_{c\cdot p}^2}{\omega^2}|Y_1(\mathbf{x_{1c\cdot p}})Y_2(\mathbf{x_{2c\cdot p}})|. \tag{3.21}$$

An alternative estimate is provided in Section 4.3, where the maximum coupling eigenvalue across an ensemble is considered.

3.4. Criterion for coupling strength

Because of the form of $\mathbf{K}$ in Eq. (3.7), the coupling appears as a succession of exchanges between the subsystems. This process converges when exchanges are decreasing with decreasing orders. This condition is by Bessac et al. interpreted as weak coupling [11].

Keane and Price [6] state that coupling is weak if the modal behaviour of the uncoupled substructure is not greatly affected when the subsystems are coupled. The modal amplitudes give one description of 'modal behaviour'. Evidently, the Keane and Price criterion for weak coupling (defined in this manner) is fulfilled when convergence of Eq. (3.9) is rapid, since then the modal amplitudes are not much different from the uncoupled ones.

Langley defines weak coupling by the criterion that Green's function of a directly excited substructure is approximately equal to that of the uncoupled substructure [7]. Green's function of the coupled structure is determined by the solution to

Eq. (3.5) for point excitation. Therefore, Langley's weak coupling criterion is fulfilled when convergence of Eq. (3.9) is rapid, as further discussed in Section 3.5.

In conclusion, the qualitative criteria for coupling strength set up by Keane and Price [6] and Langley [7] are equivalent to the quantitative criterion set up by Bessac et al. [11]. Coupling strength, defined in this manner, is measured by the magnitude of the largest eigenvalue of the non-dimensional coupling matrix **K**. For coupling by one spring, this eigenvalue is given by Eq. (3.16). For coupling by many springs that have random positions within large elements, it is estimated by Eq. (3.21) and for general spring-coupled elements, it is evaluated numerically from Eq. (3.18).

3.5. Power balance

Langley derives equations, similar to the SEA equations, for linearly vibrating systems with conservatively coupled elements having uniform density, which are randomly, rain-on-the-roof, excited [7]. This formulation is in Ref. [27] extended to apply also for a class of non-uniform elements. The equation for power balance is (see Appendix B)

$$\mathbf{C}\hat{\mathbf{E}} = \mathbf{P}, \tag{3.22}$$

where the entries to the vector **P** are the input powers to the elements, averaged in the considered frequency band Ω, and where

$$\mathbf{C} = (1/\pi)\mathbf{q}\overline{\mathbf{M}}^{-1}\mathbf{q}, \tag{3.23}$$

$$\hat{\mathbf{E}} = \pi\mathbf{r}^{-1}\mathbf{q}^{-1}\mathbf{T}. \tag{3.24}$$

The entries to the vector **T** are the frequency averaged kinetic energies in the elements. In Ref. [27] it is proposed that the variables $\hat{\mathbf{E}}$ be denoted 'vibration potentials'. The entries of the diagonal matrix **r** are the densities, or effective densities, whereas those of the symmetric matrix $\overline{\mathbf{M}}$ and the diagonal matrix **q** are given by

$$\overline{M}_{ij} = \frac{1}{2\Omega}\int_{\Omega}\int_{V_i}\int_{V_j}\omega^2\left|G_{ij}(\mathbf{x},\mathbf{y},\omega)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega, \tag{3.25}$$

$$q_i = \frac{1}{\Omega}\int_{\Omega}\int_{V_i}\mathrm{Re}(-\mathrm{i}\omega G_{ii}(\mathbf{x},\mathbf{x},\omega))\,\mathrm{d}\mathbf{x}\,\mathrm{d}\omega. \tag{3.26}$$

As in Ref. [27], the formulation above differs from Langley's in being for frequency band averages. Consequently, the matrix **C** is non-dimensional and the matrices **q** and $\overline{\mathbf{M}}$ are frequency averages. For high frequencies and large substructures, these may reach asymptotic values that are quite independent of analysis bandwidth.

3.5.1. Green's function for weakly coupled elements

Green's functions for the coupled and uncoupled system are expressed in the uncoupled modes

$$\begin{aligned} G_{ij}(\mathbf{x}|\mathbf{y}) &= \sum_n \mathbf{g_{ij}}[n]\psi_{n\cdot i}(\mathbf{x}),\\ G_{ii\cdot b}(\mathbf{x}|\mathbf{y}) &= \sum_n \mathbf{g_{ii\cdot b}}[n]\psi_{n\cdot i}(\mathbf{x}), \end{aligned} \tag{3.27}$$

where from Eq. (3.3) the entries of the vector $\mathbf{g_{ii\cdot b}}$ are given by

$$\mathbf{g_{ii\cdot b}}[n] = \psi_{n\cdot i}(\mathbf{y})/\alpha_{n\cdot i}. \tag{3.28}$$

Green's function is the solution to the equations of motion for point excitation, so for weak coupling it is found from Eq. (3.10). Neglecting terms of the order $\lambda^2_{\max}$ compared to unity, Green's function for the coupled structure is thus given by Eq. (3.27) and by

$$\begin{aligned} &\mathbf{g_{11}} = \mathbf{g_{11\cdot b}}, \quad \mathbf{g_{12}} = \mathbf{K_{12}}\mathbf{g_{22\cdot b}},\\ &\mathbf{g_{21}} = \mathbf{K_{21}}\mathbf{g_{11\cdot b}}, \quad \mathbf{g_{22}} = \mathbf{g_{22\cdot b}}. \end{aligned} \tag{3.29}$$

3.5.2. Coupling power proportionality

If coupling is conservative and dissipative power is proportional to kinetic energy, the power balance Eq. (3.22) can equally for a two element structure be written as [7]

$$\begin{bmatrix} 2\eta_1\omega & 0 \\ 0 & 2\eta_2\omega \end{bmatrix}\mathbf{T} + \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}\hat{\mathbf{E}} = \begin{bmatrix} P_{\mathrm{in}\cdot 1} \\ P_{\mathrm{in}\cdot 2} \end{bmatrix}, \tag{3.30}$$

where C is equal to any of the off diagonal terms of the matrix **C** defined in Eq. (3.23). The first term in Eq. (3.30) is the dissipated power in the elements, so the second term describes the coupling power. Eq. (3.30) has the form of the standard SEA Eq. (2.3) so one is tempted to believe that coupling power proportionality is proved. An entry of the vector $\hat{\mathbf{E}}$, however, is not a measure of the vibration energy in the corresponding element as it is, in Eq. (3.24), defined by the Green function

for the entire structure. Consequently, Eq. (3.30) does not imply that the energy flow is proportional to the difference in modal energy in two connected elements but only that it is a linear function of the energies.

For weak coupling, equations (3.29) are approximately valid. Thus, neglecting terms of the order $\lambda_{\max}^2$ compared to unity, the entries of $\mathbf{q}$ are for weak damping approximately given by

$$q_i \approx \frac{1}{\Omega}\mathrm{Re}\left(\int_\Omega\int_{V_i} -\mathrm{i}\omega G_{ii\cdot b}(\mathbf{x}|\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\omega\right)$$
$$= \frac{1}{\Omega}\mathrm{Re}\left(\int_\Omega\int_{V_i}\sum_n \frac{-\mathrm{i}\omega(\psi_{n\cdot i}(\mathbf{x}))^2}{m_i(\omega_n^2-\omega^2(1+\mathrm{i}\eta_i))}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\omega\right) \approx \frac{\pi N_i}{2\rho_i\Omega}, \tag{3.31}$$

where N_i is the number of eigenfrequencies within the frequency band Ω. The second approximate equality is asymptotically exact for resonant vibration and zero damping as in this case all energy in a mode that has resonance frequency within a band is contained in this band, see Appendix A.

From Eqs. (3.24) and (3.31) it follows that for weak coupling the entries of the vibration potentials $\hat{\mathbf{E}}$ are given by

$$\hat{\mathbf{E}}_i = \pi\mathbf{T}_i/\mathbf{q}_i\mathbf{r}_i = 2\mathbf{T}_i/(N_i/\Omega). \tag{3.32}$$

For resonant vibrations, the strain and kinetic energies are equal and the right-hand side of Eq. (3.32) is the modal energy (times bandwidth), defined in (2.1). Eqs. (3.30) and (3.32) show that coupling power proportionality is a valid hypothesis for describing the power balance in a two-element structure, when the elements are weakly coupled and their motion is determined by resonant modes. Langley demonstrates this in Ref. [7]; the contribution of the present work is the quantitative criterion by which the assumption of weak coupling may be assessed. Thus, coupling is weak if the maximum eigenvalue of the non-dimensional coupling matrix $\mathbf{K}$ is negligible compared to unity. In Section 4, it is demonstrated that this criterion may be evaluated on the basis of the standard SEA parameters discussed in Section 2.

