



Dynamic analysis of frames with viscoelastic dampers modelled by rheological models with fractional derivatives

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ABSTRACT

Frame structures with viscoelastic dampers mounted on them are considered in this paper. Viscoelastic (VE) dampers are modelled using two, three-parameter, fractional rheological models. The structures are treated as elastic linear systems. The equation of motion of the whole system (structure with dampers) is written in terms of state-space variables. The resulting matrix equation of motion is the fractional differential equation. The proposed state space formulation is new and does not require matrices with huge dimensions. The paper is devoted to determine the dynamic properties of the considered structures. The nonlinear eigenvalue problem is formulated from which the dynamic parameters of the system can be determined. The continuation method is used to solve the nonlinear eigenvalue problem. Moreover, results of typical calculations are presented.

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1. Introduction

Viscoelastic (VE) dampers have often been used in controlling the vibrations of aircrafts, aerospace and machine structures. In civil engineering VE dampers are successfully applied to reduce any excessive vibrations of buildings caused by winds and earthquakes. It was found that incorporation of VE dampers in a structure leads to significant reduction of unwanted vibrations [1]. A number of applications of VE dampers in civil engineering are listed in [2]. The VE dampers could be divided broadly into fluid and solid VE dampers. Silicone oil is used to build the fluid dampers while the solid dampers are made of copolymers or glassy substances. Good understanding of the dynamical behaviour of dampers is required for the analysis of structures supplemented with VE dampers. The dampers' behaviour depends mainly on the rheological properties of the VE material the dampers are made of and some of their geometric parameters.

In the past, several rheological models were proposed to describe the dynamic behaviour of VE materials and dampers. Both the classical and so-called fractional-derivative models of dampers and VE materials are available. Descriptions of these models are given in [3–11].

In a classic approach, mechanical models consisting of springs and dashpots are used to describe the rheological properties of VE dampers [5,11–17]. A good description of the VE dampers requires mechanical models consisting of a set of appropriately connected springs and dashpots. In this approach, the dynamic behaviour of a single damper is described by a set of differential equations (see [5,11,12]), which considerably complicates the dynamic analysis of structures with dampers because the large set of equations of motion must be solved.

The rheological properties of VE dampers are also described using the fractional calculus and the fractional mechanical models. This approach has received considerable attention and has been used in modelling the rheological behaviour of VE materials [4,18,19] and dampers [6,9]. The fractional models have an ability to correctly describe the behaviour of VE

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materials and dampers using a small number of model parameters. A single equation is enough to describe the VE damper dynamics, which is an important advantage of the discussed models. However, in this case, the VE damper equation of motion is the fractional differential equation.

The dynamic analysis of frame or building structures with dampers is presented in many papers [11,13–17,20–25] where the Maxwell [13,14,16,17] or the Kelvin model [14,15,22,23] are used to describe the dampers' dynamic behaviour. In the papers [20,24], a three-parameter fractional-derivative rheological model is used to model the dampers' behaviour. Moreover, in the paper [25] the rational polynomial approximation modelling is used for analysis of structures with VE dampers.

The methods of determination of dynamic properties of systems with damping described with the help of the fractional calculus are presented in the papers [4,19,20,25–29]. However, according to the presented formulation, a substantial linear eigenvalue problem must be solved.

The fractional derivative model of damping was applied also to describe the dynamic behaviour of viscoelastic beams [30–32]. The finite element formulation of fractional viscoelastic constitutive equations is presented in [33]. An interesting discussion of damping mechanics and models used in structural dynamics is presented in [34].

In this paper, planar frame structures with the VE dampers mounted on them are considered. The VE dampers are modelled using the fractional rheological model. Two three-parameter, fractional rheological models, i.e., the Kelvin model and the Maxwell model, are considered. The structures are treated as linear elastic systems. The equations of motion of the whole system (the structure with dampers) are written in terms of both physical and state-space variables. The proposed approach in the state space formulation is new. It is the main advantage of the proposed formulation, where matrices with huge dimensions are not required. The resulting matrix equation of motion is a fractional differential equation.

The aim of the paper is to determine the dynamic properties of the considered structures. The nonlinear eigenvalue problem is formulated from which the dynamic parameters of the system can be determined. The continuation method is used to solve the above-mentioned nonlinear eigenvalue problem. In contrast to the method presented previously (see [4,19,20,26,28,29]), the dimension of the eigenvalue problem arising here is much smaller.

The calculation results will also be presented and briefly discussed. The influence of the key parameter which describes the order of the fractional derivative on the dynamic parameters of frames with VE dampers, is also shown.

The paper is organized as follows: In Section 2 equations of motion of frame with VE dampers are derived using both the physical and state-space variables. In Section 3, the nonlinear eigenvalue problem together with the continuation method use to find the solution of the eigenvalue problem is presented. The definition of modal parameters of frame with VE dampers is given in Section 4. In Section 5, the frequency response functions of considered structures are derived. Results of sample calculation are presented in Section 6. Finally, some concluding remarks are stated in Section 7.

