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Estimation of generalized frequency response functions for quadratically and cubically nonlinear systems

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ABSTRACT

A new method of estimating the Volterra kernels in the frequency-domain is introduced based on a non-parametric algorithm. Unlike the traditional non-parametric methods using the frequency-domain formulations based on discrete Fourier transform (DFT) data, this new approach uses the time-domain measurements directly to estimate the frequency-domain response functions.

1. Introduction

The Volterra series model, first proposed by Volterra [1], is a direct generalization of the linear convolution integral and provides an intuitive representation in a simple and easy to apply way. Volterra theory quickly received a great deal of attention in the field of electrical engineering, mechanical engineering, and later in the biological field, as a powerful approach for modeling nonlinear system behaviors. From the late 1950s, there has been a continuous effort in the application of Volterra series to a nonlinear systems theory. Summaries of major contributions in the application of Volterra series modeling for the representation, analysis, and design of nonlinear systems can be found in [2–5].

The Volterra series is associated with the so-called weakly nonlinear systems, which can be well described by the first few kernels with the higher order kernels falling off rapidly. The estimation of Volterra kernels in the time-domain has been studied in a specific form [6]. Recently, the problem of estimating the Volterra kernels for special classes of nonlinear system, Wiener–Hammerstein systems, was also studied [7].

The frequency-domain version of the Volterra kernels, called generalized frequency response functions (GFRFs), which can be obtained by taking the multiple Fourier transform of the Volterra kernels, has also been extensively studied. Usually people are more interested in frequency-domain Volterra kernels over the time-domain Volterra kernels because the former have an intuitive interpretation of many important nonlinear phenomena [8]. There are generally two classes of methods for the estimation of GFRF – parametric and non-parametric methods. For the parametric method, the input–output data measurements are used to identify a NARMAX model, from which a NARX (Nonlinear AutoRegressive Moving Average with eXogenous input) model, including purely input–output terms, after discarding the terms associated with the noise model, can be derived. Then the probing method [9] can be applied on this NARX model to obtain the GFRFs. The non-parametric method usually makes use of higher order spectral analysis based on the frequency-domain Volterra model [10–14]. Boyd et al. [15] proposed a non-parametric method of estimating the GFRFs based on the separation of the contribution of each Volterra kernel, using harmonic inputs. Due to the computational complexity associated with non-parametric methods, the Volterra kernels had to be restricted to low orders, for example, up to cubic nonlinearities.

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This paper is primarily concerned with the problem of estimating the GFRFs from harmonic input–output data for quadratically and cubically nonlinear systems. By expanding the algebraic expression of the response analysis in the frequency-domain, it is shown that the GFRFs can be estimated directly from the input–output time-domain measurements, which can significantly reduce the length of the data sets needed, compared with previous purely frequency-domain based methods.

2. Preliminaries

Volterra series modeling has been widely studied for the representation, analysis and design of nonlinear systems. For an SISO nonlinear system, where $u(t)$ and $y(t)$ are the input and output, respectively, the Volterra series can be expressed as

$$y(t)=\sum_{n=1}^{\infty} y_n(t)=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)d\tau_i \tag{1}$$

where $h_n(\tau_1,\ldots,\tau_n)$ is called the 'nth-order kernel' or 'nth-order impulse response function'. If $n=1$, this reduces to the familiar linear convolution integral.

The discrete time-domain counterpart of the continuous time-domain SISO Volterra expression (1) is

$$y(k)=\sum_{n=1}^{\infty} y_n(k)=\sum_{n=1}^{\infty}\sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} h_n(\eta_1,\ldots,\eta_n)\prod_{i=1}^{n} u(k-\eta_i),\quad k\in Z \tag{2}$$

In practice only the first few kernels are used on the assumption that the contribution of the higher order kernels falls off rapidly. Systems that can be adequately represented by a Volterra series with just a few terms are called weakly nonlinear systems.

For a weakly nonlinear system up to third-order Volterra series representation, the frequency-domain expression of the discrete time Volterra model (2) is given as

$$Y(\omega)=H_1(\omega)U(\omega)+\sum_{\substack{p,q\\ p+q=\omega}} H_2(p,q)U(p)U(q)+\sum_{\substack{l,m,n\\ l+m+n=\omega}} H_3(l,m,n)U(l)U(m)U(n) \tag{3}$$

where $Y(\omega)$ and $U(\omega)$ are the Fourier transforms of the output response and input, respectively, and $H_n(\omega_1,\ldots,\omega_n)$ is called the nth order GFRF which is obtained by taking the multidimensional Fourier transform of $h_n(\cdot)$:

$$H_n(\omega_1,\ldots,\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1,\ldots,d\tau_n \tag{4}$$

The GFRFs represent an inherent and invariant property of the underlying system, and have proved to be an important analysis and design tool for characterizing nonlinear phenomena.

In practice, by taking into account the output measurement noise (generally zero-mean white noise), (3) is replaced by

$$\tilde{Y}(\omega)=Y(\omega)+\varepsilon(\omega) \tag{5}$$

where $Y(\omega)$ is in the form of (3).