3.6. Energy conductivity

The conductivity is expressed in Eqs. (3.23) and (3.30), where its evaluation requires the inversion of the matrix $\overline{\mathbf{M}}$. For weak coupling, this can be accomplished by a series expansion [16, see Eq. (9)]. Thus, upon neglecting terms of the order of $\lambda_{\max}^2$ compared to unity, the off diagonal elements of the symmetric matrix $\mathbf{C}$ are given by

$$C = \mathbf{C}[1,2] = \frac{1}{\pi}(q_1/\overline{M}_{11})\overline{M}_{12}(q_2/\overline{M}_{22}) \tag{3.33}$$

where $\overline{M}_{ij}$ is defined in Eq. (3.25) and q_i in Eq. (3.26). Both these are defined by the approximate Green's function in Eq. (3.29).

Upon noting that

$$\mathrm{Re}\left(\frac{-\mathrm{i}\omega}{\omega_{n\cdot i}^2-\omega^2(1+\mathrm{i}\eta_i)}\right) = \frac{\eta_i\omega^3}{\left|\omega_{n\cdot i}^2-\omega^2(1+\mathrm{i}\eta_i)\right|^2}, \tag{3.34}$$

and using the orthogonal relation (3.1), it is found that approximately

$$q_i/\overline{M}_{ii} = 2\rho_i\eta_i\omega. \tag{3.35}$$

This result is exact if damping is viscous or if the frequency band Ω is sufficiently narrow so that a factor $\eta_i\omega$ in the numerator may be moved outside the integral for q_i.

For weak coupling, Green's function G_{12} is given by Eqs. (3.27)–(3.29)

$$G_{12}(\mathbf{x}|\mathbf{y}) = \sum_n\sum_m \psi_{n\cdot 1}(\mathbf{x})\mathbf{K}_{\mathbf{12}}[n,m]\psi_{m\cdot 2}(\mathbf{y})/\alpha_{m\cdot 2}, \tag{3.36}$$

where the matrix $\mathbf{K}_{\mathbf{12}}$ does not depend on $\mathbf{x}$ and $\mathbf{y}$. Consequently,

$$\overline{M}_{12} = \frac{1}{2\Omega}\int_\Omega\int_{V_1}\int_{V_2}\omega^2\left|G_{12}(\mathbf{x},\mathbf{y},\omega)\right|^2\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega = \frac{V_1V_2}{2\Omega}\int_\Omega\sum_n\sum_m\omega^2\left|\frac{\mathbf{K}_{\mathbf{12}}[n,m]}{\alpha_{m\cdot 2}}\right|^2\mathrm{d}\omega. \tag{3.37}$$

For general spring coupling, $\overline{M}_{12}$ is thus given by

$$\overline{M}_{12} = \frac{V_1V_2}{2\Omega}\int_\Omega\sum_n\sum_m\left[\frac{\omega^2}{\left|\alpha_{n\cdot 1}\right|^2\left|\alpha_{m\cdot 2}\right|^2}\left|\sum_p k_{c\cdot p}\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{m\cdot 2}(\mathbf{x_{2c\cdot p}})\right|^2\right]\mathrm{d}\omega. \tag{3.38}$$

For coupling by one spring only, this simplifies to

$$\overline{M}_{12} = \frac{V_1V_2}{2\Omega}\int_\Omega\frac{k_c^2\mathrm{Re}(Y_1(\mathbf{x_{1c}}))\mathrm{Re}(Y_2(\mathbf{x_{2c}}))}{\omega^4 m_1m_2\eta_1\eta_2}\,d\omega, \tag{3.39}$$

where the mobilities Y_i are defined in (3.15).

Finally, if as in the derivation of Eq. (3.35) the terms $\eta_i\omega$ may be moved outside the integrals, the conductivity in Eq. (3.33) is found. For coupling by one spring it is

$$C = \frac{1}{\Omega}\int_\Omega \frac{2}{\pi}\frac{k_c^2}{\omega^2}\mathrm{Re}(Y_1(\mathbf{x_{1c}}))\mathrm{Re}(Y_2(\mathbf{x_{2c}}))d\omega, \tag{3.40}$$

and for general spring coupling it is

$$C = \frac{1}{\Omega}\int_\Omega \frac{2}{\pi}\sum_n\sum_m\left[\mathrm{Re}\left(\frac{-\mathrm{i}\omega}{\alpha_{n\cdot 1}}\right)\mathrm{Re}\left(\frac{-\mathrm{i}\omega}{\alpha_{m\cdot 2}}\right)\left|\sum_p \frac{k_{c\cdot p}}{\omega}\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{m\cdot 2}(\mathbf{x_{2c\cdot p}})\right|^2\right]\mathrm{d}\omega. \tag{3.41}$$

The first of these expressions is similar to the one in Eq. (2.17). Eqs. (3.40) and (3.41) are, however, deterministic expressions that may vary largely because of structural details.

To summarise, weak coupling is defined by that the solution to the coupled equations of motion can be found from a convergent series of exchanges between uncoupled modes in connected elements. The rate of convergence is given by the largest eigenvalue to the non-dimensional coupling matrix. Coupling is said to be weak if the magnitude of this eigenvalue is smaller than unity. This gives a quantitative measure of coupling strength encompassing previously reported qualitative criteria of coupling strength.

Parts of Langley's analysis in Refs. [7,16] is reiterated showing that for weak coupling an exact power balance relation is equal to SEA equations; the original contribution of the present work being the qualitative criterion for the validity of the weak coupling assumption. This qualitative criterion, using the Bessac et al. results [11], is derived for deterministic structures. In Section 4, an approximate expression for the criterion is derived for stochastic structures with reverberant motion, where the criterion is expressed in SEA parameters.

4. Ensemble averages, synthesises and discussion

In Section 3, deterministic calculations establish a criterion validating the SEA formulation for power balance. In this section, it is required that the coupling strength criterion be fulfilled for all members of the ensemble defined, where its members are equal to the investigated structure but the elements have stochastic resonance frequencies. Upon this basis, coupling strength criteria are evaluated and models for predicting ensemble averaged energy are formulated. First, to expand the understanding of the energy flow between coupled modes, a two-mode structure is investigated in detail.

4.1. Two coupled oscillators

The basis for the modal approach to SEA is Scharton and Lyon's work demonstrating that the frequency average energy flow between two oscillators is proportional to the difference in their energy [14]. This is generalised for coupled substructures described by their uncoupled resonant modes. Thus, in Ref. [1, Section 3.2] it is presumed that the energy flow is the sum of the energy flows from each of the modes in one substructure to all the modes in the other, with no direct energy flow between modes in the same substructure.

Consequently, two spring-coupled oscillators (modes in connected substructures) are investigated. A harmonic force of root mean square amplitude f drives one of the oscillators; the input power and the coupling power are given by

$$P_{\mathrm{in}\cdot 1} = \mathrm{Re}(-\mathrm{i}\omega u_1 f^*) = \frac{(r_2^2+1)+\gamma}{(r_1r_2-\gamma-1)^2+(r_1+r_2)^2}\frac{\omega|f|^2}{m_1\Delta_1}, \tag{4.1}$$

$$P_{\mathrm{coup}}^{1,2} = \mathrm{Re}(-\mathrm{i}\omega u_1 k_c(u_1-u_2)^*) = \frac{\gamma}{(r_1r_2-\gamma-1)^2+(r_1+r_2)^2}\frac{\omega|f|^2}{m_1\Delta_1} \tag{4.2}$$

where

$$\omega_i^2 = (k_i+k_c)/m_i, \quad \Delta_i = \omega^2\eta_i, \quad \chi^2 = k_c^2/m_1m_2, \tag{4.3}$$

$$r_i = (\omega_i^2-\omega^2)/\Delta_i, \tag{4.4}$$

$$\gamma = \chi^2/(\Delta_1\Delta_2), \tag{4.5}$$

and where k_i, m_i and η_i are generalised stiffness, mass and loss factor for oscillator i, k_c is the generalised coupling stiffness and u_i is the displacement. The factor $\omega|f|^2/m_1\Delta_1$ is the maximum input power to the uncoupled oscillator attained at its resonance. The non-dimensional parameter r_i measures approximately the difference between frequency and natural frequency in units of 3-dB bandwidth: $r_i \approx 2(\omega_i-\omega)/(\eta_i\omega)$. It is interesting that γ in Eq. (4.5) is equal to the 'modal interaction strength' defined in Eq. (2.23).