2. The equations of motion for frame with VE dampers

2.1. The rheological models of dampers

In this paper, two fractional rheological models, i.e., the fractional Kelvin model and the fractional Maxwell model (see Fig. 1), are used to describe the dynamic behaviour of VE dampers. The considered models of a typical damper, i , have three parameters: stiffness k_i , damping factor c_i , and fractional parameter α_i ($0 < \alpha_i \leq 1$).

The equation of motion for the Kelvin model (see Fig. 1a) could be written in the form:

$$u_i = k_i x_i + c_i D_t^{\alpha_i} x_i, \tag{1}$$

where u_i is the damper force and x_i is the relative damper displacement. Moreover, $D_t^{\alpha_i}(\bullet)$ denotes the Riemann–Liouville fractional derivative of the order α_i with respect to time, t . The Riemann–Liouville fractional derivative is defined as

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau, \tag{2}$$

where Γ is the gamma function. For a precise definition of the Riemann–Liouville fractional derivative, Podlubny [35] may be consulted.

The equations of motion for the Maxwell model could be written using the so-called relative internal variable v_i (compare Fig. 1b). The above-mentioned equations of motion for damper are as follows:

$$u_i = c_i D_t^{\alpha_i} (x_i - v_i), \quad u_i = k_i v_i. \tag{3}$$

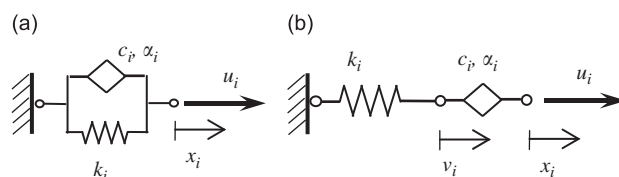


Fig. 1. Rheological models of damper: (a) fractional Kelvin–Voigt model; (b) fractional Maxwell model.

A damper of which the behaviour is described by the Kelvin model or the Maxwell model will be shortly referred to as the Kelvin damper or the Maxwell damper, respectively.

More information concerning the fractional rheological models can be found in [6,29]. The equation of motion of the classical Kelvin and Maxwell models could be obtained after introducing $\alpha_i = 1$ into Eqs. (1) and (3).

2.2. The equations of motion of structures expressed in physical coordinates

The frame with VE dampers is treated as the elastic linear system and their model could be the shear frame shown in Fig. 2a. The masses of the system are lumped at the level of storeys. The frame can be also modelled as a structure with flexible beams. In this case we assume that beams and columns are axially inextensible. Moreover, the static condensation is used to eliminate the rotational nodal parameters from the equations of motion. Finally, the equation of motion of such a structure can be written as follows:

$$\mathbf{M}_s \ddot{\mathbf{q}}_s(t) + \mathbf{C}_s \dot{\mathbf{q}}_s(t) + \mathbf{K}_s \mathbf{q}_s(t) = \mathbf{s}(t) + \mathbf{p}(t), \tag{4}$$

where the symbols \mathbf{M}_s , \mathbf{C}_s and \mathbf{K}_s denote the mass, damping and stiffness ($n \times n$) matrices, respectively. Moreover, $\mathbf{q}_s(t) = \text{col}(q_{s,1}, \dots, q_{s,j}, \dots, q_{s,n})$ and $\mathbf{p}(t) = \text{col}(p_1, \dots, p_j, \dots, p_n)$ denote the vector of displacements of the structure and the vector of excitation forces, respectively. The $\mathbf{s}(t) = \text{col}(s_1, s_2, \dots, s_n)$ vector is the ($n \times 1$) vector of interaction forces between the frame and dampers (compare Fig. 2b).

First of all, the structure with one damper, denoted as the damper number i , which is mounted between two successive storeys j and $j+1$ (shown in Fig. 2a), is considered. If the Kelvin damper is considered, the force interaction vector $\mathbf{s}(t)$ could be written as follows:

$$\mathbf{s}(t) \equiv \mathbf{s}_i(t) = \text{col}(0, \dots, s_j = u_i, s_{j+1} = -u_i, \dots, 0) = \mathbf{e}_i u_i(t), \tag{5}$$

where $\mathbf{e}_i = \text{col}(0, \dots, e_j = 1, e_{j+1} = -1, \dots, 0)$ is the i th damper allocation vector of dimension ($n \times 1$), $u_i(t)$ is the damper force given in (1). It is assumed the brace systems used to connect the dampers with the successive storeys are rigid.