The excitations used in the application of non-parametric methods can be either Gaussian white noise or non-Gaussian harmonic inputs. For instance, Ref. [12] investigated the problem of estimation of GFRFs and system identification of a cubically nonlinear system based on (5), subject to a non-Gaussian input. Although (5) is nonlinear between the spectrum of the measured input/output $u(k)$ and $y(k)$, i.e., $U(\omega)$ and $Y(\omega)$, it is linear between $Y(\omega)$ and the unknown GFRFs $H_1(\cdot)$, $H_2(\cdot)$, and $H_3(\cdot)$, and the standard least squares type algorithms can be readily applied to obtain different orders of transfer functions. The possible disadvantages of the above purely frequency-domain based approaches are that large data sets are often needed and also the frequency-domain noisy term $\varepsilon(\omega)$, obtained from time-domain white Gaussian noise $e(k)$, may no longer be white, potentially resulting in bias in the estimates obtained from a least squares procedure.

Alternatively, the steady-state response of the nonlinear system that can be adequately represented by up to third-order Volterra kernels, excited by a harmonic signal at frequency ω, is given [2] by

$$\begin{aligned}y(k)=\hat{y}(k)+e(k)=A\,\mathrm{Re}\left\{H_1(\omega)e^{j\omega k}\right\}+2\left(\frac{A}{2}\right)^2\mathrm{Re}\left\{H_2(\omega,\omega)e^{j2\omega k}\right\}+2\left(\frac{A}{2}\right)^2\mathrm{Re}\{H_2(\omega,-\omega)\}\\ +2\left(\frac{A}{2}\right)^3\mathrm{Re}\left\{H_3(\omega,\omega,\omega)e^{j3\omega k}\right\}+6\left(\frac{A}{2}\right)^3\mathrm{Re}\left\{H_3(\omega,\omega,-\omega)e^{j\omega k}\right\}+e(k)\end{aligned} \tag{6}$$

where A is the amplitude of the input signal and $e(k)$ is a zero-mean Gaussian white noise. 'Re' represents the real part of a complex number.

Eq. (6) forms the basis of the current study which suggests an alternative and simple non-parametric approach of estimating GFRFs directly from noisy time-domain data. The study begins with quadratic nonlinear systems in Section 3, followed by cubic nonlinear systems in Section 4.

3. Estimation of frequency response functions of quadratically nonlinear systems

Because it is the simplest special case of the finite Volterra series model, the quadratic Volterra model class has been studied fairly extensively.

By considering the first two Volterra kernels in (6), the output response $y(k)$ can be expressed as

$$y(k)=\hat{y}(k)+e(k)=A\,\mathrm{Re}\left\{H_1(\omega)e^{j\omega k}\right\}+2\left(\frac{A}{2}\right)^2\mathrm{Re}\left\{H_2(\omega,\omega)e^{j2\omega k}\right\}+2\left(\frac{A}{2}\right)^2\mathrm{Re}\left\{H_2(\omega,-\omega)e^{j0k}\right\}+e(k) \tag{7}$$

Defining

$$H_1(\omega)=R_1(\omega)+jI_1(\omega) \tag{8}$$

where $R_1(\omega)$ and $I_1(\omega)$ are the real and imaginary parts of $H_1(\omega)$, respectively. For simplicity, $R_1(\omega)$and $I_1(\omega)$ will be written in abbreviated form as R_1 and I_1.

Then the first term in the right-hand side of (7) can be expanded as

$$A\,\mathrm{Re}\{H_1(\omega)e^{j\omega k}\}=A\,\mathrm{Re}\{[R_1+jI_1][\cos(\omega k)+j\sin(\omega k)]\}=AR_1\cos(\omega k)-AI_1\sin(\omega k) \tag{9}$$

Similarly by defining

$$H_2(\omega,\omega)=R_2(\omega,\omega)+jI_2(\omega,\omega) \tag{10}$$

and noting that $H_2(\omega,-\omega)=R_0$ is a constant, the second and the third terms on the right-hand side of (7) can be expanded as

$$2\left(\frac{A}{2}\right)^2\mathrm{Re}\left\{H_2(\omega,\omega)e^{j2\omega k}\right\}+2\left(\frac{A}{2}\right)^2\mathrm{Re}\left\{H_2(\omega,-\omega)e^{j0k}\right\}=2\left(\frac{A}{2}\right)^2R_2\cos(2\omega k)-2\left(\frac{A}{2}\right)^2I_2\sin(2\omega k)+2\left(\frac{A}{2}\right)^2R_0 \tag{11}$$

Combining (9) and (11) yields

$$y(k)=R_1\left[A\cos(\omega k)\right]+I_1\left[-A\sin(\omega k)\right]+R_2\left[2\left(\frac{A}{2}\right)^2\cos(2\omega k)\right]+I_2\left[-2\left(\frac{A}{2}\right)^2\sin(2\omega k)\right]+R_0 2\left(\frac{A}{2}\right)^2+e(k) \tag{12}$$

