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Robust non-fragile dynamic vibration absorbers with uncertain factors

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ABSTRACT

In this paper, the design problem for non-fragile dynamic vibration absorbers (DVAs) is investigated. Due to the imprecision of the manufacturing process or the variation during the operation, uncertainty in the parameters of the DVA is unavoidable. The uncertainty may degrade the performance of the designed DVA or even deteriorate the system. Hence, it is practically demanding to propose a design method for a non-fragile DVA, i.e., when the parameters of the DVA vary in an admissible range, an expected vibration suppression level should be guaranteed. The uncertainty of the DVA is feasibly assumed to be norm-bounded. Then, the design problem for the DVA is converted into a static output feedback (SOF) control problem. Sufficient condition for the existence of the non-fragile DVA with a prescribed H_∞ level is derived by using a bilinear matrix inequality (BMI). An iterative linear matrix inequality (ILMI) method is employed to solve the BMI condition. Finally, a design example is given to show the effectiveness of the proposed approach.

1. Introduction

Vibrations are inevitable in practical mechanical systems. As vibrations affect the safety of structures and their service life, the suppression of the vibrations of machines is very important during their operation. Therefore, how to suppress vibrations is always a key issue in the machine design and applications.

The dynamic vibration absorber (DVA) which consists of mass and spring components is attached on a machine (primary system). When the primary system is excited by external disturbances, the vibration of the machine can be attenuated by the DVA. The DVA is a very simple yet effective device to suppress the vibration of the machine especially for the vibration at frequencies close to the natural frequency of the system. It has been applied to various applications since its invention by Frahm in 1909 [1,2].

As new applications emerge, more stringent requirements arise on the design of DVAs. Research on the DVA has remained very active. In general, there are two main approaches to improve the performance of the DVA.

- The first approach is to introduce an extra feedback control system to form the active DVA or semi-active DVA. By using linear quadratic Gaussian (LQG) control, the authors in [3] presented an active control scheme of saw blade vibrations. As pointed out in [4], semi-active vibration absorbers have some advantages over conventional passive vibration absorbers. Several different algorithms by making use of the information required on the impact load and system states were proposed in [4].

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- The other main approach is to configure new structures. In traditional passive DVAs, there are generally an inertia, a spring, and a damper. Nowadays, in order to have a better performance, passive DVAs may have various new structures. Advanced control theories have been applied to achieve optimal design for different structures. In [5], an additional spring element was proposed to be connected with the damper in series; in order to minimize the disturbed output signal subject to a unit white noise input, the resulting optimization problem of the three-element type absorber was solved based on the H_2 performance index. In [6], an added mass onto the structure was studied. The innovative multi-DOF DVA was proposed in [7] by Zuo and Nayfeh. Based on this concept, further research on the performance analysis and comparison studies were further discussed in [8].

In practice, uncertainties existing in the stiffness and the damping of the DVA are unavoidable. However, in the literature as discussed above, this important and practical issue on uncertainties has not been considered, to the best of our knowledge. The uncertainty is generally caused by the lack of precision in the manufacturing process or variations during operation. As unavoidable uncertainty arises in the DVA, the performance of the DVA using the existing methods cannot be guaranteed. A particularly bad situation occurs when the DVA does not suppress the vibration of the primary system, but aggravates the vibration due to the uncertainty. Hence it is practically demanding to consider the uncertainty in the DVA when designing the system. A DVA is called non-fragile if it still works well when the parameters of the stiffness and damping vary. To propose such a non-fragile DVA concept is motivated by robust non-fragile control that has recently attracted lots of attention, see [9–12], to name a few.

In this paper, we investigate the design problem of non-fragile passive DVAs. As mentioned in [7,8], the design problem of passive DVAs can be converted into a static output feedback (SOF) controller design problem. We assume that the excitation of the primary system is L_2-norm bounded (the energy of the excitation is bounded), and the uncertainties in the parameters of the DVA are norm-bounded. Our objective is to minimize the energy of the amplitude of the machine's vibration, which can be cast as an H_∞ optimization problem. By employing the Lyapunov theory, sufficient condition for the existence of H_∞ non-fragile DVA design is expressed in terms of a bilinear matrix inequality (BMI). The BMI problem is complicated and there lack efficient solvers for it. Though various methods have been proposed to deal with the BMI problem, it is hard to guarantee the convergence. An improved iterative linear matrix inequality (ILMI) has been presented recently in [13] in which the algorithm converges quickly. Inspired by the work in [13–15], by solving the H_∞ performance criterion, the derived BMI assures the existence of the non-fragile DVA design.