Taking into account that the relative damper displacement, written in terms of structure displacements, is

$$x_i(t) = q_{s,j+1}(t) - q_{s,j}(t) = -\mathbf{e}_i^T \mathbf{q}_s(t) \tag{6}$$

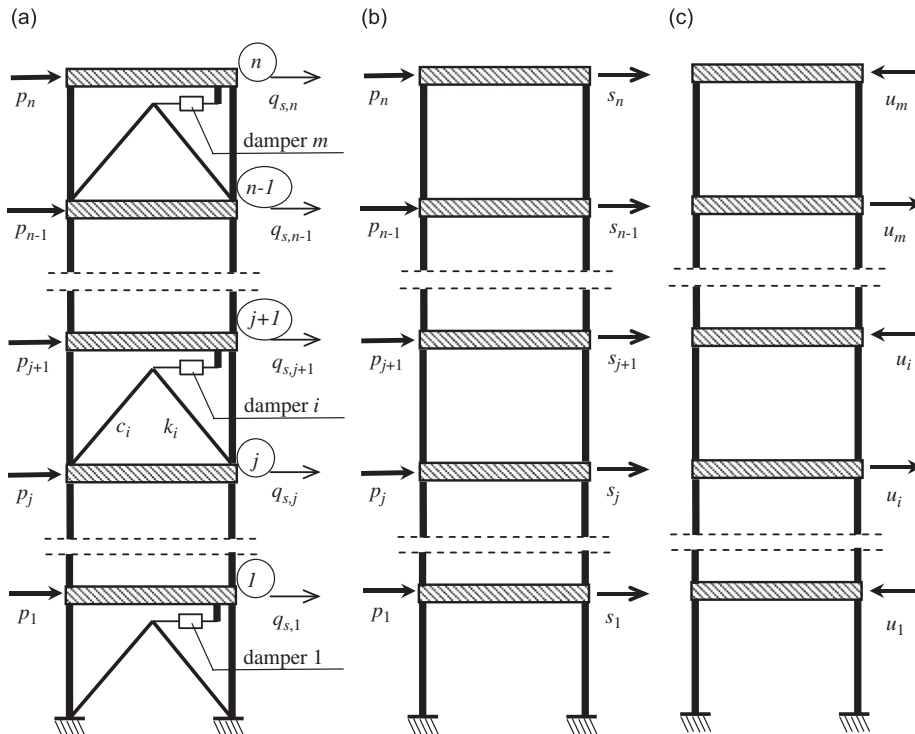


Fig. 2. Diagram of a frame with VE dampers: (a) a frame with dampers; (b) explanation of elements of the \mathbf{s} vector; (c) a frame with dampers' forces.

the damper force and the vector of interactive forces could be written as follows:

$$u_i(t) = -k_i \mathbf{e}_i^T \mathbf{q}_s(t) - c_i \mathbf{e}_i^T D_t^{\alpha_i} \mathbf{q}_s(t), \quad (7)$$

$$\mathbf{s}_i(t) = -\mathbf{e}_i k_i \mathbf{e}_i^T \mathbf{q}_s(t) - \mathbf{e}_i c_i \mathbf{e}_i^T D_t^{\alpha_i} \mathbf{q}_s(t). \quad (8)$$

For a structure with m dampers, the vector of interactive forces is given by:

$$\mathbf{s}(t) = \sum_{i=1}^m \mathbf{s}_i(t) = -\sum_{i=1}^m \mathbf{e}_i k_i \mathbf{e}_i^T \mathbf{q}_s(t) - \sum_{i=1}^m \mathbf{e}_i c_i \mathbf{e}_i^T D_t^{\alpha_i} \mathbf{q}_s(t), \quad (9)$$

and the equation of motion (4) could be rewritten in the form:

$$\mathbf{M}_s D_t^2 \mathbf{q}_s(t) + \mathbf{C}_s D_t^1 \mathbf{q}_s(t) + \sum_{i=1}^m \mathbf{e}_i c_i \mathbf{e}_i^T D_t^{\alpha_i} \mathbf{q}_s(t) + \left(\mathbf{K}_s + \sum_{i=1}^m \mathbf{e}_i k_i \mathbf{e}_i^T \right) \mathbf{q}_s(t) = \mathbf{p}(t), \quad (10)$$

where, in order to be consistent with the notation, a symbol such as $D_t^1(\bullet)$ is introduced to denote the first derivative with respect to time.

Eq. (10) is the matrix fractional differential equation which describes the dynamic behaviour of the considered frame with the Kelvin dampers. In this approach each damper can have its own values of parameters, different from others. Eq. (10) is much simplified when all the fractional parameters are equal, i.e., $\alpha_i = \alpha = \text{const.}$ (Appendix A).

Proceeding to considering the structure with the Maxwell dampers, Eq. (3) is used to describe the Maxwell damper behaviour. The vector of interactive forces $\mathbf{s}(t)$ is treated as a sum of two vectors, i.e., $\mathbf{s}(t) = \mathbf{s}_1(t) + \mathbf{s}_2(t)$. The $\mathbf{s}_1(t)$ vector contains interactive forces which are reactions of the elastic part of the Maxwell dampers to the frame, while the $\mathbf{s}_2(t)$ vector contains the interactive forces which are reactions of the dashpot part of the dampers. It is assumed that the dashpot part of the Maxwell model is jointed with the upper storey while the elastic part is jointed with the lower storey. Moreover, the brace stiffness could be taken into account in the stiffness parameter of the Maxwell model.