Figs. 1–3 show the normalised coupling power, $P_{\mathrm{coup}}^{1,2}/(\omega|f|^2/m_1\Delta_1)$, for different values of the modal interaction strength as function of r_1 and r_2. From Eq. (4.2), the shapes of these curves depend only on γ. Thus, each figure displays the coupling power for an ensemble of coupled oscillators with the same modal interaction strength. A point in the figure gives

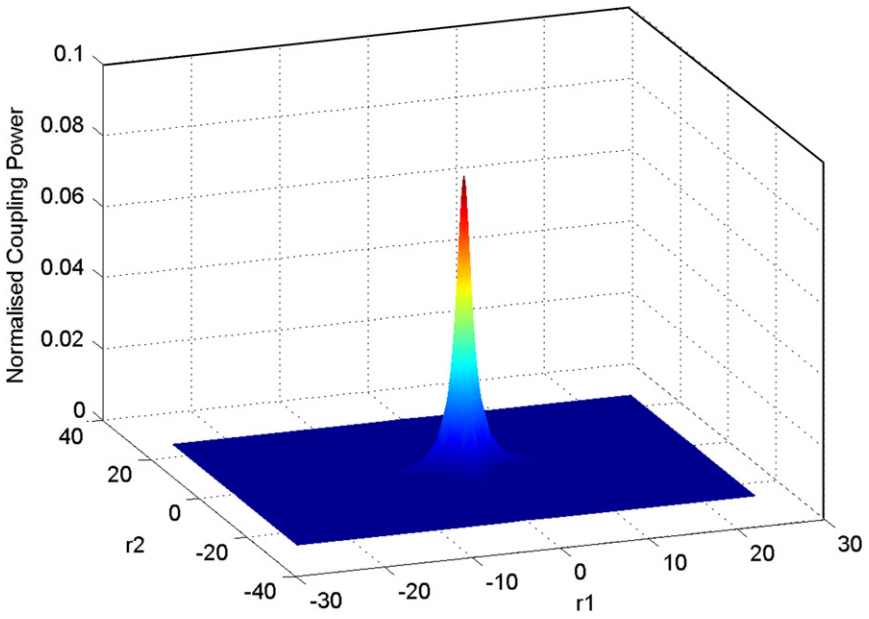

Fig. 1. Normalised coupling power, $P^{1,2}_{\mathrm{coup}}/(\omega|f|^2/m_1\Delta_1)$, for $\gamma=0.1$. The non-dimensional parameter r_i measures the separation of uncoupled oscillator frequency ω_i and frequency of vibration ω in units of oscillator damping band width, see Eq. (4.4).

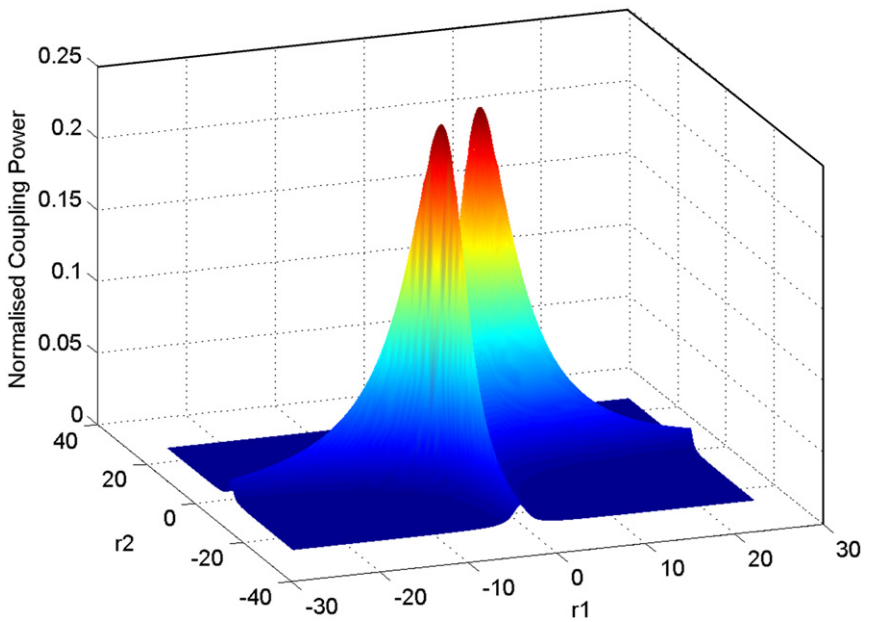

Fig. 2. Normalised coupling power, $P^{1,2}_{\mathrm{coup}}/(\omega|f|^2/m_1\Delta_1)$, for $\gamma=10$.

the coupling power for a particular system vibrating at a specified frequency. If the frequency is varied, a (slightly curved) line gives the coupling power.

For $\gamma=0.1$ the coupling power is substantial only when the frequency and the two oscillator frequencies are all equal. In contrast to this, for $\gamma=10$ maximum coupling power frequencies are shifted and the area in the r_1-r_2 plane where the coupling power is substantial is increased. Thus, for strong coupling there are, for all members of the ensemble, two frequencies for which the coupling power is large, except for those that have elements with very different uncoupled frequencies.

Figs. 4–6 show the normalised input power, $P_{\mathrm{in}\cdot 1}/(\omega|f|^2/m_1\Delta_1)$. For $\gamma<1$, the input power is approximately equal to that which applies for the uncoupled oscillator and is independent of the other oscillator's frequency. In contrast, for $\gamma>1$ the input power is influenced by the coupled oscillator and the frequencies for maximum input power are shifted.

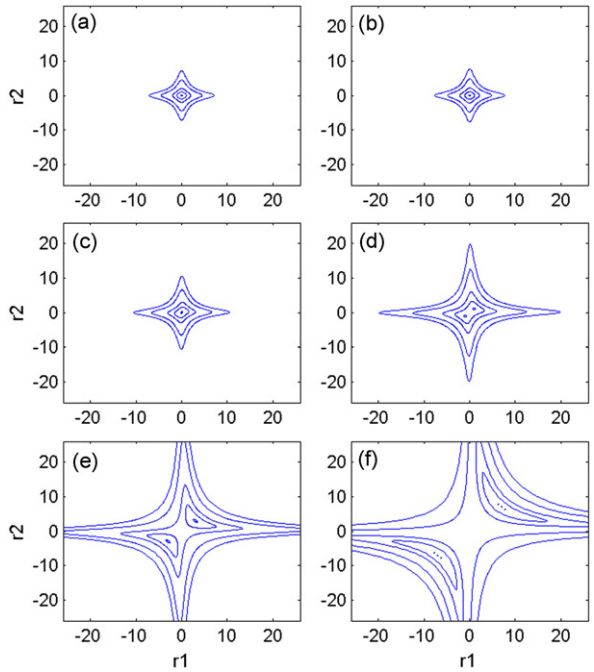

Fig. 3. Normalised coupling power, $P^{1,2}_{\mathrm{coup}}/(\omega|f|^2/m_1\Delta_1)$: (a) γ=0.02; (b) γ=0.1; (c) γ=0.5; (d) γ=2; (e) γ=10; (f) γ=50. The contours are drawn for 2%, 5%, 20%, 50% and 98% of the maximum coupling power.

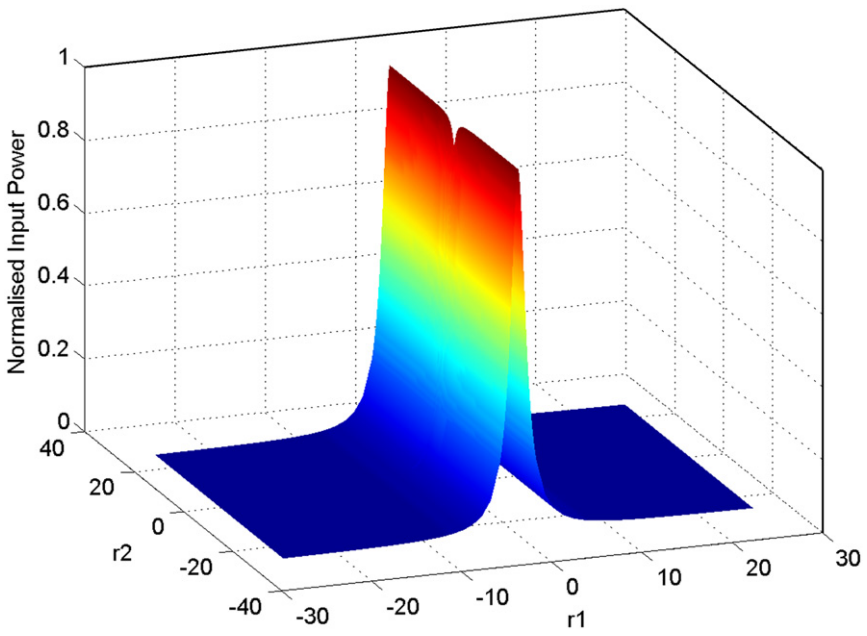

Fig. 4. Normalised input power, $P_{\mathrm{in}1}/(\omega|f|^2/m_1\Delta_1)$, for γ=0.1.