2. Problem formulation and preliminaries

Consider a machine with a passive DVA as shown in Fig. 1. The mass of the machine is m_s and the machine is placed on a spring–damper system. k_s and c_s are the stiffness and damping of the spring–damper system, respectively. X_0 is the vibration from the ground and is assumed to be bounded, and X_s is the vibration of the primary system. In order to protect the machine, the magnitude of X_s is expected to be as smaller as possible when the machine is excited by an external noise (vibration from the ground). A DVA, consisting of a mass m_1, a spring k_1 and a damper c_1 is attached on the primary system to suppress the vibration of the machine. In general, the mass of the DVA should be much smaller than the mass of the machine, for example 1–10 percent of the mass of the primary system. Moreover, X_1 denotes the vibration of the DVA. The stiffness k_1 and the damping c_1 of the DVA are parameters which should be appropriately chosen such that the vibration of the machine is suppressed. Note that the variations in k_1 and c_1 are needed to be taken into account. We assume that the

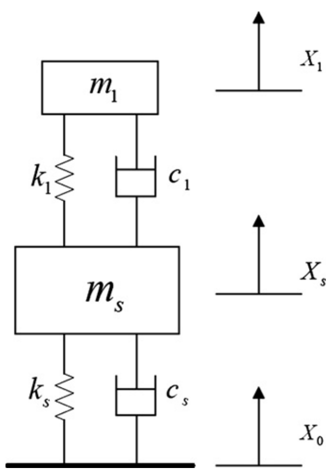

Fig. 1. A primary system with a passive DVA.

actual stiffness $k=k_1+\Delta k$ and the virtual damping $c=c_1+\Delta c$. Here, Δk and Δc are unknown variables representing variations in the spring and damper, respectively. Moreover, the variations are practically assumed to be norm-bounded and they satisfy

$$[\Delta k \;\; \Delta c]=\mathbf{L}N(t)\mathbf{E}, \tag{1}$$

where $\mathbf{L}$ is a constant and $\mathbf{E}$ is a 1×2 matrix, and $N(t)$ is an unknown time-varying variable satisfying $N^{\mathrm{T}}(t)N(t)\leq 1$.

Under the excitation of the base vibration X_0, the equations of motion for the primary system with a DVA are

$$\begin{cases} m_1\ddot{X}_1+c(\dot{X}_1-\dot{X}_s)+k(X_1-X_s)=0, \\ m_s\ddot{X}_s+c(\dot{X}_s-\dot{X}_1)+c_s(\dot{X}_s-\dot{X}_0)+k(X_s-X_1)+k_s(X_s-X_0)=0. \end{cases} \tag{2}$$

Eq. (2) can be rewritten in the following compact matrix form:

$$\mathbf{M}\ddot{\mathbf{X}}+\mathbf{C}\dot{\mathbf{X}}+\mathbf{K}\mathbf{X}=\mathbf{B}_{e1}\dot{X}_0+\mathbf{B}_{e2}X_0+\mathbf{B}_u u(t), \tag{3}$$

where

$$\mathbf{X}=\begin{bmatrix} X_1 \\ X_s \end{bmatrix}, \quad \mathbf{M}=\begin{bmatrix} m_1 & 0 \\ 0 & m_s \end{bmatrix}, \quad \mathbf{C}=\begin{bmatrix} 0 & 0 \\ 0 & c_s \end{bmatrix},$$

$$\mathbf{K}=\begin{bmatrix} 0 & 0 \\ 0 & k_s \end{bmatrix}, \quad \mathbf{B}_{e1}=\begin{bmatrix} 0 \\ c_s \end{bmatrix}, \quad \mathbf{B}_{e2}=\begin{bmatrix} 0 \\ k_s \end{bmatrix}, \quad \mathbf{B}_u=\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$u=c(\dot{X}_1-\dot{X}_s)+k(X_1-X_s).$$