For $k=1$ to N, (12) can be arranged in the matrix form as

$$\mathbf{Y}=\mathbf{X}\boldsymbol{\theta}+\mathbf{E} \tag{13}$$

where $\mathbf{Y}=[y(N)\cdots y(1)]^T$, $\boldsymbol{\theta}=[R_1I_1R_2I_2R_0]^T$, $\mathbf{E}=[e(N)\cdots e(1)]$, and $\mathbf{X}=[\mathbf{x_1}\mathbf{x_2}\mathbf{x}_3\mathbf{x}_4\mathbf{x}_5]$ with

$$\begin{aligned}\mathbf{x_1}&=[A\cos(\omega N)\cdots A\cos(\omega)]^T\\ \mathbf{x}_2&=[-A\sin(\omega N)\cdots -A\sin(\omega)]^T\\ \mathbf{x}_3&=\left[2\left(\frac{A}{2}\right)^2\cos(2\omega N)\cdots 2\left(\frac{A}{2}\right)^2\cos(2\omega)\right]^T\\ \mathbf{x}_4&=\left[-2\left(\frac{A}{2}\right)^2\sin(2\omega N)\cdots -2\left(\frac{A}{2}\right)^2\sin(2\omega)\right]^T\\ \mathbf{x}_5&=\left[2\left(\frac{A}{2}\right)^2\cdots 2\left(\frac{A}{2}\right)^2\right]^T\end{aligned} \tag{14}$$

The estimation of the unknown frequency response functions $\hat{\boldsymbol{\theta}}=[R_1I_1R_2I_2R_0]^T$can now be derived from (13) using a standard least squares procedure. If the truncation error is sufficiently small, on the assumption that third and higher orders of Volterra kernels make a negligible contribution to the output, then the estimation $\hat{\boldsymbol{\theta}}$ will be unbiased. Unlike the previous complex estimator based on (5), the new estimator is in the real domain based on time-domain measurements. The new approach is illustrated using a simulation example.

Consider a system described as

$$\ddot{y}+a\dot{y}+by+cy^2=u(t) \tag{15}$$

where $u(t)=A\cos(\omega t)$.

In the continuous time-domain, the GFRFs can be derived [16] as

$$\begin{aligned}H_1(s)\big|_{s=j\omega}&=\frac{1}{s^2+as+b}\\ H_2(s_1,s_2)\Big|_{\substack{s_1=j\omega_1\\ s_2=j\omega_1}}&=-cH_1(s_1)H_1(s_2)H_1(s_1+s_2)\end{aligned} \tag{16}$$

System (15), which has a quadratic nonlinear term y^2, is not a quadratically but infinitely nonlinear system in terms of the Volterra series representation because the nonlinearity is on the output. However for a considerable range of input level the system can be adequately represented by up to second-order Volterra kernels, and importantly all the coefficients

Table 1
Estimation of $\hat{H}_1$ and $\hat{H}_2$ of system (15) using the new approach.

	From Eq. (15)—true	From new approach
$H_1(\omega)=R_1+jI_1$	$-0.6151-0.1262j$	$-0.6100-0.1250j$
$H_2(\omega,\omega)=R_2+jI_2$	$0.0038+0.0019\,j$	$0.0043+0.0022j$
$H_2(\omega,-\omega)=R_0$	-0.0394	-0.0397

Table 2
Estimation of the original continuous time system (15) using (17).

	a	*b*	*c*
True value	0.2	1.0	0.1
Estimates from new approach	0.2015	0.9867	0.1011

of the underlying system (15) can be fully characterized by the first two orders of Volterra kernels or the associated frequency response functions.

To illustrate the new GFRF estimation algorithm, system (15) was excited by a single sinusoidal input at A=2 and$\omega=1.6\,\text{rad}/s$ for a=0.2, b=1, and c=0.1. A total amount of 2000 input and output data were collected at a sampling time T_s=0.01 s and a zero-mean white noise was added to the output with SNR=40 dB.

The proposed new procedure (12)–(14) is applied to obtain the $\hat{H}_1$and $\hat{H}_2$ using a least squares estimator. The results are shown in Table 1 which are very satisfactory.

To induce a further verification of the accuracy of the above estimation, the original continuous time system parameters, from a simple quadratic format as in (15), can be extracted from the estimates of $\hat{H}_1$ and $\hat{H}_2$ in Table 1 using (16), as

$$\hat{a}=\frac{1}{\omega}\text{Im}\left[\frac{1}{\hat{H}_1(\omega)}\right]$$
$$\hat{b}=\text{Re}\left[\frac{1}{\hat{H}_1(\omega)}\right]+\omega^2$$
$$\hat{c}=\frac{-\hat{H}_2(\omega,-\omega)}{\hat{H}_1(\omega)\hat{H}_1(-\omega)\hat{H}_1(0)}=\frac{-\hat{b}\hat{H}_2(\omega,-\omega)}{\hat{H}_1(\omega)\hat{H}_1(-\omega)} \tag{17}$$

The reconstructed parameters of the continuous time model (15) from the estimates of $\hat{H}_1$ and $\hat{H}_2$ in Table 1 using (17) are shown in Table 2, which are very satisfactory compared with the true parameters.

4. Estimation of frequency response functions of cubically nonlinear systems

The literature associated with the cubic Volterra series model is substantially smaller compared with that associated with the quadratic Volterra model due to the significantly greater complexity induced. In terms of frequency response function estimation for up to third-order Volterra representation, this procedure is not as straightforward as the quadratic case in Section 3. In fact, the main complication is that, unlike the quadratic Volterra model case where the first harmonics in the output are all due to the linear response function, for a cubic Volterra model both first and third-order frequency response functions make contributions to the first harmonics. Therefore, additional efforts are needed to solve the separation of contributions between H_1 and H_3.