To quantify these observations, the maximum of the coupling power as a function of the oscillator frequencies is calculated. The frequencies for maximum coupling power are given by

$$\frac{\partial P^{1,2}_{\mathrm{coup}}}{\partial \omega_i} = \frac{\partial P^{1,2}_{\mathrm{coup}}}{\partial r_i}\frac{\partial r_i}{\partial \omega_i} = 0, \quad i = 1,2. \tag{4.6}$$

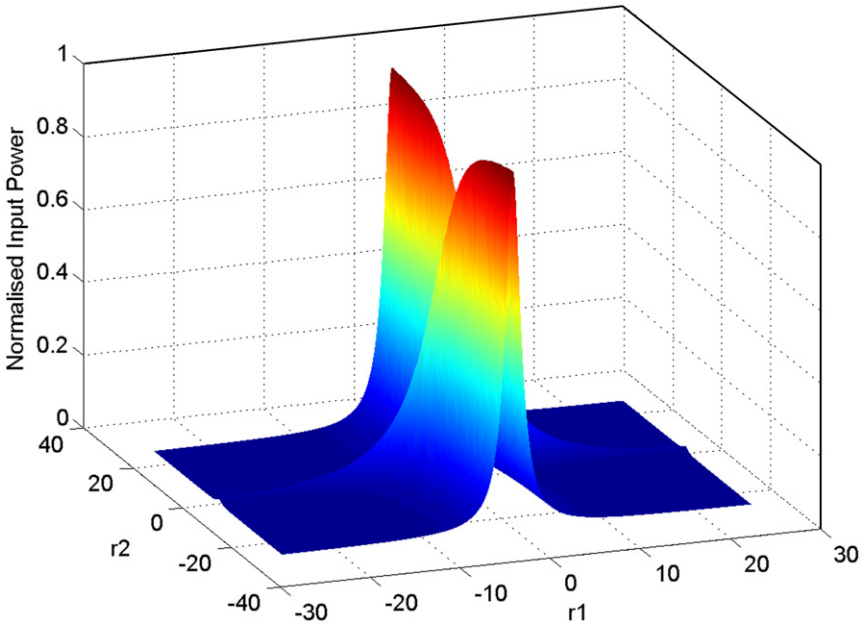

Fig. 5. Normalised input power, $P_{in1}/(\omega|f|^2/m_1\Delta_1)$, for $\gamma=10$.

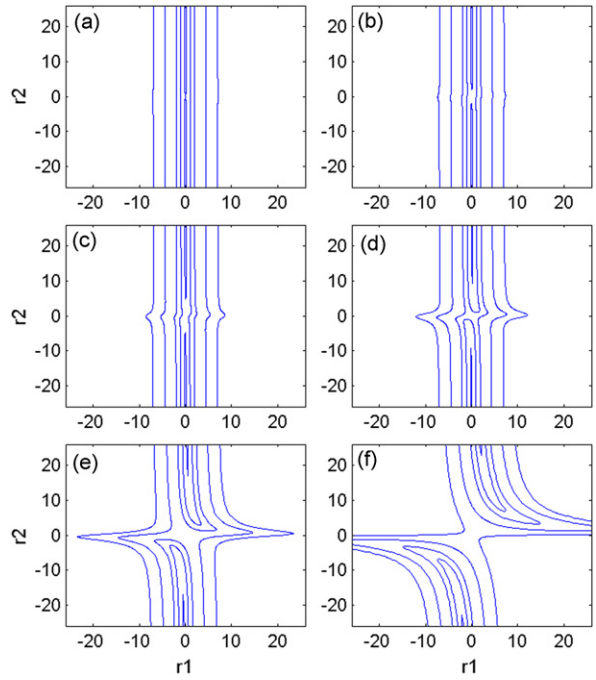

Fig. 6. Normalised input power, $P_{in1}/(\omega|f|^2/m_1\Delta_1)$: (a) $\gamma=0.02$; (b), $\gamma=0.1$; (c) $\gamma=0.5$; (d) $\gamma=2$; (e), $\gamma=10$; (f) $\gamma=50$. The contours are drawn for 2%, 5%, 20%, 50% and 98% of the maximum input power.

This is equally written

$$r_1(r_2^2+1)-\gamma r_2=0, \quad r_2(r_2^2+1)-\gamma r_1=0, \tag{4.7}$$

These equations have the trivial solution $r_1=r_2=0$, i.e., $\omega_1=\omega_2=\omega$. For $\gamma<1$ this is the only real solution. For $\gamma>1$, however, there are two additional solutions given by

$$r_1=r_2=\pm\sqrt{\gamma-1}, \tag{4.8}$$

so that the natural frequencies for maximum energy flow are given by

$$\left\{\begin{array}{l}\omega_1=\omega\sqrt{1-\eta_1\sqrt{\gamma-1}}\\ \omega_2=\omega\sqrt{1-\eta_2\sqrt{\gamma-1}}\end{array}\right\}\quad\text{and}\quad\left\{\begin{array}{l}\omega_1=\omega\sqrt{1+\eta_1\sqrt{\gamma-1}}\\ \omega_2=\omega\sqrt{1+\eta_2\sqrt{\gamma-1}}\end{array}\right\}. \tag{4.9}$$

Consequently, when the 'modal interaction strength', γ, is smaller than unity the only maxima to the coupling power is at $\omega_1=\omega_2=\omega$ whereas for $\gamma>1$ there are two maxima given by Eq. (4.9), plus a saddle point at $\omega_1=\omega_2=\omega$.

The maximum coupling power is given by

$$\mathrm{Max}\left(\frac{P^{1,2}_{\mathrm{coup}}}{\omega|f|^2/m_1\Delta_1}\right)=\begin{cases}\gamma/(\gamma+1)^2, & \gamma<1,\\ 1/4, & \gamma\geq 1,\end{cases} \tag{4.10}$$

while the input power at the frequencies for maximum coupling power is given by

$$\frac{P_{\mathrm{in}\cdot 1}}{\omega|f|^2/m_1\Delta_1}=\begin{cases}1/(\gamma+1), & \gamma<1,\\ 1/2, & \gamma\geq 1.\end{cases} \tag{4.11}$$

The maximum input power, regardless of coupling strength, is found at $r_1=0$, $r_2\to\pm\infty$ and is: $\max(P_{\mathrm{in}\cdot 1})=\omega|f|^2/m_1\Delta_1$.

Langley states that coupling is weak if Green's function for a directly excited substructure is approximately equal to that for the uncoupled substructure [7]. It is seen that this criterion is fulfilled for two coupled oscillators if the modal interaction strength γ is less than unity.

4.2. Ensemble averages

4.2.1. Two coupled modes

For random structures, the average coupling power is proportional to the probability for substantial coupling power, which, for two-oscillator structures, increases with the area enclosed by the contours in Fig. 3. (N.B. It is a special frequency scale.) It is seen in Fig. 3, and is evident from Eq. (4.2), for small modal interactions strength, this area is independent of γ. Thus, the number of ensemble members that contribute largely to the ensemble average coupling power is independent of γ, if $\gamma\ll 1$. This is in stark contrast to behaviour when modal interaction strength is greater than unity. For such structures, maximum coupling power is independent of γ whereas the number of ensemble members that contributes to the average increases with γ. It follows, the modal interaction strength, γ, is the non-dimensional parameter that characterises the ensemble energy flow between two coupled modes.

4.2.2. Multi-modal structures

The vibration potentials $\hat{\mathbf{E}}$ are given as linear functions of input powers by the power balance equation (3.22). Inspection of the derivation Appendix B, reveals that this equation predicts ensemble average values of vibration potentials, if the matrices $\mathbf{q}$ and $\overline{\mathbf{M}}$ are ensemble averaged. That is, if Eqs. (B.4) and (B.6) are ensemble averaged, the power balance is given by

$$\begin{aligned}\mathbf{C}\langle\hat{\mathbf{E}}\rangle&=\langle\mathbf{P}\rangle\\ \mathbf{C}&=(1/\pi)\langle\mathbf{q}\rangle\langle\overline{\mathbf{M}}\rangle^{-1}\langle\mathbf{q}\rangle\end{aligned} \tag{4.12}$$

where $\langle\ \rangle$ denotes ensemble averaging.

A simple example of an ensemble is provided by the set of structures of known properties but their resonance frequencies are uniformly probable within the frequency band Ω and the coupling springs have random positions within the elements. These two conditions aren't fully independent as the natural frequency of the uncoupled sub structures depend on the strength of the coupling springs and their precise locations. This dependency is here neglected. This is asymptotically exact for vanishing coupling strength and for moderate coupling strength it can be motivated by ignorance.

For weak coupling, the expected values of q_i and $\overline{M}_{ii}$ are, from Equations (3.31) and (3.34) and the first integral in the Appendix A, given by

$$\langle q_i\rangle=\frac{\pi}{2}\frac{N_i}{\rho_i\Omega},\quad\langle\overline{M}_{ii}\rangle=\frac{\pi}{4}\frac{N_i}{\rho_i^2\eta_i\omega\Omega} \tag{4.13}$$

The expected value of $\overline{M}_{12}$ is, from Eq. (3.38), given by

$$\langle\overline{M}_{12}\rangle=\frac{V_1V_2}{2\Omega}\left\langle\int_\Omega\omega^2\sum_n\sum_m\left|\sum_p\frac{\psi_{n\cdot 1}(\mathbf{x}_{1\mathbf{c}\cdot\mathbf{p}})^2}{\alpha_{n\cdot 1}}k^2_{c\cdot p}\frac{\psi_{m\cdot 2}(\mathbf{x}_{2\mathbf{c}\cdot\mathbf{p}})}{\alpha_{m\cdot 2}}\right|\mathrm{d}\omega\right. \tag{4.14}$$

Now, if it happens that the coupling springs are located at well separated random locations, they are, on average, independent paths for the energy transport and Eq. (4.14) simplifies to

$$\begin{aligned}\langle \overline{M}_{12} \rangle &= \frac{V_1 V_2}{2\Omega} \left\langle \int_\Omega \omega^2 \sum_n \sum_m \sum_p \left| \frac{\psi_{n\cdot 1}(\mathbf{x_{1c}\cdot p})}{\alpha_{n\cdot 1}} k_{c\cdot p}^2 \frac{\psi_{m\cdot 2}(\mathbf{x_{2c}\cdot p})}{\alpha_{m\cdot 2}} \right|^2 \mathrm{d}\omega \right\rangle \\ &= \frac{1}{2\Omega} \int_\Omega \sum_p \frac{k_{c\cdot p}^2}{\omega^2} \frac{\langle \mathrm{Re}(Y_1(\mathbf{x_{1c}\cdot p})) \rangle \langle \mathrm{Re}(Y_2(\mathbf{x_{2c}\cdot p})) \rangle}{\rho_1 \rho_2 \omega \eta_1 \omega \eta_2} \, d\omega \\ &\approx \frac{1}{2} \sum_p \frac{k_{c\cdot p}^2}{\omega^2} \frac{Y_{c\cdot 1} Y_{c\cdot 2}}{\rho_1 \rho_2 \omega \eta_1 \omega \eta_2} \end{aligned} \tag{4.15}$$

where Eqs. (3.15) and (3.34) were used. In the last approximate equality the ensemble averaged point mobility at the connection points is equated to the characteristic mobility, evaluated at the band's nominal frequency.