Now, in order to convert the passive DVA design problem into an SOF design problem, we define a new state vector $\mathbf{x}^{\mathrm{T}}=[\mathbf{X}^{\mathrm{T}}(\dot{\mathbf{X}}-\mathbf{M}^{-1}\mathbf{B}_{e1})^{\mathrm{T}}]$ and obtain

$$\dot{\mathbf{x}}(\mathbf{t})=\mathbf{A}\mathbf{x}(\mathbf{t})+\mathbf{B}_1\omega(t)+\mathbf{B}_2u(t). \tag{4}$$

Here,

$$\mathbf{A}=\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \omega(t)=X_{0,}$$

$$\mathbf{B}_1=\begin{bmatrix} \mathbf{M}^{-1}\mathbf{B}_{e1} \\ \mathbf{M}^{-1}\mathbf{B}_{e2}-\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{B}_{e1} \end{bmatrix}, \quad \mathbf{B}_2=\begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B}_u \end{bmatrix}.$$

The output vector of interest is

$$\mathbf{y}(\mathbf{t})=\begin{bmatrix} X_1-X_s \\ \dot{X}_1-\dot{X}_s \end{bmatrix}=\mathbf{C}_2\mathbf{x}(\mathbf{t})+\mathbf{D}_{21}\omega(t), \tag{5}$$

where

$$\mathbf{C}_p=[1 \;\; -1], \quad \mathbf{C}_2=\begin{bmatrix} \mathbf{C}_p & 0 \\ 0 & \mathbf{C}_p \end{bmatrix}, \quad \mathbf{D}_{21}=\begin{bmatrix} 0 \\ \mathbf{C}_p\mathbf{M}^{-1}\mathbf{B}_{e1} \end{bmatrix}.$$

The control signal $u(t)$ can be expressed as

$$u(t)=\mathbf{F}\mathbf{y}(\mathbf{t}). \tag{6}$$

Here, $\mathbf{F}:=[k \;\; c]$ denotes the SOF gain.

The function of the DVA is to suppress the vibration of the primary system when it is subject to disturbance. In fact, the controlled output is the vibration of the machine. Hence the controlled output is defined as

$$z(t)=\mathbf{C}_1\mathbf{x}(\mathbf{t}), \tag{7}$$

where $\mathbf{C}_1=[0 \;\; 1 \;\; 0 \;\; 0]$.

Remark 1. In order to transform the design problem to an SOF control problem, it is regarded that the spring of the DVA feedbacks the relative displacement and the damper of the DVA feedbacks the relative velocity. Then, the optimal design problem can be converted to a control problem, and hence, the control forces will be determined by the spring and damper parameters. Such a conversion procedure was first proposed in [16]. Due to its flexibility and effectiveness, it has been further studied in [7,8].

Remark 2. In this paper, the main goal is to suppress the vibration of the primary system. Therefore, the controlled output is chosen as X_s. For other applications, if concerning with the acceleration, the controlled output can be chosen as the fourth state (acceleration).

Remark 3. In fact, the machine is also subject to system uncertainties. Yet in this work, our attention is focused on dealing with the uncertainties of the DVA, which has not been studied in this area. It is worthwhile noting that the method proposed in this work can be readily extended to the design problem of DVAs considering the machine uncertainties. It is also interesting to further consider the uncertainty on the mass of the DVA. In this case, we can characterize the uncertainty by a two-vertex polytope such that the closed-loop system can be accordingly represented by two sub-systems [21].

Remark 4. It is quite interesting to discuss how to choose values of the uncertainties in the DVA. Since the stiffness and the damping of the DVA are to be designed, it is not easy to precisely consider the uncertainties during the problem formulation procedure. However, we can have a rough prediction on the manufacturing error or the variation, such as the uncertainty ratio the tolerable uncertainty/the value for the nominal DVA $\times$ 100percent $= \pm \mathbf{LE}/\overline{\mathbf{F}} \times 100$percent is $\pm$10–30 percent. Moreover, we can derive a rough range of the parameters of the DVA by using the existing method without considering any uncertainties in the DVA. Then we get an interval of the product of **L** and **E**. The idea of choosing the values for **L** and **E** is to ensure that the product of **L** and **E** covers the predicted interval. After choosing the values of **L** and **E**, we can design the DVA by our proposed method below. Once the DVA is designed, the actual uncertainty ratio can be computed. If the computed actual uncertainty ratio is too small, enlarge the product of **L** and **E** and redesign the DVA until the requirements are satisfied.