First, by defining

$$H_3(\omega,\omega,\omega)=R_3(\omega,\omega,\omega)+jI_3(\omega,\omega,\omega) \tag{18}$$

the fourth term in (6) can be expanded as

$$2\left(\frac{A}{2}\right)^3\text{Re}\left\{H_3(\omega,\omega,\omega)e^{j3\omega k}\right\}=2\left(\frac{A}{2}\right)^3R_3\cos(3\omega k)-2\left(\frac{A}{2}\right)^3I_3\sin(3\omega k) \tag{19}$$

which makes a contribution purely to the third harmonics (3ω) in the response.

The problem arises with the fifth term in (6). By defining

$$H_3(\omega,\omega,-\omega)=R_{31}(\omega,\omega,-\omega)+jI_{31}(\omega,\omega,-\omega) \tag{20}$$

the fifth term in (6) can be expanded as

$$6\left(\frac{A}{2}\right)^3 \mathrm{Re}\left\{H_3(\omega,\omega,-\omega)e^{j\omega k}\right\} = 6\left(\frac{A}{2}\right)^3 R_{31}\cos(\omega k) - 6\left(\frac{A}{2}\right)^3 I_{31}\sin(\omega k) \tag{21}$$

which makes a contribution to the first harmonics, mixed up with the contribution from the linear kernel, shown in (9).

The overall expansion of (6) for the first three Volterra kernel terms, by combining (12), (19), and (21), is given as

$$y(k) = \left[AR_1 + 6\left(\frac{A}{2}\right)^3 R_{31}\right]\cos(\omega k) + \left[AI_1 + 6\left(\frac{A}{2}\right)^3 I_{31}\right][-\sin(\omega k)] + R_2\left[2\left(\frac{A}{2}\right)^2\cos(2\omega k)\right] + I_2\left(-2\left(\frac{A}{2}\right)^2\sin(2\omega k)\right)$$
$$+R_0 2\left(\frac{A}{2}\right)^2 + 2\left(\frac{A}{2}\right)^3 R_3\cos(3\omega k) - 2\left(\frac{A}{2}\right)^3 I_3\sin(3\omega k) + e(k) \tag{22}$$

It is possible to define

$$AR_1 + 6\left(\frac{A}{2}\right)^3 R_{31} = R$$
$$AI_1 + 6\left(\frac{A}{2}\right)^3 I_{31} = I \tag{23}$$

Then for k=1–N, (22) can be arranged in the matrix form as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\theta} + \mathbf{E} \tag{24a}$$

where $\mathbf{Y} = [y(N)\cdots y(1)]^T$, $\boldsymbol{\theta} = [RIR_2I_2R_0R_3I_3]^T$, $\mathbf{E} = [e(N)\cdots e(1)]$, and $\mathbf{X} = [\mathbf{x_1}\mathbf{x_2}\mathbf{x}_3\mathbf{x_4}\mathbf{x}_5\mathbf{x_6}\mathbf{x}_7]$ with

$$\mathbf{x_1} = [\cos(\omega N)\cdots\cos(\omega)]^T$$
$$\mathbf{x}_2 = [-\sin(\omega N)\cdots-\sin(\omega)]^T$$
$$\mathbf{x}_3 = \left[2\left(\frac{A}{2}\right)^2\cos(2\omega N)\cdots 2\left(\frac{A}{2}\right)^2\cos(2\omega)\right]^T$$
$$\mathbf{x}_4 = \left[-2\left(\frac{A}{2}\right)^2\sin(2\omega N)\cdots -2\left(\frac{A}{2}\right)^2\sin(2\omega)\right]^T$$
$$\mathbf{x}_5 = \left[2\left(\frac{A}{2}\right)^2\cdots 2\left(\frac{A}{2}\right)^2\right]^T$$
$$\mathbf{x}_6 = \left[2\left(\frac{A}{2}\right)^3\cos(3\omega N)\cdots 2\left(\frac{A}{2}\right)^3\cos(3\omega)\right]^T$$
$$\mathbf{x}_7 = \left[-2\left(\frac{A}{2}\right)^3\sin(3\omega N)\cdots -2\left(\frac{A}{2}\right)^3\sin(3\omega)\right]^T \tag{24b}$$

from which the unknown frequency response function data $\hat{\boldsymbol{\theta}} = [R\ \ I\ \ R_2\ \ I_2\ \ R_0\ \ R_3\ \ I_3]^T$ can be derived using a standard Least Squares procedure. In order to separate in (23) the R_1 and R_{31} from R, and the I_1 and I_{31} from I, two tests at different input levels, denoted by $A^{(1)}$ and $A^{(2)}$, at the same frequency, are required to obtain two sets of estimates, $R^{(1)}$, $R^{(2)}$, and $I^{(1)}$ and $I^{(2)}$, respectively. This results in the following expressions:

$$A^{(1)}\hat{R}_1 + 6\left(\frac{A^{(1)}}{2}\right)^3\hat{R}_{31} = R^{(1)}, \quad A^{(1)}\hat{I}_1 + 6\left(\frac{A^{(1)}}{2}\right)^3\hat{I}_{31} = I^{(1)}$$
$$A^{(2)}\hat{R}_1 + 6\left(\frac{A^{(2)}}{2}\right)^3\hat{R}_{31} = R^{(2)}, \quad A^{(2)}\hat{I}_1 + 6\left(\frac{A^{(2)}}{2}\right)^3\hat{I}_{31} = I^{(2)} \tag{25}$$