In cases when the springs are not independent paths for the energy transmission, ensemble averaging the natural frequencies of the two elements and, as in Appendix A, neglecting the small difference between the natural frequencies in the considered band and the nominal frequency, yields

$$\begin{aligned}\langle \overline{M}_{12} \rangle &= \frac{V_1 V_2}{2\Omega} \int_\Omega \sum_{n'} \sum_{m'} \left(\frac{\pi^2}{4\eta_1 \eta_2 \Omega^2 \omega^4} \right) \left\langle \left| \sum_p \psi_{n\cdot 1}(\mathbf{x_{1c}\cdot p}) \frac{k_{c\cdot p}}{m_1 m_2} \psi_{m\cdot 2}(\mathbf{x_{2c}\cdot p}) \right|^2 \right\rangle \mathrm{d}\omega \\ &\approx \left(\frac{\pi^2 N_1 N_2}{8\rho_1 \rho_2 \eta_1 \eta_2 \Omega^2 \omega^2} \right) \frac{1}{N_1 N_2} \sum_{n'} \sum_{m'} \left\langle \left| \sum_p \psi_{n\cdot 1}(\mathbf{x_{1c}\cdot p}) \frac{k_{c\cdot p}}{\omega \sqrt{m_1 m_2}} \psi_{m\cdot 2}(\mathbf{x_{2c}\cdot p}) \right|^2 \right\rangle \end{aligned} \tag{4.16}$$

where the remaining ensemble averaging considers the spring locations only and n' and m' denotes that the summations consider the modes within the band only.

Upon this basis, the conductivity that relates the ensemble average energy flow to the ensemble average modal energies is, when Eq. (4.15) applies, given by

$$C = \frac{1}{\pi} \frac{\langle q_1 \rangle \langle \overline{M}_{12} \rangle \langle q_2 \rangle}{\langle \overline{M}_{11} \rangle \langle \overline{M}_{22} \rangle} = \frac{2}{\pi} \sum_p \frac{k_{c\cdot p}^2}{\omega^2} Y_{c\cdot 1} Y_{c\cdot 2}. \tag{4.17}$$

This value of C is equal to the standard SEA value, for weak dynamic coupling, see Section 2. If instead, Eq. (4.16) applies, the conductivity is given by

$$C = \left(\frac{\pi N_1 N_2}{2\Omega^2} \right) \frac{1}{N_1 N_2} \sum_{n'} \sum_{m'} \left\langle \left| \sum_p \psi_{n\cdot 1}(\mathbf{x_{1c}\cdot p}) \frac{k_{c\cdot p}}{\omega \sqrt{m_1 m_2}} \psi_{m\cdot 2}(\mathbf{x_{2c}\cdot p}) \right|^2 \right\rangle. \tag{4.18}$$

In conclusion, the Langley model for power balance determines ensemble average vibration potentials, if the matrices $\mathbf{q}$ and $\overline{\mathbf{M}}$ are ensemble averaged. For resonant vibration and weak coupling, defined where the largest coupling eigenvalue has a magnitude of less than unity, the vibration potentials equals the modal energy times analysis bandwidth. Also, for weak coupling, and if the elements have independent stochastic properties, the ensemble averages are calculated independently for the connected elements. Upon this basis, the standard SEA model discussed in Section 2 is recovered. In particular, the "standard SEA" model for two coupled oscillators with random properties given by Eq. (2.24) is verified by either of Eqs. (4.17) and (4.18).

4.3. Modal interaction strength and coupling eigenvalues

Coupling is weak if the coupling eigenvalues magnitudes are less than unity. For weak coupling, defined in this manner, the series expansion in Eq. (3.9) converges and Green's function of the coupled structure as well as the "modal behaviour" is approximated by those applying for uncoupled motion. In what follows, relations between coupling eigenvalues, SEA parameters and modal interaction strength are investigated.

4.3.1. Two oscillators

The coupling eigenvalue for a two-mode structure are, from Eq. (3.16), given by

$$|\lambda_{\max}^2| = \frac{k_c^2}{|\alpha_1 \alpha_2|} = \frac{\gamma}{|(r_1 - i)(r_2 - i)|} \leq \gamma, \tag{4.19}$$

where the modal receptance, α_i, is defined in Eq. (3.4) and r_i and the modal interaction strength γ are defined in Eqs. (4.4) and (4.5). Coupling is weak if the magnitude of the coupling eigenvalue is less than unity. It is seen in Eq. (4.19) that this

criterion is fulfilled for all frequencies and for all members of the ensemble of 'similar' oscillators, if the modal interaction strength, γ, is less than unity.

The modal interaction strength is thus an ensemble property, which does not depend on the precise properties of the oscillators. This is in contrast to Smith's coupling strength criterion based on the coupling loss factor derived by Scharton and Lyon [14], see below, which is very sensitive to the difference between the oscillator frequencies.

4.3.2. The Smith criterion for weak coupling and the modal interaction strength

Smith states that coupling is weak provided that [5]

$$\eta_{\mathrm{coup}}^{1,2}/\eta_1 \ll 1 \quad \text{and} \quad \eta_{\mathrm{coup}}^{2,1}/\eta_2 \ll 1, \tag{4.20}$$

where $\eta_{\mathrm{coup}}^{ij}$ are the coupling loss factors. The Smith criterion is frequently used for assessing the validity of assumptions in SEA, see, e.g., Ref. [1]. This application of the criterion is questioned in Refs. [9,10]. It is doubtless, however, that it determines the character of the solutions to the SEA equation [5].

The two criteria (4.20) are equally written as one single criterion:

$$\eta_{\mathrm{coup}}^{1,2}/\eta_1 + \eta_{\mathrm{coup}}^{2,1}/\eta_2 \ll 1. \tag{4.21}$$

For two coupled oscillators, Scharton and Lyon derived the fundamental equation for the modal approach to SEA, where the frequency averages of coupling power and oscillator energies are related [14]

$$P_{\mathrm{coup}}^{1,2} = \beta(E_1 - E_2). \tag{4.22}$$

For two spring-coupled oscillators, β is given by Ref. [1, Eq. (3.1.15)]

$$\beta = \omega\eta_{\mathrm{coup}}^{1,2} = \omega\eta_{\mathrm{coup}}^{2,1} = \frac{\chi^2(c_1+c_2)}{(\omega_1^2-\omega_2^2)^2 + (c_1+c_2)(c_1\omega_2^2 + c_2\omega_1^2)}, \tag{4.23}$$

where c_i is the specific viscous damping coefficient: $c_i = \eta_i\omega$. Note, as is common practise in SEA, Eq. (4.22) is generalised to apply for a frequency band, with mid-frequency ω, which contains both oscillator frequencies.

Now, it is required that the Smith criterion (4.21) be valid for all members of an ensemble of similar two-oscillator structures, having random oscillator frequencies. Thus, it must also be fulfilled when β has its maximum value, which (upon neglecting terms of the order of η_i^2 compared to unity) is found for $\omega_2 = \omega_1$. This value of ω_2 is used in Eq. (4.23), and, upon neglecting the presumed small difference between ω and ω_1, we have

$$\mathrm{Max}\left(\eta_{\mathrm{coup}}^{1,2}/\eta_1 + \eta_{\mathrm{coup}}^{2,1}/\eta_2\right) = \frac{\chi^2(c_1+c_2)}{(c_1+c_2)^2\omega^2}\left(\frac{1}{c_1}+\frac{1}{c_2}\right) = \gamma. \tag{4.24}$$

Consequently, requiring that the Smith criterion be fulfilled for all members of an ensemble of coupled oscillators, having random frequencies within the considered frequency band, is equivalent to requiring that the modal interaction strength, γ, be much less than unity.