Throughout the paper, we use the following definition.

Definition 1. Given a scalar $\gamma > 0$, a DVA is said to be non-fragile with an H_∞ performance γ if the following condition holds:

$$\|z\|_2 < \gamma \|\omega\|_2 \tag{8}$$

for all the variations in (1) and all nonzero $\omega(t) \in L_2[0,\infty)$, where $\|\omega\|_2$ and $\|z\|_2$ represent the 2-norm of the disturbance and the controlled output, respectively.

The objective of this paper is to develop a DVA as shown in Fig. 1 such that, for all admissible variations in (1), the designed DVA is non-fragile with an H_∞ performance γ.

3. Main results

The closed-loop system for the control problem is expressed as

$$\begin{cases} \dot{\mathbf{x}}(\mathbf{t}) = (\mathbf{A}+\mathbf{B}_2\mathbf{F}\mathbf{C}_2)\mathbf{x}(\mathbf{t}) + (\mathbf{B}_1+\mathbf{B}_2\mathbf{F}\mathbf{D}_{21})\omega(t), \\ z(t) = \mathbf{C}_1\mathbf{x}(\mathbf{t}). \end{cases} \tag{9}$$

Note that the uncertainty appears in the feedback gain. In order to deal with the uncertainty, we introduce the following lemma.

Lemma 1 (*Xie and Soh [17] and Jiang and Han [18]*). *Let $\Theta = \Theta^{\mathrm{T}}$, $\overline{\mathbf{L}}$ and $\overline{\mathbf{E}}$ be real matrices with compatible dimensions, and $N(t)$ be time-varying and satisfy* (1), *then the following condition*:

$$\Theta + \overline{\mathbf{L}}N(t)\overline{\mathbf{E}} + \overline{\mathbf{E}}^{\mathrm{T}}N^{\mathrm{T}}(t)\overline{\mathbf{L}}^{\mathrm{T}} < 0, \tag{10}$$

holds if and only if there exists a positive scaler $\varepsilon > 0$ such that

$$\begin{bmatrix} \Theta & \overline{\mathbf{L}} & \varepsilon\overline{\mathbf{E}}^{\mathrm{T}} \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0 \tag{11}$$

is satisfied. Here, we use an asterisk (*) as an ellipsis for the terms that are introduced by symmetry.

In the following, we analyze the H_∞ criterion and develop an algorithm to design the optimal non-fragile DVA.

3.1. H_∞ performance analysis

The H_∞ criterion minimizes the L_2 gain from external disturbance to the controlled output, i.e., the infinity norm of the transfer function. It has been a powerful tool in optimal control and filtering [19–22]. The conditions for H_∞ control can be expressed by LMIs, see the following lemma.

Lemma 2 (*Gahinet and Apkarian [19]*). *Suppose the parameters of the DVA are given. The system in* (9) *is asymptotically stable with a given H_∞ performance γ if there exists a positive definite $\mathbf{P}=\mathbf{P}^{\mathrm{T}}$ such that the following matrix inequality holds*:

$$\begin{bmatrix} \mathrm{sym}(\mathbf{P}(\mathbf{A}+\mathbf{B}_2\mathbf{F}\mathbf{C}_2)) & \mathbf{P}(\mathbf{B}_1+\mathbf{B}_2\mathbf{F}\mathbf{D}_{21}) & \mathbf{C}_1^{\mathrm{T}} \\ * & -\gamma\mathbf{I} & 0 \\ * & * & -\gamma\mathbf{I} \end{bmatrix} < 0. \tag{12}$$

Here, $\mathrm{sym}(\mathbf{P}(\mathbf{A}+\mathbf{B}_2\mathbf{F}\mathbf{C}_2)) := \mathbf{P}(\mathbf{A}+\mathbf{B}_2\mathbf{F}\mathbf{C}_2)+(\mathbf{A}+\mathbf{B}_2\mathbf{F}\mathbf{C}_2)^{\mathrm{T}}\mathbf{P}$.