Then the final estimates of $\hat{H}_1 = \hat{R}_1 + j\hat{I}_1$ and $\hat{H}_3(\omega,\omega,-\omega) = \hat{R}_{31} + j\hat{I}_{31}$ can be calculated from (25) as

$$\hat{R}_1 = \frac{6(A^{(2)}/2)^3R^{(1)} - 6(A^{(1)}/2)^3R^{(2)}}{A^{(1)}6(A^{(2)}/2)^3 - A^{(2)}6(A^{(1)}/2)^3} \quad \hat{R}_{31} = \frac{A^{(2)}R^{(1)} - A^{(1)}R^{(2)}}{A^{(2)}6(A^{(1)}/2)^3 - A^{(1)}6(A^{(2)}/2)^3}$$
$$\hat{I}_1 = \frac{6(A^{(2)}/2)^3I^{(1)} - 6(A^{(1)}/2)^3I^{(2)}}{A^{(1)}6(A^{(2)}/2)^3 - A^{(2)}6(A^{(1)}/2)^3} \quad \hat{I}_{31} = \frac{A^{(2)}I^{(1)} - A^{(1)}I^{(2)}}{A^{(2)}6(A^{(1)}/2)^3 - A^{(1)}6(A^{(2)}/2)^3} \tag{26}$$

The above procedure will be illustrated using the well-known Duffing oscillator.

Consider a Duffing oscillator, with cubic nonlinearity, subject to a sinusoidal excitation as

$$m\ddot{y} + c\dot{y} + k_1y + k_3y^3 = A\cos(\omega t) \tag{27}$$

where m, c, k_1, and k_3 are the mass, the damping, the linear stiffness, and nonlinear stiffness, respectively. The nonlinear stiffness parameter k_3 in (27) needs to stay small in order to be 'weakly' nonlinear for the existence of the Volterra series representation. The corresponding GFRFs from (27) are

$$H_1(s)\Big|_{s=j\omega} = \frac{1}{s^2+cs+k_1}$$

$$H_2(s_1,s_2)\Big|_{\substack{s_1=j\omega_1\\ s_2=j\omega_2}} = 0$$

$$H_3(s_1,s_2,s_3)\Big|_{\substack{s_1=j\omega_1\\ s_2=j\omega_2\\ s_3=j\omega_3}} = -k_3H_1(s_1)H_1(s_2)H_1(s_3)H_1(s_1+s_2+s_3) \tag{28}$$

System (27) was excited at ω=1.5 rad/s and T_s=0.02 s with the parameters

$$m=1, \quad c=1.5, \quad k_1=0.5, \quad k_3=0.1 \tag{29}$$

The excitation is chosen as $A^{(i)}\cos(\omega t)$, $i=1,2$ where $A^{(1)}$=2 and $A^{(2)}$=3. A zero-mean white noise was added to each of the outputs with a SNR=40 dB. The length of the data was 2000. The estimation results for $\hat{H}_1$and $\hat{H}_3$ using the new algorithm are given in Table 3.

Again, for the simple cubic format presented in (27), the original continuous time system parameters can be extracted from $\hat{H}_1$ and $\hat{H}_3$ using the relationship in (28), as

$$\hat{c} = \frac{1}{\omega}\mathrm{Im}\left(\frac{1}{\hat{H}_1(\omega)}\right)$$

$$\hat{k}_1 = \mathrm{Re}\left(\frac{1}{\hat{H}_1(\omega)}\right)+\omega^2$$

$$\hat{k}_3 = \frac{-\hat{H}_3(\omega,\omega,-\omega)}{\hat{H}_1(\omega)\hat{H}_1(\omega)\hat{H}_1(\omega)\hat{H}_1(-\omega)} \tag{30}$$

which are shown in Table 4. It is clear from Table 4 that again the reconstructed continuous time system parameters, compared with the true system parameters in (29), are very satisfactory, which further confirms the effectiveness of the new approach.

In [12], a simulation study on the motion of a simple pendulum (expressed in the form of Duffing's equation) was presented using the higher order spectral analysis based approach using (5) to estimate the GFRFs and subsequently the continuous time equation parameters subject to a single tone sinusoidal input. The Duffing equation was expressed as

$$\ddot{y}+a\dot{y}+by+cy^3 = A\cos(\omega t) \tag{31}$$

where a=3, b=2, and c= − 1/3.