4.3.3. General spring coupled structures

For elements coupled by randomly positioned springs, the maximum coupling eigenvalue is given by

$$\left|\lambda_{\mathrm{max}}^2\right| = \sum_p \frac{k_{c\cdot p}^2}{\omega^2}\left|Y_1(\mathbf{x_{1c\cdot p}})Y_2(\mathbf{x_{2c\cdot p}})\right|. \tag{3.21}$$

Again, it is required that the coupling eigenvalue be less than unity for all frequencies and for all members of the ensemble of 'similar' structures. Thus, the criterion must apply also when there is resonance in both elements. Applying, for reverberant motion ($M<1$), Skudrzyk's relation between the characteristic mobility and the maximum mobility in Eq. (2.10), the criterion is seen to be fulfilled provided that

$$|\lambda_{\mathrm{max}}^2| = \sum_p \frac{k_{c\cdot p}^2}{\omega^2}|Y_1Y_2| \le \sum_p \frac{k_{c\cdot p}^2}{\omega^2}\frac{Y_{c\cdot 1}Y_{c\cdot 2}}{(\pi/2)^2 M_1M_2} = \frac{2}{\pi}\frac{C}{M_1M_2} = \gamma < 1, \tag{4.25}$$

where the conductivity C is given in Eq. (2.19), and equally in Eq. (4.17) and the modal interaction strength γ is defined by Eq. (2.15). The criterion (4.25) is equal to that which governs the validity of standard SEA for prediction of ensemble average energies in one-dimensional waveguides [8–10].

The coupling eigenvalue for general spring coupling is given by Eq. (3.18). An estimate of the largest coupling eigenvalue across an ensemble of elements is derived assuming that: (i) the responses of both connected elements are resonant; (ii) the maximum is defined when there is resonance in both connected elements; (iii) these resonant modes

have typical mode shapes. Upon this basis, an estimate of the largest coupling eigenvalue is given by

$$\begin{aligned}\max(\lambda_{\max}^2) &= \max\left(\sum_n \left|\sum_m\sum_p\sum_q \left[k_{c\cdot p}\frac{\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{m\cdot 2}(\mathbf{x_{2c\cdot p}})}{\alpha_{n\cdot 1}}\right]\left[k_{c\cdot q}\frac{\psi_{m\cdot 2}(\mathbf{x_{2c\cdot q}})\psi_{n\cdot 1}(\mathbf{x_{1c\cdot q}})}{\alpha_{m\cdot 2}}\right]\right|\right)\\ &\approx \frac{1}{m_1\eta_1\omega^2}\frac{1}{m_2\eta_2\omega^2}\frac{1}{N_1N_2}\\ &\quad\times\left\langle\sum_{n'}\left|\sum_{m'}\left(\sum_p[k_{c\cdot p}\psi_{n'\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{m'\cdot 2}(\mathbf{x_{2c\cdot p}})]\sum_q\left[k_{c\cdot q}\psi_{m'\cdot 2}(\mathbf{x_{2c\cdot q}})\psi_{n'\cdot 1}(\mathbf{x_{1c\cdot q}})\right]\right)\right|\right\rangle\\ &= \frac{1}{\eta_1\eta_2\omega^2}\frac{1}{N_1N_2}\sum_{n'}\left|\sum_{m'}\left\langle\left(\sum_p\frac{k_{c\cdot p}}{\omega\sqrt{m_1m_2}}\psi_{n\cdot 1}(\mathbf{x_{1c\cdot p}})\psi_{m\cdot 2}(\mathbf{x_{2c\cdot p}})\right)^2\right\rangle\right|\\ &= \frac{2}{\pi}\frac{C}{M_1M_2} = \gamma\end{aligned} \tag{4.26}$$

where, in this case, C is given by Eq. (4.18).

From Eqs. (4.25) and (4.26) follows the important result that the validity of the weak coupling assumption can be approximately assessed by the modal interaction strength. It is given by comparatively simple SEA parameters, derived for random structures and upon the assumption of weak coupling. Thus, the validity of such an SEA model may be assessed by its input data.

5. Remarks

The reviews in Sections 1 and 2 concern mostly the development up until 2000. Additionally, the last decade has shown some substantial development in SEA; particularly on the interaction of SEA and FEA, the expected variance in the response of the members of an ensemble and, also, on the understanding of the method. Some of this progress is discussed below. First, two complications, tacitly omitted in the preceding analysis are discussed.

5.1. Moderately weak coupling

In the present work, the deterministic analysis is simplified upon the assumption that the coupling eigenvalues are much less than unity. Upon this basis, SEA results in Section 3 are derived without using any statistical assumptions or without accounting for the smoothing effect of frequency averaging. Then, in Section 4, simple ensembles are considered showing the similarity to standard SEA formulations, demonstrating that the maximum coupling eigenvalue for an ensemble is equal to the coupling strength measure γ, here denoted 'modal interaction strength'. This parameter governs the application of SEA for prediction of ensemble averaged energies in one-dimensional waveguides [8–10]. Refs. [9,10], however, demonstrate that standard SEA procedures estimate ensemble averages accurately if only the modal interaction strength is less than unity. This indicates that the present results may be valid not only for very weak coupling, $\gamma \ll 1$, but also for moderately weak coupling, $\gamma \leq 1$.

In the present work, we are content to derive results for very weak coupling and to demonstrate a criterion by which the validity of this assumption can be assessed. Quantitative estimates of the errors resulting for moderate weak or strong coupling, or the quantitative implications of stochastic assumptions in SEA, are not dealt with.

5.2. Comment on other coupling conditions

Iteration schemes similar to Eq. (3.9) can be formulated also for coupling mechanisms other than spring coupling. Thus, coupling may be weak if there is a large difference in wave impedance between the connected elements. Maxit and Guyader present a technique for formulating the coupled equations of motion for such structures, which can be used with the present methods [28,29]. Coupling may also be weak if both elements' wave impedances are very different to the impedance of the coupling mechanism, for instance if there is a blocking mass or an elastic interlayer. (In the present work coupling may be weak if the spring mobility is much larger than the elements' input mobilities.)

When applying the iterative method, note that the rapid convergence of Eq. (3.9) is only a sufficient condition for weak coupling—not a necessary condition. For instance, consider two large thin flexurally vibrating plates that are connected via a rigid, very heavy, mass. These elements are, from an energetic point-of-view, weakly coupled. However, if the free plate modes are used in Eq. (3.6) to express the solution to the coupled equations of motion, for each frequency a large number of modes are required to fulfil the condition of almost blocked motion at the heavy mass. Thus, the series expansion (3.9) converges slowly (if at all) although the elements are weakly coupled. If, instead, an equation similar to Eq. (3.5) is formulated with fixed-base modes, having blocked motion at the mass, convergence should be rapid.

The implication is that when calculating SEA coupling factors using a weak coupling assumption, as in Eq. (3.33) or in Ref. [16, Eq. (43)], care must be taken so that the uncoupled modes obey appropriate boundary conditions, since coupling eigenvalues, as well as conductivities calculated by a first order approximation, depend not only on the investigated structure but also on the choice of boundary conditions for the uncoupled modes.

For structures studied in academic research, the choice of boundary conditions is usually obvious but it is not always so for engineering structures. Therefore, it is vital that future research focuses upon the criteria for subdividing structures into SEA elements and for choosing uncoupled boundary conditions for these elements. To this end, the recent works by Maxit, Totaro and Guyader are interesting [28–30].

5.3. SEA, FEA and one-way methods

Sometimes, it is possible to solve the equations of motion for the entire structure of interest. Shorter and Mace calculate the global modes of the structure and formulate the frequency band averaged energy conservation for substructures on this basis [31–33]. The formulation is similar to the power injection method (PIM) used by Fredö [34] but more efficient. It expresses the energy conservation with the subsystem's energies as dependent variable and yields an un-symmetric system matrix and is an 'SEA-like model'. Ref. [35] contains a valuable discussion on the various SEA models that can be devised by deterministic FE calculations. Here, SEA-like signifies that the model describes the law of energy conservation, as SEA does, but it applies for an individual member of the ensemble considered by SEA [34]. Green's function may be calculated with the FEM upon which Langley's formalism, as presented in this work, yields an SEA-like model where the system matrix is symmetric but the dependent energy variable is defined by the entire structures properties. Provided that the subsystem's space and frequency averaged point mobility can be approximated by Eq. (2.9) (Langley's definition of weak coupling [7] and [33, Eq. (30)]), both methods yield a symmetric matrix and local energy variables defined by the subsystem's properties. Carbonelli compares the two formulations based on measurements and FE calculation of the response of a three plate structure and notes some differences: in this case, Langley's method was somewhat more efficient and robust [36].

Often it is not practical, or interesting, to solve the fully coupled equations of motion for the entire structure while the modes proper to the un-coupled subsystems are attainable. The present work (as well as [19]) is concerned with SEA models based on such local modes. It is found that, provided the modal interaction strength γ is small, the coupling appears on a mode-to-mode basis, as in Fig. 3.4 in Ref. [1]. Moreover, the conductivity (or coupling loss factors) are calculated by a one-way procedures, where the first element's energy is defined and the action of this element is calculated without concern of the reaction. For such one-way calculations, the ensemble averages of coupling power and energies are calculated independently for the two elements, as in Eq. (4.15).