If a structure with only one damper, denoted as the damper number i , mounted between two successive storeys j and $j+1$ is considered (see Fig. 2), then the vectors $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ could be written in the following form:

$$\mathbf{s}_1(t) \equiv \mathbf{s}_1^{(i)}(t) = \text{col}(0, \dots, s_j = u_i, \dots, 0) = \tilde{\mathbf{e}}_i u_i(t), \quad (11)$$

$$\mathbf{s}_2(t) \equiv \mathbf{s}_2^{(i)}(t) = \text{col}(0, \dots, s_{j+1} = -u_i, \dots, 0) = \hat{\mathbf{e}}_i u_i(t) \quad (12)$$

where

$$\tilde{\mathbf{e}}_i = \text{col}(0, \dots, \tilde{e}_j = 1, \tilde{e}_{j+1} = 0, \dots, 0), \quad \hat{\mathbf{e}}_i = \text{col}(0, \dots, \hat{e}_j = 0, \hat{e}_{j+1} = -1, \dots, 0).$$

Taking into account that $q_{s,j}(t) = \tilde{\mathbf{e}}_i^T \mathbf{q}_s(t)$ and $q_{s,j+1}(t) = -\hat{\mathbf{e}}_i^T \mathbf{q}_s(t)$, the damping force $u_i(t)$ of the Maxwell damper could be shown in two equivalent forms:

$$u_i(t) = k_i(v_i(t) - q_{s,j}(t)) = k_i v_i(t) - k_i \tilde{\mathbf{e}}_i^T \mathbf{q}_s(t), \quad (13)$$

$$u_i(t) = c_i(D_t^{\alpha_i} q_{s,j+1}(t) - D_t^{\alpha_i} v_i(t)) = -c_i D_t^{\alpha_i} v_i(t) - c_i \hat{\mathbf{e}}_i^T D_t^{\alpha_i} \mathbf{q}_s(t), \quad (14)$$

and the interaction force vectors $\mathbf{s}_1^{(i)}(t)$ and $\mathbf{s}_2^{(i)}(t)$ are given by

$$\mathbf{s}_1^{(i)}(t) = \tilde{\mathbf{e}}_i k_i v_i(t) - \tilde{\mathbf{e}}_i k_i \tilde{\mathbf{e}}_i^T \mathbf{q}_s(t) = \tilde{\mathbf{e}}_i k_i \mathbf{h}_i^T \mathbf{q}_r(t) - \tilde{\mathbf{e}}_i k_i \tilde{\mathbf{e}}_i^T \mathbf{q}_s(t), \quad (15)$$

$$\mathbf{s}_2^{(i)}(t) = -\hat{\mathbf{e}}_i c_i D_t^{\alpha_i} v_i(t) - \hat{\mathbf{e}}_i c_i \hat{\mathbf{e}}_i^T D_t^{\alpha_i} \mathbf{q}_s(t) = -\hat{\mathbf{e}}_i c_i \mathbf{h}_i^T D_t^{\alpha_i} \mathbf{q}_r(t) - \hat{\mathbf{e}}_i c_i \hat{\mathbf{e}}_i^T D_t^{\alpha_i} \mathbf{q}_s(t), \quad (16)$$

where the vector of internal variables $\mathbf{q}_r(t) = \text{col}(v_1(t), \dots, v_i(t), \dots, v_m(t))$ and the vector $\mathbf{h}_i = \text{col}(0, \dots, h_i = 1, \dots, 0)$ have the dimension $(m \times 1)$.

When m dampers are present in the frame then the interaction force vectors are:

$$\mathbf{s}_1(t) = \sum_{i=1}^m \tilde{\mathbf{e}}_i k_i \mathbf{h}_i^T \mathbf{q}_r(t) - \sum_{i=1}^m \tilde{\mathbf{e}}_i k_i \tilde{\mathbf{e}}_i^T \mathbf{q}_s(t), \quad (17)$$

$$\mathbf{s}_2(t) = -\sum_{i=1}^m \hat{\mathbf{e}}_i c_i \mathbf{h}_i^T D_t^{\alpha_i} \mathbf{q}_r(t) - \sum_{i=1}^m \hat{\mathbf{e}}_i c_i \hat{\mathbf{e}}_i^T D_t^{\alpha_i} \mathbf{q}_s(t). \quad (18)$$

The dimensions of matrices \mathbf{K}_{sr}^d and \mathbf{K}_{ss}^d are $(n \times m)$ and $(n \times n)$, respectively.