For comparison, system (31) was studied using the new approach proposed in this study, with the simulation parameters $\omega = 1\,\mathrm{rad/s}$ and $T_s = \pi/60$, at two different excitations $A^{(1)}$=1 and $A^{(2)}$=1.5. 2000 data points were used in each realization. Three different levels of noise were added to the responses. The estimated final results are listed in Table 5. It can be seen that the results from the new approach produce slightly more accurate estimates than those in [12] (Section 5) where a large number of realizations at different amplitude are employed, each of the realization needing 600 segments of 64 data points. This suggests that the proposed new approach is potentially more efficient in terms of data length. This is likely to be especially useful in situations where it is not possible to obtain a sufficiently large set of data samples such that conventional frequency-domain methods can be applied. Apart from the large data samples needed, the conventional

Table 3
Estimation of H_1 and H_3 for the Duffing equation (27) using the new approach.

	Eq. (27)—true	From new approach
$H_1(\omega)=R_1+jI_1$	$-0.2154–0.2769j$	$-0.2159–0.2776j$
$H_3(\omega,\omega,\omega)=R_3+jI_3$	2.0616e − 04–1.7152e − 05j	2.011e − 04–1.8124e − 04j
$H_3(\omega,\omega,-\omega)=R_{01}+jI_{31}$	0.3729e − 03–0.1468e − 02j	0.4595e-03–0.1579e − 02j

Table 4
Estimation of parameters of original continuous time system (27) using (30).

	c	k_1	k_3
True value from (29)	1.5	0.5	0.1
Estimates from (30)	1.4960	0.5037	0.0991

Table 5
Comparison of the estimation of system (31) using the new and the conventional approaches.

Noise level (SNR) (dB)	True values of the parameters	Estimation results in [12]	Estimation result using the new approach
30	$a=3$	2.939	3.0068
	$b=2$	1.956	1.9911
	$c=-1/3$	N/A	−0.3403
20	$a=3$	2.939	3.0119
	$b=2$	1.955	1.9836
	$c=-1/3$	N/A	−0.3421
10	$a=3$	2.940	3.0209
	$b=2$	1.954	1.9703
	$c=-1/3$	N/A	−0.3455

approach involves additional procedures in mapping the time-domain data into the frequency-domain and separating discrete Fourier transforms at different frequency components, whereas the new approach avoids the frequency-domain manipulations. The results in Table 5 also suggest that this new approach is quite robust to different levels of white noise.

5. The selection of level of excitation using OLS

The new algorithm introduced in the previous sections is based on the assumption that the underlying system is weakly nonlinear in the sense that it can be well described by the first few Volterra kernels, with the higher order kernels fading off rapidly. The accuracy of the results of the new procedure is largely dependant on this assumption as are all other non-parametric methods, and the significance of the nonlinearity of the system, reflected by the Volterra kernel order, is associated with the level of the excitation. It is generally agreed that estimating the rapidly decreasing coefficients of a noisy polynomial is inherently difficult [15, 17–20]. Either under-excitation or over-excitation may result in inaccuracy in the estimates. It is therefore essential to have a measure that can provide an indication of the order of the Volterra kernels under certain levels of excitation before the final application of the new algorithm. One possible measure is based on using the orthogonal least squares method (OLS) [21], which is briefly reviewed below.

Consider a general system expressed by the linear-in-the-parameters model

$$z=\sum_{i=1}^{M}\theta_i p_i+\varepsilon \tag{32}$$

where θ_i,$i=1,\ldots,M$ are unknown parameters.

Reformulating Eq. (32) in the form of an auxiliary model yields

$$z=\sum_{i=1}^{M}g_i w_i+\varepsilon \tag{33}$$

where g_i,$i=1,\ldots,M$ are the auxiliary parameters and w_i, $i=1,\ldots,M$ are constructed to be orthogonal over the data record such that

$$\sum_{t=1}^{N}w_j(t)w_{k+1}(t)=0,\quad j=0,1,\ldots,k \tag{34}$$

where N is the length of the data record.

Multiplying the auxiliary model (33) by itself, using the orthogonal property (34) and taking the time average gives

$$\frac{1}{N}\sum_{t=1}^{N}z^2(t)=\frac{1}{N}\sum_{t=1}^{N}\left\{\sum_{i=0}^{M}g_i^2 w_i^2(t)\right\}+\frac{1}{N}\sum_{t=1}^{N}\varepsilon^2(t) \tag{35}$$

Finally define

$$ERR_i=\frac{\sum_{t=1}^{N}g_i^2 w_i^2(t)}{\sum_{t=1}^{N}z^2(t)-1/N\{\sum_{t=1}^{N}z(t)\}^2}\times 100 \tag{36}$$

for $i=1,2,\ldots,M$. The quantity ERR_i is called the error reduction ratio and provides an indication of which terms should be included in the model in accordance with the contribution each term makes to the energy of the dependent variable. Terms with associated ERR values which are less than a pre-defined threshold value can be considered to be insignificant and negligible.

For simplicity, the quadratic system (15), with $a=0.2$, $b=1$, and $c=0.1$, was used as an example to illustrate the use of OLS in the selection of the amplitude of excitation A. Two tests were conducted at different excitation amplitudes at frequency $\omega=2$ rad/s. First, the amplitude of the input was chosen at $A=0.3$ and the response was corrupted by a

Table 6
Estimation of H_1 and H_2 of system (15) at A=0.3 using OLS.