Maxit and Guyader [28,29] and Mace and Ji [37] also consider systems described by local modes. They postulate that the coupling power is given on a mode-to-mode basis while considering fully coupled mode pairs. Maxit and Guyader develop a generalised gyroscopic coupling which applies to a great variety of coupling mechanisms. As in SEA they calculate frequency band averages of the response but the system properties are deterministic and they attribute the individual modal amplitudes, i.e., they do not assume modal equipartition and this appears to improve results in low mode count frequency regimes and, probably, also whenever the excitation or coupling discriminates some of the modes.

Mace and Ji consider spring coupled oscillators (local modes) that have random properties defined by rectangular distributed natural uncoupled frequencies, as in the present work. The results are identical to those in Eq. (2.24) except for two factors that appear for strong coupling; thus the coupling loss factor η_{12} given by Eq. (2.24) is

$$\omega\eta_{12}^{(c)} = \frac{\pi}{2}\frac{\chi^2}{\omega^2\Omega}, \tag{4.27}$$

while Eq. (2.15) of Ref. [37] is

$$\omega\eta_{12}^{(a)} = \frac{\omega\eta_{12}^{(c)}/\sqrt{1+\gamma}}{1-\omega\eta_{12}^{(c)}/\omega\eta_d/\sqrt{1+\gamma}}; \quad \eta_d = \frac{\eta_1\eta_2}{\eta_1+\eta_2}. \tag{4.28}$$

γ is defined equally in this work and in Refs. [19,37]. The denominator in the first of Eqs. (4.28) is in the later analysis, see Ref. [37, Eq. (3.11)], substituted with a factor of unity, without any explanation. Mace and Ji state that previous works have calculated the coupling parameters that relates ensemble averaged energies and input powers by ensemble averaging the coupling loss factor, which is not correct. However, correct ensemble averaging is made in Ref. [19], as in this work. Also, the ensemble averaged coupling power that results from Eqs. (2.9), (2.14) and (2.15) of Ref. [37] is identical to the one in Ref. [1, Eq. (3.1.20)], indicating that the same analysis is made.

The current work shows that, for spring coupled structures, a sufficient condition for the mode-to-mode coupling and one-way calculations of coupling parameters is that the modal interaction strength γ is small. It might be, though it is not proven, that this is also a necessary condition. If so, the correction factor $\sqrt{1+\gamma}$ in Eq. (4.28) is inconsistent for multi modal structures, meaning that it only matters when the mode-to-mode analysis fails. This surmise is supported by the results in Figs. 4 and 5 of [37] and more significantly in Ref. [28, Section IV.2] revealing that if incorrect boundary conditions are chosen for the uncoupled modes, the mode-to-mode approach provides incorrect results.

As mentioned above, some of the most important later developments in SEA are the variance predictions by Langley, Cotoni and Brown, e.g., Ref. [38] and the Shorter and Langley reciprocity relation [39] that is the basis for the new hybrid method [40,41]. Here it is simply noted that the analyses leading up to these important results are partly based on one-way calculations and weak coupling assumptions. It is, therefore, suggested that future practitioners of the

methods check the magnitude of the modal interaction strength to gain further insight to the range of validity of the methods.

Finally it is noted that Le Bot and Cotoni [42] have braved to list the criteria for the validity of SEA; these are (i) large mode count; (ii) small energy decay as a wave travel across an element; (iii) large modal overlap factors; (iv) weak coupling as measured by Smith's criterion (coupling loss factor/damping loss factor=conductivity/modal overlap factor). Now, if the modal overlap factor is large and the coupling losses are smaller than the damping losses, the modal interaction strength γ is small. Thus the findings in this work agree with commonly used criteria.

6. Conclusions

Coupling strength is a central concept in SEA. It is only for weak coupling that the coupling power hypothesis can be approximately valid for multi-element structures, since it is not valid for three strongly coupled oscillators [15]. It follows that it is only for weak coupling that the parameters in the SEA equations for power balance may be evaluated for large structures while considering only two coupled elements at a time. Despite the significance of the weak coupling assumption, there are no universally agreed criteria by which this assumption may be assessed; though, several definitions are found in the literature, as discussed in the introduction. The present work investigates the response of two general spring-coupled elements in an attempt to develop a unifying approach to the weak coupling criterion for applying Statistical Energy Analysis (SEA).

The coupled equations of motion are expressed in the bases given by the uncoupled elements' eigenmodes. An iterative solution to these equations is expressed as a succession of exchanges between elements, where the uncoupled motion provides the start approximation. This procedure is similar to the one employed by Newland [4,25]. Bessac, Gagliardini and Guyader demonstrate that the process converges if the maximum 'coupling eigenvalue' (the largest magnitude of the eigenvalues to the non-dimensional coupling matrix) is less than unity, in which case coupling is said to be weak [11].

Keane and Price define weak coupling where the 'modal behaviour' of the coupled structure approximates that of the uncoupled [6]. The modal amplitudes provide one description of modal behaviour. Consequently, the Keane and Price criterion for weak coupling, defined in this manner, is fulfilled if the coupling eigenvalue is small.

Langley formulates equations for the power balance in built-up structures based only upon the assumptions of uniform density within the elements, linear motion and rain on the roof excitation [7]. The formulation is, for conservative coupling, approximately equal to the SEA formulation, if the element motion is resonant and if Green's function for a directly excited element is approximately equal to that for the uncoupled element; in which case coupling is said to be weak [7]. Green's function is evaluated by the iterative process applied for point excitation, becoming immediately clear that for small coupling eigenvalues it is asymptotically equal to that for the uncoupled element. Concluding, the coupling power hypothesis is valid for stochastically excited, albeit deterministic, structures if the coupling eigenvalue is much less than unity.

SEA considers the average response of an ensemble of similar structures. Here, investigations are made considering the ensemble defined when its members are equal to the investigated structure but the elements have stochastic frequencies.

Two oscillators (modes in connected structures) are considered first. The coupling power is written in uni-dimensional form, where the non-dimensional 'modal interaction strength' γ characterises the coupling. A value of unity of this parameter marks a bifurcation of the coupling power characteristics across the ensemble. For $\gamma<1$ coupling power is substantial, only when the frequency of vibration and the two uncoupled resonance frequencies are all equal and the maximum coupling power across the ensemble increases with increasing modal interaction strength. For $\gamma>1$, however, the maximum coupling power is independent of γ while the number of ensemble members that contribute to the ensemble averaged coupling power increases with increasing γ.

Ensemble averages of multi-modal structures are then considered. For small coupling eigenvalues, the SEA model describing the ensemble averaged energies is evaluated independently for the connected elements, if these have independent statistical properties. For a simple stochastic model where the uncoupled frequencies are assumed uniformly probable within a frequency band, the standard SEA conductivity is recovered. Also, so is the "standard SEA" formulation for two spring-coupled oscillators, which is presented in Ref. [19], repeated in Eq. (2.24) and demonstrated in Section 4.

The standard modal approach in SEA is based upon the assumption that the energy flow between sets of modes is given on a mode-to-mode basis [1, Fig. 3.4]. This is a valid assumption, if the coupling eigenvalue is less than unity; i.e., it is a valid assumption, for all members across an ensemble, if the modal interaction is small.

It is believed that SEA applies if: (I) the response in each element is governed by its resonances, i.e., by its local modes. (II) Coupling Power Proportionality holds. In this study, it is shown that, for spring coupled two-element structures; the second criterion is a consequence of the first. It is speculated: if the first assumption holds, the response of each element is given by its local solutions to the equations of motion (unless the forcing or coupling applies restraints—as, e.g., in periodic structures). The local solutions act independently and interact weakly, otherwise the first assumption would not hold. If, in addition, local solutions are statistically defined, the interaction is of a random nature and the situation resembles the one treated in statistical mechanics. Hence, the heat conduction analogy for SEA, i.e., coupling power proportionality.

To sum up, the coupling eigenvalues determine whether response in coupled structures is local or global. For small coupling eigenvalues, the coupling strength criteria set up by Newland [4,25], Bessac, Gagliardini and Guyader [11], Keane and Price [5] and Langley [7] are all fulfilled. Upon this basis, the SEA coupling power hypothesis is validated for general,

deterministic, spring coupled structures that are stochastically excited. A scheme for evaluating the SEA parameters based on modal data found, e.g., by the FEM is presented.

The modal interaction strength is the largest coupling eigenvalue found across an ensemble of structures that are similar to the investigated structure but they have random natural frequencies. The modal interaction strength characterises the ensemble averaged energy flow in uni-dimensional waveguides [8–10]. It characterises the energy flow between two coupled oscillators. Requiring the modal interaction strength to be small for two coupled oscillators is equal to requiring the Smith criterion [5] be fulfilled for all members of the ensemble.

The modal interaction strength is an ensemble property describing the character of mode coupling not the actual strength of coupling between particular modes which is, as shown by Scharton and Lyon [14], very sensitive to the difference in uncoupled frequencies. In contrast, the modal interaction strength depends on gross descriptors of the structure: the modal overlaps of the elements and the SEA conductivity found by considering the coupling of elements with random properties or, equally, semi-infinite elements. Thus, an SEA model can be assessed by its input data.