Taking into account that $\mathbf{s}(t) = \mathbf{s}_1(t) + \mathbf{s}_2(t)$ and introducing Eqs. (17) and (18) into (4) we obtain the following equation of motion for frame with the Maxwell dampers

$$\mathbf{M}_s D_t^2 \mathbf{q}_s(t) + \mathbf{C}_s D_t^1 \mathbf{q}_s(t) + \sum_{i=1}^m \hat{\mathbf{e}}_i c_i \hat{\mathbf{e}}_i^T D_t^{\alpha_i} \mathbf{q}_s(t) + (\mathbf{K}_s + \sum_{i=1}^m \tilde{\mathbf{e}}_i k_i \tilde{\mathbf{e}}_i^T) \mathbf{q}_s(t) + \sum_{i=1}^m \hat{\mathbf{e}}_i c_i \mathbf{h}_i^T D_t^{\alpha_i} \mathbf{q}_r(t) - \sum_{i=1}^m \tilde{\mathbf{e}}_i k_i \mathbf{h}_i^T \mathbf{q}_r(t) = \mathbf{p}(t). \quad (19)$$

Eq. (19) represents a set of n equations with $n+m$ unknowns which are elements of vectors $\mathbf{q}_s(t)$ and $\mathbf{q}_r(t)$. Additional m equations in the following form:

$$-c_i D_t^{\alpha_i} q_{s,j+1}(t) + c_i D_t^{\alpha_i} v_i(t) - k_i q_{s,j}(t) + k_i v_i(t) = 0, \tag{20}$$

where $i = 1, 2, \dots, m$ are obtained from the equilibrium condition of the internal node of the Maxwell model of damper.

In the matrix notation, Eq. (20) for $i = 1, 2, \dots, m$ may be rewritten in the form:

$$c_i \hat{\mathbf{e}}_i^T D_t^{\alpha_i} \mathbf{q}_s(t) + c_i \mathbf{h}_i^T D_t^{\alpha_i} \mathbf{q}_r(t) - k_i \tilde{\mathbf{e}}_i^T \mathbf{q}_s(t) + k_i \mathbf{h}_i^T \mathbf{q}_r(t) = 0 \tag{21}$$

The final form of Eq. (21) is obtained by pre-multiplying Eq. (20) by \mathbf{h}_i and summing up all equations with respect to i . As the result, we have

$$\sum_{i=1}^m \mathbf{h}_i c_i \hat{\mathbf{e}}_i^T D_t^{\alpha_i} \mathbf{q}_s(t) + \sum_{i=1}^m \mathbf{h}_i c_i \mathbf{h}_i^T D_t^{\alpha_i} \mathbf{q}_r(t) - \sum_{i=1}^m \mathbf{h}_i k_i \tilde{\mathbf{e}}_i^T \mathbf{q}_s(t) + \sum_{i=1}^m \mathbf{h}_i k_i \mathbf{h}_i^T \mathbf{q}_r(t) = \mathbf{0} \tag{22}$$

Eqs. (19) and (22) constitute a set of equations from which the dynamic response of structure with Maxwell dampers can be determined. It is a set of fractional differential equations. In this formulation each damper can have its own values of parameters, different from others. For the case where all the fractional parameters are equal ($\alpha_i = \alpha = const.$) Eqs. (21) and (22) are presented in Appendix A.

2.3. The equations of motion of structures expressed in the state space

In many cases it is very convenient to use the equation of motion expressed in the state space. When the Kelvin model is used to describe dampers' behaviour, then the vector of state variables and the vectors of their derivatives could be defined as $\mathbf{z}(t) = col(\mathbf{q}_s(t), D_t^1 \mathbf{q}_s(t))$, $D_t^1 \mathbf{z}(t) = col(D_t^1 \mathbf{q}_s(t), D_t^2 \mathbf{q}_s(t))$, $D_t^2 \mathbf{z}(t) = col(D_t^2 \mathbf{q}_s(t), D_t^{\alpha_i+1} \mathbf{q}_s(t))$.

Moreover, when the following additional matrix equation

$$\mathbf{M}_s D_t^1 \mathbf{q}_s(t) - \mathbf{M}_s D_t^1 \mathbf{q}_s(t) = \mathbf{0} \tag{23}$$

is appended to Eq. (10) we get the set of Eqs. (10) and (23) which could be rewritten using the state variables defined above. The resulting matrix equation is in the form:

$$\mathbf{A} D_t^1 \mathbf{z}(t) + \sum_{i=1}^m \mathbf{A}_i D_t^{\alpha_i} \mathbf{z}(t) + \mathbf{B} \mathbf{z}(t) = \tilde{\mathbf{p}}(t), \tag{24}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}_s & \mathbf{M}_s \\ \mathbf{M}_s & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_i = \begin{bmatrix} \mathbf{e}_i c_i \mathbf{e}_i^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} (\mathbf{K}_s + \sum_{i=1}^m \mathbf{e}_i k_i \mathbf{e}_i^T) & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_s \end{bmatrix}, \quad \tilde{\mathbf{p}}(t) = \begin{pmatrix} \mathbf{p}(t) \\ \mathbf{0} \end{pmatrix}. \tag{25}$$

When all the fractional parameters are equal, i.e., $\alpha_i = \alpha = const.$, Eq. (24) has the form presented in Appendix B.