	Result by Eq. (15)—true	Result by OLS	ERR_i (%)
R_1	−0.3275	−0.3276	97.3370
I_1	−0.04367	−0.04349	1.6999
R_2	6.902e−04	−9.519e−04	1.867e−05
I_2	2.275e−04	1.980e−03	7.8622e−05
R_0	−0.01092	−0.01459	8.704e−03

Table 7
Estimation of H_1 and H_2 of system (15) at A=5 using OLS.

	Result by Eq. (15)—true	Result by OLS	ERR_i (%)
R_1	−0.3275	−0.3246	96.117
I_1	−0.04367	−0.04315	1.5499
R_2	6.902e−04	5.753e−04	1.892e−03
I_2	2.275e−04	2.485e−03	3.437e−04
R_0	−0.01092	−0.01096	1.3518

zero-mean white noise with SNR=40 dB. The OLS was applied to obtain the estimates of linear and quadratic frequency response functions, together with the values ERR_i, shown in Table 6.

It can be seen from Table 5 that at this level of excitation, the contributions by the quadratic term H_2, i.e., R_2, I_2, and R_0, are extremely small compared with the contributions from the linear term. This means that the quadratic kernel contribution has a very small SNR, consequently the least squares estimation results for the quadratic terms are more affected by the presence of noise, leading to unreliability in the estimates. The almost negligible sum of ERR of the quadratic terms, i.e., from terms R_2, I_2, and R_0, suggests that the system can be regarded as linear at this level of excitation, therefore this can be considered as under-excited.

Now the amplitude of input was chosen at A=5 and again a zero-mean white noise was added to the response with SNR=40 dB. The OLS results are given in Table 7.

The sum of the values ERR_i of all the linear and quadratic terms is 99.0214%, indicating that this system at this amplitude level can be sufficiently described by up to second-order Volterra kernels. It is also clear by looking at the ERR_i values from Table 7 that the overall contributions by the quadratic terms, especially the R_0 term are no longer negligible. As a result, the accuracy of the estimates of the quadratic response functions becomes very satisfactory, but the accuracy of the estimates of the linear response function is a little bit poorer compared with the lower excitation result in Table 6.

The above analysis suggests that in real applications of the proposed new approach, a sequential procedure could be employed, by firstly estimating the linear response function at a relatively low level of excitation, and then estimating the quadratic frequency response function at a higher level of excitation where the nonlinear part plays a more significant role, etc. The appropriate choice of the level of excitation in each step can be guided by the values of ERR_i.

6. Generalization to multitone inputs

It needs to be pointed out that the approach presented in the above sections can be easily extended to admit 2-tone or multitone sinusoidal inputs. The complexity, however, will inevitably increase dramatically as the number of the tones and the order of the nonlinearity grow, like the cases in all other Volterra kernel related estimation problems. For simplicity, assume the input is a 2-tone signal in the form

$$u(k)=\sum_{i=1}^{2}A_i\cos(\omega_i k),\quad \omega_2>\omega_1>0 \tag{37}$$

and the system can be sufficiently described by first and quadratic Volterra kernels, then the single-tone response expression (7) can be extended to

$$\begin{aligned} y(k)=&\sum_{i=1}^{2}A_i\,\mathrm{Re}\{H_1(\omega_i)e^{j\omega_i k}\}+\sum_{i=1}^{2}2\left(\frac{A_i}{2}\right)^2\mathrm{Re}\left\{H_2(\omega_i,\omega_i)e^{j2\omega_i k}\right\}+\sum_{i=1}^{2}2\left(\frac{A_i}{2}\right)^2\mathrm{Re}\{H_2(\omega_i,-\omega_i)\}\\ &+4\left(\frac{A_1}{2}\frac{A_2}{2}\right)\mathrm{Re}\left\{H_2(\omega_1,\omega_2)e^{j(\omega_1+\omega_2)k}\right\}+4\left(\frac{A_1}{2}\frac{A_2}{2}\right)\mathrm{Re}\left\{H_2(-\omega_1,\omega_2)e^{j(\omega_2-\omega_1)k}\right\}+e(k) \end{aligned} \tag{38}$$

Eq. (38) can be further expanded to a form similar to (12) involving real and imaginary parts of the individual GFRFs, from which $H_1(\omega_1)$, $H_1(\omega_2)$, $H_2(\omega_1,\omega_1)$, $H_2(\omega_2,\omega_2)$, and $H_2(\omega_1,\omega_2)$, etc. can be estimated using the least squares procedure illustrated in Section 3. If the separation of the contributions by $H_2(\omega_1,-\omega_1)$ and $H_2(\omega_2,-\omega_2)$, which are two d.c.