Finally, the significance of the modal interactions strength as an indicator of coupling strength is enhanced by that it can be measured: for spring coupled structures, C_s, defined by Fahy and James [12], is effectively an indicator for the modal interaction strength γ [19].

Acknowledgements

This work started at the ISVR, the University of Southampton, with support from the EPSRC (UK) and was continued at the MWL, Department of Vehicle Engineering, KTH, with support from TFR (Sweden). It is a pleasure to acknowledge the helpful and stimulating discussions with Roger Pinnington and Frank Fahy. The work was concluded with support from the Swedish Research Council (621-2005-5754) and The European Commission (Mid-Mod, Grant agreement no: 218508).

Appendix A. Two integrals

Much of the magic in SEA appears from the approximate evaluation of two integrals. These integrals appear frequently in the present work and a discussion of the approximations involved seems motivated. The first integral, $I1$, is used for calculations of the input power to a single oscillator (a mode in a substructure) excited by a white force. $I1$ is given by

$$I1 = \int_\Omega \mathrm{Re}\left(\frac{\mathrm{i}\omega\, d\omega}{(\omega_1^2-\omega^2)+\mathrm{i}\omega\omega_1\eta}\right) = \int_\Omega \frac{\omega^2\omega_1\eta\, d\omega}{(\omega_1^2-\omega^2)^2+(\omega\omega_1\eta)^2} \tag{A.1}$$

where ω_1 is the oscillator frequency, η is the loss factor and Ω is the frequency band. The frequency dependence of damping is chosen in the most convenient form, being a standard trick in SEA. It is allowed since it is very rare to have more detailed information about losses than an average loss factor in frequency bands.

The integral above is evaluated upon two further assumptions, both valid for weak damping: $\eta \ll 1$. First, it is assumed that the frequency band is wide enough to contain most of the power input. This is valid, with a relative error that is less than 10%, if the lower and upper limits of the frequency band Ω are at least three 3-dB bandwidths below and above the oscillator frequency, ω_1 [20]. Second, it is assumed that the oscillator bandwidth is so small that the value of the integral will not be altered by the following approximations:

$$\omega_1^2-\omega^2 = (\omega_1+\omega)(\omega_1-\omega) \approx 2\omega_1(\omega_1-\omega) \approx 2\omega(\omega_1-\omega). \tag{A.2}$$

Upon this basis, the integral, $I1$, is given by

$$I1 \approx \int_\Omega \frac{1}{4}\frac{\omega_1\eta\, d\omega}{(\omega_1-\omega)^2+(\omega_1\eta/2)^2} = \left[\frac{1}{2}atan\left(2\frac{\omega_1-\omega}{\omega_1\eta}\right)\right]_\Omega \approx \frac{\pi}{2}, \tag{A.3}$$

where the first approximate equality is given by the 'narrowness' of the oscillator resonance and the last one by the 'wideness' of the frequency band Ω.

The second integral, $I2$, is used for calculations of oscillator energy. $I2$ is, using the approximations above, given by

$$I2 = \int_\Omega \frac{(\omega^2+\omega_1^2)/2\, d\omega}{(\omega_1^2-\omega^2)^2+(\omega\omega_1\eta)^2} \approx \int_\Omega \frac{1}{4}\frac{d\omega}{(\omega_1-\omega)^2+(\omega_1\eta/2)^2} = \left[\frac{1}{2\omega_1\eta}atan\left(2\frac{\omega_1-\omega}{\omega_1\eta}\right)\right]_\Omega \approx \frac{\pi}{2\omega_1\eta}. \tag{A.4}$$

This result could, of course, have been reached from the value of $I1$ and upon requiring energy conservation for the oscillator. It is notable that upon the approximation (A.2), the integrals $I1$ and $I2$ are equally evaluated if the frequency ω is given while the natural frequency ω_1 is a rectangle distributed random variable.

Appendix B. Power balance

For completeness, Langley's power balance equations are derived in this Appendix. Linear theory and harmonic motion are assumed; without further limitation, the response of the coupled system can be expressed in the form [7]

$$w_i(\mathbf{x},\omega) = \sum_j \int_{V_j} G_{ij}(\mathbf{x},\mathbf{y},\omega) F_j(\mathbf{y},\omega)\,\mathrm{d}\mathbf{y}, \tag{B.1}$$

where w_i is the displacement in subsystem i, F_j is the applied force in subsystem j, V_j is the 'volume' of subsystem j and the Green function $G_{ij}(\mathbf{x},\mathbf{y},\omega)$ represents the response at location $\mathbf{x}$ on subsystem i to a harmonic point load situated at location $\mathbf{y}$ on subsystem j. The frequency averaged kinetic energy in element i is given by

$$T_i = \frac{1}{2\Omega}\int_\Omega \sum_j \sum_k \int_{V_i}\int_{V_j}\int_{V_k} \left(\rho_i(\mathbf{x})\omega^2 G_{ij}^*(\mathbf{x},\mathbf{y},\omega) G_{ik}(\mathbf{x},\mathbf{z},\omega) F_j^*(\mathbf{y},\omega) F_k(\mathbf{z},\omega)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\omega\right) \tag{B.2}$$

where ρ is the 'density' of subsystem i, Ω is the frequency band and F^* is the complex conjugate of F.

Assuming rain on the roof excitation, the applied forces are statistically independent and delta correlated in space. Thus the cross-spectrum, S_{jk}, of F_j and F_k has the form

$$S_{jk}(\mathbf{y},\mathbf{z},\omega) = \delta_{jk}\delta(\mathbf{y}-\mathbf{z})\alpha_j \tag{B.3}$$

This also incorporates the assumption that the cross-spectrum is independent of ω over the frequency band Ω. Upon these assumptions, the statistical expectations of the frequency averaged kinetic energies in the subsystems can be calculated [7]. The result is conveniently expressed in matrix form,

$$\mathbf{T} = \mathbf{M}\boldsymbol{\alpha}, \tag{B.4}$$

where the vector $\mathbf{T}$ contains the frequency averaged kinetic energies of the subsystem and the entries of $\mathbf{M}$ are given by

$$M_{ij} = \frac{1}{2\Omega}\int_\Omega \int_{V_j}\int_{V_i} \rho_i(\mathbf{x})\omega^2 \left|G_{ij}(\mathbf{x},\mathbf{y},\omega)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega. \tag{B.5}$$

By similar calculations, the frequency averages of the input powers are given by

$$\mathbf{P} = \mathbf{q}\boldsymbol{\alpha}, \tag{B.6}$$

where the diagonal matrix $\mathbf{q}$ has the entries

$$q_i = \frac{1}{\Omega}\mathrm{Re}\left(\int_\Omega \int_{V_i} -i\omega G_{ii}(\mathbf{x},\mathbf{x},\omega)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\omega\right). \tag{3.26}$$

From Eqs. (B.4) and (B.6) follows the relation between frequency averaged kinetic energies and input powers

$$\mathbf{T} = \mathbf{M}\mathbf{q}^{-1}\mathbf{P}. \tag{B.7}$$

To further simplify this, Langley considers subsystems that have constant density, however, simplification may be found for more general systems [27]. Thus, an average, or effective, density is defined for each element and for each excitation. These effective densities are collected in the matrix $\mathbf{R}$ with entries

$$R_{ij} = \frac{1}{2\Omega}\int_\Omega \int_{V_j}\int_{V_i} \rho_i\omega^2 \left|G_{ij}(\mathbf{x},\mathbf{y},\omega)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega/\overline{M}_{ij}, \tag{B.8}$$

where

$$\overline{M}_{ij} = \frac{1}{2\Omega}\int_\Omega \int_{V_j}\int_{V_i} \omega^2 \left|G_{ij}(\mathbf{x},\mathbf{y},\omega)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\omega. \tag{3.25}$$

Now it is assumed that the effective density is independent of which substructure is excited. This is valid when either the density is uniform within each subsystem or the form of the vibrations, the relative amplitudes of the modes or of the vibration waves, is independent of excitation. The first assumption is Langley's; the latter is in accordance with the SEA assumptions of equipartition of modal energies or of diffuse wave fields within each subsystem. Consequently, in mathematical terms, it is assumed that

$$R_{ij} = R_{ii} \quad \forall j. \tag{B.9}$$

Finally, to arrive at a symmetric equation similar to the one used in SEA, variables $\hat{\mathbf{E}}$ are introduced

$$\hat{\mathbf{E}} = \pi\mathbf{r}^{-1}\mathbf{q}^{-1}\mathbf{T}, \tag{3.24}$$

where the entries of the positive and diagonal matrix $\mathbf{r}$ are R_{ii}, Eq. (B.8). Upon this basis, Eq. (B.7) can be inverted to yield

$$\mathbf{C}\hat{\mathbf{E}} = \mathbf{P} \tag{3.22}$$

where the non-dimensional symmetric matrix $\mathbf{C}$ is given by

$$\mathbf{C} = (\pi\mathbf{q}^{-1}\overline{\mathbf{M}}\mathbf{q}^{-}1)^{-1} = (1/\pi)\mathbf{q}\overline{\mathbf{M}}^{-1}\mathbf{q}. \tag{3.23}$$

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