The equations of motion in the state space can also be derived for frames with Maxwell dampers. In this case, the vector of state variables and vectors of state variables' derivatives are defined as

$$\mathbf{z}(t) = col(\mathbf{q}_r(t), \mathbf{q}_s(t), D_t^1 \mathbf{q}_s(t)), \quad D_t^1 \mathbf{z}(t) = col(D_t^1 \mathbf{q}_r(t), D_t^1 \mathbf{q}_s(t), D_t^2 \mathbf{q}_s(t)) \\ D_t^{\alpha_i} \mathbf{z}(t) = col(D_t^{\alpha_i} \mathbf{q}_r(t), D_t^{\alpha_i} \mathbf{q}_s(t), D_t^{\alpha_i+1} \mathbf{q}_s(t)). \tag{26}$$

At this point Eqs. (22), (19) and (23) can be treated as a set of equations which can be written in the form of Eq. (24) where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_s & \mathbf{M}_s \\ \mathbf{0} & \mathbf{M}_s & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_i = \begin{bmatrix} \mathbf{h}_i c_i \mathbf{h}_i^T & \mathbf{h}_i c_i \hat{\mathbf{e}}_i^T & \mathbf{0} \\ \hat{\mathbf{e}}_i c_i \mathbf{h}_i^T & \hat{\mathbf{e}}_i c_i \hat{\mathbf{e}}_i^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{27}$$

$$\mathbf{B} = \begin{bmatrix} \sum_{i=1}^m \mathbf{h}_i k_i \mathbf{h}_i^T & -\sum_{i=1}^m \mathbf{h}_i k_i \tilde{\mathbf{e}}_i^T & \mathbf{0} \\ -\sum_{i=1}^m \tilde{\mathbf{e}}_i k_i \mathbf{h}_i^T & \mathbf{K}_s + \sum_{i=1}^m \tilde{\mathbf{e}}_i k_i \tilde{\mathbf{e}}_i^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{M}_s \end{bmatrix}, \quad \tilde{\mathbf{p}}(t) = \begin{pmatrix} \mathbf{0} \\ \mathbf{p}(t) \\ \mathbf{0} \end{pmatrix}. \tag{28}$$

For the fractional parameters $\alpha_i = \alpha = const.$, the above matrices are much simplified (Appendix B).

The above approach to the state space formulation is new. In comparison with previous ones, such as those given in [20,26], matrices with huge dimensions were not required, which is the main advantage of the proposed formula. Moreover, all matrices appearing in Eq. (24) are symmetrical. For example, when the ten storey frame with dampers at each storey is considered the above formulas consist of matrices of dimensions 20×20 . The same problem, solved using

the methods presented by Chang in [20], leads to the matrices of dimensions 100×100 ($\alpha = 0.6$). Moreover, the fractional parameter α given more precisely raises the matrices' dimension, i.e., $\alpha = 0.63$ leads to 2000×2000 matrices.

3. The eigenvalue problem and the continuation method

Applying the Laplace transform, taking into account that $\dot{\mathbf{p}}(t) = \mathbf{0}$ and (see [40]):

$$\mathbf{L}[\mathbf{z}(t)] = \mathbf{Z}, \quad \mathbf{L}[D_t^{\alpha_i} \mathbf{z}(t)] = s^{\alpha_i} \mathbf{Z}, \quad \mathbf{L}[D_t^1 \mathbf{z}(t)] = s \mathbf{Z}, \quad (29)$$

the equation of motion (24) or (B1) can be written as

$$\left(s \mathbf{A} + \sum_{i=1}^m s^{\alpha_i} \mathbf{A}_i + \mathbf{B} \right) \mathbf{Z} = \mathbf{0}. \quad (30)$$

Eq. (30) constitutes a nonlinear eigenproblem, which can be solved using the continuation method. Methods for the solution of the eigenproblem appearing in the dynamic analysis of viscoelastic structures or systems where the damping forces are modelled using the fractional derivative are considered in [19,41–44].

The continuation method, also termed as the path following method, is frequently used to solve nonlinear equations with parameter, occurring in many problems of modern mechanics. The static analysis of geometrically or/and physically nonlinear structures (see [36,37]) and the analysis of large-amplitude free and steady state vibrations [38–40] are examples of such problems. A general description of the continuation method can be found, for example, in [45]. In the continuation method, the set of nonlinear equations with one parameter, also called the main parameter, is usually considered. In this paper, Eq. (30) is viewed as an equation with m main parameters which are all the fractional parameters of dampers. In the investigated case, without loss of generality of consideration, the following linear dependence between the fractional parameters is introduced:

$$\bar{\alpha}_i(\alpha) = v_i \alpha + \kappa_i, \quad (31)$$

where

$$v_i = \frac{\alpha_i - 1}{\alpha_p - 1}, \quad \kappa_i = \frac{\alpha_p - \alpha_i}{\alpha_p - 1} \quad (32)$$