It is noted that the uncertainty has been incorporated in Lemma 2. In order to eliminate the time-varying variable, we employ Lemma 1 and derive the following theorem.

Theorem 1. *Suppose the parameters of the DVA are given. The system in* (9) *is asymptotically stable with a given H_∞ performance γ, if there exist a positive definite $\mathbf{P}=\mathbf{P}^{\mathrm{T}}$ and a positive ε such that the following matrix inequality holds*:

$$\begin{bmatrix} \Lambda_{11} & \mathbf{P}(\mathbf{B}_1+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{D}_{21}) & \mathbf{C}_1^{\mathrm{T}} & \mathbf{P}\mathbf{B}_2L & \varepsilon(\mathbf{E}\mathbf{C}_2)^{\mathrm{T}} \\ * & -\gamma\mathbf{I} & 0 & 0 & \varepsilon(\mathbf{E}\mathbf{D}_{21})^{\mathrm{T}} \\ * & * & -\gamma\mathbf{I} & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \tag{13}$$

where $\Lambda_{11}=\mathbf{P}(\mathbf{A}+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{C}_2)+(\mathbf{A}+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{C}_2)^{\mathrm{T}}\mathbf{P}$ *and* $\overline{\mathbf{F}}=[k_1\ \ c_1]$.

Proof. According to Lemma 1, the system in (9) is asymptotically stable with a given H_∞ performance γ if the condition (12) holds. Moreover the condition (12) can be written in the following form:

$$\Theta+\overline{\mathbf{L}}N(t)\overline{\mathbf{E}}+\overline{\mathbf{E}}^{\mathrm{T}}N^{\mathrm{T}}(t)\overline{\mathbf{L}}^{\mathrm{T}}<0,$$

where

$$\Theta=\begin{bmatrix} \Lambda_{11} & \mathbf{P}(\mathbf{B}_1+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{D}_{21}) & \mathbf{C}_1^{\mathrm{T}} \\ * & -\gamma\mathbf{I} & 0 \\ * & * & -\gamma\mathbf{I} \end{bmatrix}<0,$$

$$\overline{\mathbf{L}}^{\mathrm{T}}=[(\mathbf{P}\mathbf{B}_2\mathbf{L})^{\mathrm{T}}\ \ 0\ \ 0],$$

$$\overline{\mathbf{E}}=[\mathbf{E}\mathbf{C}_2\ \ \mathbf{E}\mathbf{D}_{21}\ \ 0].$$

By using Lemma 1, we obtain the matrix inequality (13). The proof is completed. □

3.2. DVA design

It is necessary to mention that, in Theorem 1, Condition (13) is not an LMI but a BMI. Thus far, there lack efficient solvers for BMI problems due to its highly complexity. As the analysis part, the parameters of the DVA are supposed to be given in Theorem 1. Now we turn to the synthesis problem by proposing a design method for determining the DVA parameters.

Here, the main challenge is to decouple the bilinear terms. One approach is that the parameter $\overline{\mathbf{F}}$ can be easily obtained if we can find the Lyapunov weighting matrix $\mathbf{P}$. Let $\mathbf{W}_1$ denote the product of $\mathbf{P}\mathbf{B}_2\overline{\mathbf{F}}$. Then the matrix inequality (13) is equivalent to

$$\begin{bmatrix} \overline{\Lambda}_{11} & \mathbf{P}\mathbf{B}_1+\mathbf{W}_1\mathbf{D}_{21} & \mathbf{C}_1^{\mathrm{T}} & \mathbf{P}\mathbf{B}_2L & \varepsilon(\mathbf{E}\mathbf{C}_2)^{\mathrm{T}} \\ * & -\gamma\mathbf{I} & 0 & 0 & \varepsilon(\mathbf{E}\mathbf{D}_{21})^{\mathrm{T}} \\ * & * & -\gamma\mathbf{I} & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \tag{14}$$

where $\overline{\Lambda}_{11}=\mathbf{P}\mathbf{A}+\mathbf{W}_1\mathbf{C}_2+\mathbf{A}^{\mathrm{T}}\mathbf{P}+(\mathbf{W}_1\mathbf{C}_2)^{\mathrm{T}}$.