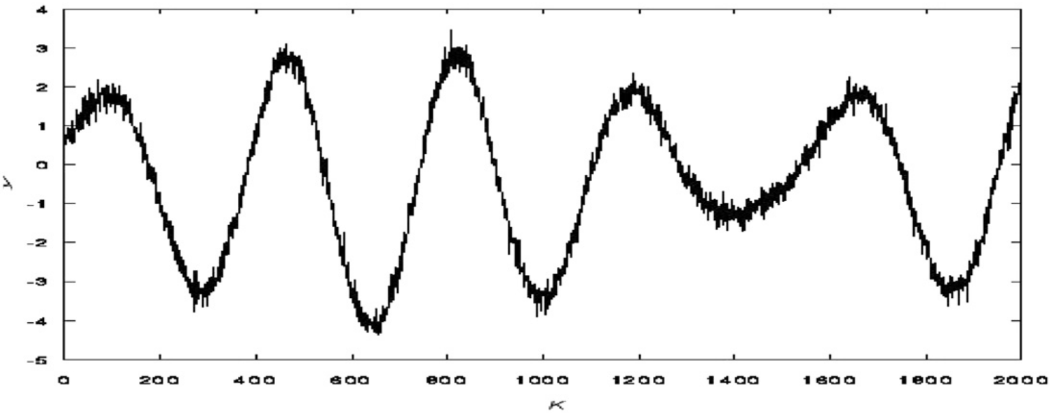

Fig.1. Response of system (15) under the excitation of a 2-tone input.

Table 8
Estimation of $\hat{H}_1$ and $\hat{H}_2$ of system (15) using the new approach.

	From Eq. (15)—true	Estimates from new approach
$H_1(\omega_1)$	$-0.6151-0.1262j$	$-0.5958-0.1119j$
$H_1(\omega_2)$	$-0.3275-0.0437j$	$-0.3191-0.0418j$
$H_2(\omega_1,\omega_1)$	$0.0038+0.0019j$	$0.0052+0.0014j$
$H_2(\omega_2,\omega_2)$	$0.0007+0.0002j$	$0.0008+0.0002j$
$H_2(\omega_1,\omega_2)$	$0.0016+0.0007j$	$0.0017+0.0008j$
$H_2(-\omega_1,\omega_2)$	$-0.0243+0.0040j$	$-0.0233+0.0025j$
d.c. component	-0.2892	-0.2784

components, are of interest, then the 2-tone signal at 2 levels for each frequency will be applied and the similar procedure like those in (25)–(26) in Section 4 can be followed to separate $H_2(\omega_1,-\omega_1)$ and $H_2(\omega_2,-\omega_2)$.

System (15) is used as an example to demonstrate the direct estimation of $H_2(\omega_1,\omega_2)$, etc. by using a 2-tone input.

System (15), with a=0.2, b=1, and c=0.1, was excited by a 2-tone sinusoidal input in the form of (37) with $A_1=3.2$, $\omega_1=1.6\,\mathrm{rad}/s$ and $A_2=4$, $\omega_2=2\,\mathrm{rad}/s$. A total amount of 2000 input and output data were collected at a sampling time T_s=0.01 s and a zero-mean white noise was added to the output with SNR=40 dB.

The observation on the collected response samples, shown in Fig. 1, reveals that the response contains significant noise and contains a number of frequency components.

The proposed new procedure is applied to obtain the $\hat{H}_1$and $\hat{H}_2$ using a least squares estimator. The results are shown in Table 8 which are quite satisfactory. Please note that the d.c. component in Table 7 is the mixed contribution of $H_2(\omega_1,-\omega_1)$ and $H_2(\omega_2,-\omega_2)$. A separation of the $H_2(\omega_1,-\omega_1)$ and $H_2(\omega_2,-\omega_2)$contribution would require another 2-tone excitation at different levels.

For a cubically nonlinear systems, a 3-tone excitation can be applied, which will generate a response possessing up to 22 different frequencies at 3 orders of kernels. In order to obtain the full spectrum estimation of GFRFs for a cubically nonlinear system, at least 3 levels of excitation at each harmonics are needed for the separation of the GFRFs at different orders.

7. Conclusions

A new non-parametric algorithm, which directly uses the time-domain input–output measurements but avoids the direct differentiation of these data, has been derived to estimate up to the third-order GFRFs and subsequently identify the associated continuous time model. This is achieved by expanding the algebraic expression in the analysis of the output response using the real and imaginary parts of each order of Volterra kernels. The new algorithm has the advantage of admitting smaller data sets than the traditional spectral analysis based frequency-domain non-parametric methods.

Like many other general parameter estimation problems, the accuracy of the estimation of the GFRFs in this new approach depends on the fact that the level of excitation is appropriate. That is, the relevant order of nonlinearity needs to be adequately excited. The OLS method has superior numerical properties compared with the ordinary least squares method, in the sense that it can provide vital information on the suitability of the excitation level, indirectly from the contribution indicators (ERR) for each Volterra order.

The new approach illustrated using single-tone sinusoidal inputs can be readily extended to accommodate 2-tone or even multitone inputs, to directly obtain the higher order GFRFs in the full spectrum. Nevertheless in those cases the complexity of the procedure will inevitably grow dramatically as the number of tones in the input and the order of Volterra kernels are increased.

An important aspect of the new method is that the continuous time model can be reconstructed from the estimated GFRFs. A direct extraction of the continuous time model for the simple low dynamic order system is possible from the GFRFs at one single frequency point, as illustrated in this paper. When the continuous time model structure is not known *a priori*, or is in a more complicated form, this new procedure can be repeated at different excitation frequencies until a sufficient number of points have been collected, from which the identification of the general form of continuous time model can be derived using the approach in [22].

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