In relations (31) and (32) α_p is the chosen fractional parameter, say $\alpha_p = \alpha_1$. In the context of the continuation method α_i is the final value of fractional parameter. Symbol $\bar{\alpha}_i$ is used to denote the current value of this parameter. Moreover, the current value of the relative parameter α_p is denoted by α . In this way the number of main parameters is reduced to one main parameter α and Eq. (30) could be rewritten in the form:

$$\mathbf{g}_1 \equiv \left(s \mathbf{A} + \sum_{i=1}^m s^{v_i \alpha + \kappa_i} \mathbf{A}_i + \mathbf{B} \right) \mathbf{Z} = \mathbf{0}. \quad (33)$$

A new space configuration, i.e., the space s, \mathbf{Z} , different from the previously introduced state space, is now introduced. In this new space the solution of the considered nonlinear equation with parameter could be shown as a curve (see [45] for details). The first point on this curve is obtained for $\alpha_i = 1$ because, in this case, Eq. (30) is the linear eigenvalue problem of the form:

$$\left[s \left(\mathbf{A} + \sum_{i=1}^m \mathbf{A}_i + \mathbf{B} \right) \right] \mathbf{Z} = \mathbf{0}. \quad (34)$$

From (34) a set of solutions denoted as $s^{(j)}(\alpha_i = 1)$, $\mathbf{Z}^{(j)}(\alpha_i = 1)$, $j = 1, 2, \dots, J$ are obtained. The number of solutions J depends on the model of dampers, i.e., $J = 2n$ and $J = 2n + m$ for the Kelvin model and the Maxwell model, respectively. The eigenvalues $s^{(j)}$ and eigenvectors $\mathbf{Z}^{(j)}$ could be both, the complex conjugate or real numbers. This means that, in general, the solutions to the investigated nonlinear eigenvalue problem could be shown as J curves in the space configuration. These curves will be referred to as the response curves. Moreover, one single point on each curve is known.

At this point the authors are interested in determination of successive points on the chosen curve, say the j th curve. Below, notation like $s_r(\alpha)$, $\mathbf{Z}_r(\alpha)$ will be used to denote the coordinate of the r th point on this curve. In our problem we are interested in determination of the considered response curve for $\alpha \in (\alpha_p, 1)$ and the most interesting solution is for $\alpha = \alpha_p$.

To the set of equations (33), which consists of $J + 1$ unknowns, i.e., the vector \mathbf{Z} of dimension $(J \times 1)$ and the parameter s , an additional condition in the following form is introduced:

$$g_2 = \frac{1}{2} \mathbf{Z}^T \mathbf{G}_s - a = \frac{1}{2} \mathbf{Z}^T \left[\sum_{i=1}^m (v_i \alpha + \kappa_i) s^{v_i \alpha + \kappa_i - 1} \mathbf{A}_i + \mathbf{A} \right] \mathbf{Z} - a = 0, \quad (35)$$

where a is of a given value. Eq. (35) may be considered as a way of normalization of the eigenvector \mathbf{Z} . Moreover, in this way, the symmetry of incremental equations which will be derived below is preserved.

Next, the authors will proceed to finding a solution to the eigenproblem (30) for a chosen value of $\alpha \in (\alpha_p, 1)$, using the incremental-iteration method. Based on the solution obtained for a certain value of parameter $\alpha = \alpha_r$ the solution is searched for a new value of this parameter $\alpha_{r+1} = \alpha_r + \Delta\alpha$, where $\Delta\alpha$ is the assumed increment of parameter α . The

approximate solution to a new value of parameter α obtained at the iteration step i will be denoted $s_{r+1}^{(i)}$ and $\mathbf{Z}_{r+1}^{(i)}$. In the first iteration step the solution obtained for α_r is used, which means $s_{r+1}^{(0)} = s_r$ and $\mathbf{Z}_{r+1}^{(0)} = \mathbf{Z}_r$.

The incremental equations of the Newton method, associated with Eqs. (33) and (35), are in the following form:

$$\mathbf{G}_z d\mathbf{Z} + \mathbf{G}_s ds = -\mathbf{g}_1, \quad \mathbf{G}_s^T d\mathbf{Z} + G_s ds = -g_2, \tag{36}$$