It is noted that the variable $\overline{\mathbf{F}}$ is eliminated in matrix inequality (14). To keep the variable $\overline{\mathbf{F}}$, we perform a congruence transformation to (13) by $\mathbf{J}_1 = \mathrm{diag}\{\mathbf{P}^{-1},\mathbf{I},\mathbf{I},\mathbf{I}\}$, and obtain

$$\begin{bmatrix} \tilde{\Lambda}_{11} & (\mathbf{B}_1+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{D}_{21}) & \mathbf{P}^{-1}\mathbf{C}_1^{\mathrm{T}} & \mathbf{B}_2 L & \varepsilon\mathbf{P}^{-1}(\mathbf{E}\mathbf{C}_2)^{\mathrm{T}} \\ * & -\gamma\mathbf{I} & 0 & 0 & \varepsilon(\mathbf{E}\mathbf{D}_{21})^{\mathrm{T}} \\ * & * & -\gamma\mathbf{I} & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \tag{15}$$

where $\tilde{\Lambda}_{11} = (\mathbf{A}+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{C}_2)\mathbf{P}^{-1}+\mathbf{P}^{-1}(\mathbf{A}+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{C}_2)^{\mathrm{T}}$.

By setting $\mathbf{PQ=I}$ and $\overline{\mathbf{F}}\mathbf{C}_2\mathbf{Q} = \mathbf{W}_2$, Condition (15) is converted into

$$\begin{bmatrix} \hat{\Lambda}_{11} & (\mathbf{B}_1+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{D}_{21}) & \mathbf{Q}\mathbf{C}_1^{\mathrm{T}} & \mathbf{B}_2\mathbf{L} & \varepsilon\mathbf{Q}(\mathbf{E}\mathbf{C}_2)^{\mathrm{T}} \\ * & -\gamma\mathbf{I} & 0 & 0 & \varepsilon(\mathbf{E}\mathbf{D}_{21})^{\mathrm{T}} \\ * & * & -\gamma\mathbf{I} & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \tag{16}$$

where $\hat{\Lambda}_{11} = \mathbf{AQ}+\mathbf{B}_2\mathbf{W}_2+\mathbf{QA}^{\mathrm{T}}+(\mathbf{B}_2\mathbf{W})^{\mathrm{T}}$. Another constraint is that the parameters of the DVA cannot be negative. Hence, constraints on Condition (16) are given as

$$\{\overline{\mathbf{F}}\}_i \geq 0, \quad \forall i = 1,2. \tag{17}$$

Here, for a row matrix $\overline{\mathbf{F}}$, $\{\overline{\mathbf{F}}\}_i$ represents the ith element. It was shown in [15,13] that the condition $\mathbf{PQ=I}$ is a rank constraint, which can be relaxed by minimizing the trace of $\mathbf{PQ}$ subject to following LMI:

$$\begin{bmatrix} \mathbf{P} & \mathbf{I} \\ * & \mathbf{Q} \end{bmatrix} \geq 0. \tag{18}$$

Now, we can employ an ILMI method [13,14] to solve the design problem. The main idea of the iterative method is to find an initial value for the Lyapunov weighting matrix $\mathbf{P}$ by using the Cone-complement linearization (CCL) [15] algorithm and iterate the value for $\mathbf{P}$ until Condition (13) is satisfied.

The CCL algorithm for the initial value $\mathbf{P}$ in this paper is addressed as

Algorithm 1. *Step* 1: Set i=1, $\mathbf{P}_{00}$=*randn*(4,4), $\mathbf{P}_0=\mathbf{P}_{00}^{\mathrm{T}}\mathbf{P}_{00}$, $\mathbf{Q}_{00}$=*randn*(4,4), and $\mathbf{Q}_0=\mathbf{Q}_{00}^{\mathrm{T}}\mathbf{Q}_{00}$. Choose a constant ε and a prescribed H_∞ performance index γ.

Step 2: Solve the following LMI problems: Minimize trace $(\mathbf{P}_i\mathbf{Q}_{i-1}+\mathbf{Q}_i\mathbf{P}_{i-1})$ such that (14), (16), (17) and (18) hold.

Step 3: Check the trace of $\mathbf{P}_i\mathbf{Q}_i$. If $|\mathrm{trace}(\mathbf{P}_i\mathbf{Q}_i)-4| < \delta$ for some sufficiently small scalar $\delta > 0$, i.e., 10^{-5}, then we obtain the initial value for $\mathbf{P}=\mathbf{P}_i$. EXIT.

Step 4: If $i > TN$ where TN is the maximum number of allowed iterations, no feasible initial value for $\mathbf{P}$. EXIT.

Step 5: Set i=i+1 and goto Step 2.

After obtaining an initial value for $\mathbf{P}$ in Algorithm 1, another ILMI algorithm for the non-fragile DVA design, as shown below, will be employed.

Algorithm 2. *Step* 1: Set j=1 and $\mathbf{P}_1=\mathbf{P}$ where $\mathbf{P}$ is obtained from Algorithm 1.

Step 2: Solve the following LMI optimization problem for $\overline{\mathbf{F}}$ with the given $\mathbf{P}_j$:

Minimize α_j subject to the LMI constraints (17) and

$$\begin{bmatrix} \check{\Lambda}_{11} & \mathbf{P}_j(\mathbf{B}_1+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{D}_{21}) & \mathbf{C}_1^{\mathrm{T}} & \mathbf{P}_j\mathbf{B}_2 L & \varepsilon(\mathbf{E}\mathbf{C}_2)^{\mathrm{T}} \\ * & -\gamma\mathbf{I} & 0 & 0 & \varepsilon(\mathbf{E}\mathbf{D}_{21})^{\mathrm{T}} \\ * & * & -\gamma\mathbf{I} & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \tag{19}$$

where $\check{\Lambda}_{11} = \mathbf{P}_j(\mathbf{A}+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{C}_2)+(\mathbf{A}+\mathbf{B}_2\overline{\mathbf{F}}\mathbf{C}_2)^{\mathrm{T}}\mathbf{P}_j-\alpha_j\mathbf{P}_j$.

Step 3: If $\alpha_j < 0$, the parameter $\overline{\mathbf{F}}$ for the robust non-fragile DVA is obtained. EXIT.

Step 4: Set j=j+1. Solve the following LMI optimization problem for $\mathbf{P}_j$ with the $\overline{\mathbf{F}}$ obtained at Step 2:

Minimize α_j subject to the LMI constraints (17) and (19).

Step 5: If $\alpha_j < 0$, the parameter $\overline{\mathbf{F}}$ for the robust non-fragile DVA is obtained. EXIT.

Step 6: Solve the following LMI optimization problem for $\mathbf{P}_j$ with the $\overline{\mathbf{F}}$ and α_j obtained at Step 4:

Minimize trace($\mathbf{P}_j$) subject to the LMI constraints (17) and (19).

Step 7: If $j > TN$ where TN is the maximum number of allowed iterations, it concludes that, by this algorithm, the parameters of the robust non-fragile DVA not solvable with the prescribed H_∞ index γ are not solvable. EXIT.

Step 8: Set $j=j+1$, $\mathbf{P}_j=\mathbf{P}_{j-1}$ and goto Step 2.

Remark 5. Though the results in [13] shed the light on our development in this work, it focused on the SOF control problem for a numerical example system without uncertainty. In this paper, we extend their results by incorporating the uncertainty in the designed controller. In addition, we have considered the practical non-fragile DVA design, which moves a step further towards the practical application of robust control theory. The parameters of the DVA are practically required to be non-negative, which is one additional constraint for the design problem in this work.

Remark 6. We introduce the robust non-fragile DVA in the passive suppression design problem. As we mentioned, the uncertainty in the designed DVA is inevitable [23] so that the performance of the designed DVA cannot be guaranteed if it is not considered. Therefore, to consider the uncertainty when designing the DVA is of not only the theoretical merits but also of practical application perspective. The presented design procedures and method can be extended to multiple degree-of-freedom DVA design problems readily.

Remark 7. BMI problems are difficult to solve due to the NP hardness [15,16,20,24]. Hence, they have attracted considerable attention during the past decades. Although there is not any algorithm which is effective for all the problems, there are lots of methods which are suitable for many cases, such as the homotopy approach in [16] and the gradient-based algorithm in [8]. Among these approaches, the CCL algorithm is the most famous and has been shown the powerfulness for many applications [24,25]. Recently, the authors in [13] proposed a new ILMI method which is essentially based on the CCL algorithm. From our experience, the newly proposed method in [13] is fast as demonstrated from the example of [13]. Therefore, we adopt and modify the ILMI method for our design problem of robust non-fragile passive DVA.

4. Design example

In this section, an example will be given to compare and show the effectiveness of the proposed design method.

Suppose that the mass of the primary system m_s=1.0 kg, the damping c_s=0.1 N s/m, the stiffness of the spring k_s=1.0 N/m, and the mass of the DVA m_1=0.1 kg. For the variations, it is assumed that $\mathbf{L}$=0.025, and $\mathbf{E}$=[1, 0.4]. By utilizing the proposed design method, we obtain $\overline{\mathbf{F}} = [0.0789\ 0.0491]$ when $\varepsilon = 0.02$ and $\gamma = 5.0$, i.e., k_1=0.0789 N/m and c_1=0.0491 N s/m. The stiffness of the nominal DVA (with k_1 and c_1) is 0.0789 N/m and the damping of the nominal DVA is 0.0491 N s/m. Recall that we devote to design non-fragile DVAs which can maintain the performance when subject to admissible manufacturing errors and operation variations. In our example, the tolerable uncertainty in the stiffness is $\pm$0.025 N/m and in the damping is $\pm$0.01 N s/m. We obtain the admissible uncertainty ratio is $\pm$31.69 percent for the spring and $\pm$20.37 percent for the damper when the required H_∞ performance index is no more than 5. Though there are two ILMI methods in the design procedure, there are only three iterations (i=2 in Algorithm 1, j=1 in Algorithm 2 and $\alpha = -6.1217 \times 10^{-7}$) in this example.

In the interest of comparing the suppressing effect and illustrating the correctness of the designed DVA, Fig. 2 shows the frequency response of the system with different DVAs (different parameters for the spring and damper). Note that the peak value of the frequency response is the L_2 gain from the input to the output. The L_2 gains for different DVAs are all less than

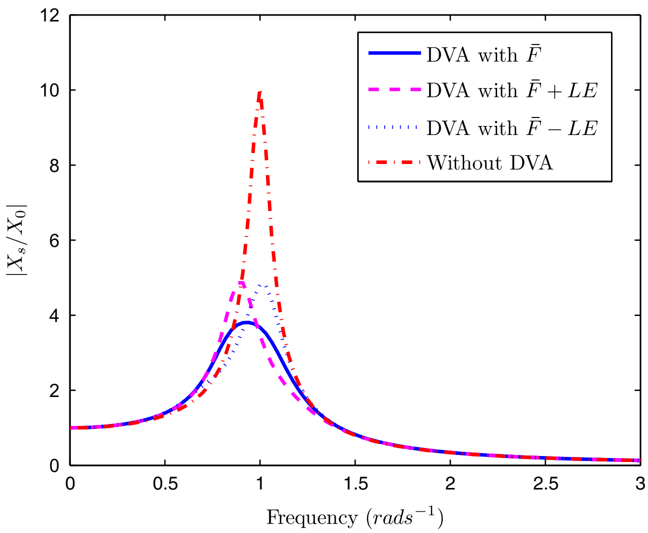

Fig. 2. Frequency response of the system with different DVAs and without DVA.

the prescribed H_∞ attenuation level γ. Moreover, compared with the response of the system without DVA, the designed DVA shows significant advantage in suppressing machine vibrations.

5. Conclusions

In this paper, we have studied the design problem for non-fragile H_∞ DVAs. The design problem was first transformed into an SOF controller design problem. Sufficient condition under which the closed-loop system is asymptotically with a prescribed H_∞ attenuation level was derived. An ILMI method was employed to solve the derived condition. The given example illustrates the validity and the effectiveness of the proposed design method. Moreover, the designed DVA shows significant improvement in suppression of vibrations over the primary system without a DVA.

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