where

$$\begin{aligned} \mathbf{g}_1 &\equiv \mathbf{g}_1(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}), & g_2 &= g_2(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}), \\ \mathbf{G}_z &\equiv \mathbf{G}_z(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}) = \frac{\partial \mathbf{g}_1}{\partial \mathbf{Z}} = \sum_{i=1}^m s^{v_i \alpha + \kappa_i} \mathbf{A}_i + s \mathbf{A} + \mathbf{B}, \\ \mathbf{G}_s &\equiv \mathbf{G}_s(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}) = \frac{\partial \mathbf{g}_1}{\partial s} = \left[\sum_{i=1}^m (v_i \alpha + \kappa_i) s^{v_i \alpha + \kappa_i - 1} \mathbf{A}_i + \mathbf{A} \right] \mathbf{Z}, \\ G_s &\equiv G_s(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}) = \frac{\partial g_2}{\partial s} = \frac{1}{2} \mathbf{Z}^T \left[\sum_{i=1}^m (v_i \alpha + \kappa_i)(v_i \alpha + \kappa_i - 1) s^{v_i \alpha + \kappa_i - 2} \mathbf{A}_i \right] \mathbf{Z}. \end{aligned} \tag{37}$$

The new approximation of the solution is obtained after solving the set of equations (36) with respect to $d\mathbf{Z}$ and ds and using the following formulae:

$$s_{r+1}^{(i+1)} = s_{r+1}^{(i)} + ds, \quad \mathbf{Z}_{r+1}^{(i+1)} = \mathbf{Z}_{r+1}^{(i)} + d\mathbf{Z} \tag{38}$$

The iteration process may be finished when

$$|ds| \leq \varepsilon_1 \left| s_{r+1}^{(i+1)}(\alpha_{r+1}) \right|, \quad \|d\mathbf{Z}\| \leq \varepsilon_2 \|\mathbf{Z}_{r+1}^{(i+1)}(\alpha_{r+1})\|, \tag{39}$$

where ε_1 and ε_2 are the assumed accuracies of calculations.

Eqs. (36) and (37) are much simplified (Appendix C) when all the fractional parameters are equal, i.e., $\alpha_i = \alpha = \text{const}$.

The proposed method has good convergence properties. Usually, one incremental step and three or four iterations are enough to reach a solution providing the final value of the fractional parameter α_p . However, the proposed method has one important drawback. The method fails in attempts to determine the response curves $s(\alpha)$ and $\mathbf{Z}(\alpha)$ starting with real values of $s(\alpha)$ for $\alpha = 1$.

The computational method derived above enables determination of eigenvalues s_i . When the structural damping of a system is sufficiently small and the Kelvin fractional model is used to describe dampers then all eigenvalues are complex and conjugate. For the Maxwell fractional model of dampers some eigenvalues are real.

In this work, the authors propose to characterize the dynamic behaviour of frame with viscoelastic dampers by the natural frequency ω_i and the non-dimensional damping parameter γ_i . Like in the case of viscous damping, the above-mentioned properties are defined as

$$\omega_i^2 = \mu_i^2 + \eta_i^2, \quad \gamma_i = -\mu_i / \omega_i, \tag{40}$$

where $\mu_i = \text{Re}(s_i)$, $\eta_i = \text{Im}(s_i)$. These formulas refer to complex eigenvalues only.

In literature one may find different definitions of above-mentioned dynamic characteristics which, among others, can be found in papers [34,42–44].

4. Frequency response functions

In this section, the authors focus on steady state harmonic responses of the structures governed by Eqs. (10) or (19) and (22). For the harmonic external forces described by

$$\mathbf{p}(t) = \mathbf{P} \exp(i\lambda t), \tag{41}$$

where λ is the frequency of excitation, the displacement response of structure and the vector of state variables can be expressed as

$$\mathbf{q}_s(t) = \mathbf{Q}_s(\lambda) \exp(i\lambda t), \quad \mathbf{q}_r(t) = \mathbf{Q}_r(\lambda) \exp(i\lambda t), \tag{42}$$

$$\mathbf{z}(t) = \mathbf{Z}(\lambda) \exp(i\lambda t). \tag{43}$$

Substituting (41) and (43) into the state equation (24) yields the input–output relationship using the frequency response function $\tilde{\mathbf{H}}(\lambda)$

$$\mathbf{Z}(\lambda) = \tilde{\mathbf{H}}(\lambda) \tilde{\mathbf{P}}, \tag{44}$$

where $\tilde{\mathbf{P}} = \text{col}(\mathbf{P}, \mathbf{0})$. The matrix frequency response function $\tilde{\mathbf{H}}(\lambda)$ in terms of systems parameters is defined as

$$\tilde{\mathbf{H}}(\lambda) = \left[(i\lambda) \mathbf{A} + \sum_{j=1}^m (i\lambda)^{\alpha_j} \mathbf{A}_j + \mathbf{B} \right]^{-1}. \tag{45}$$

Alternatively, after substituting relationships (41) and (A2.2) into Eqs. (10), written in terms of physical coordinates, for the frame with Kelvin dampers the following equation is obtained:

$$\mathbf{Q}_s(\lambda) = \mathbf{H}(\lambda)\tilde{\mathbf{P}}, \tag{46}$$

where the frequency response function is defined as

$$\mathbf{H} = \left[-\lambda^2 \mathbf{M}_s + i\lambda \mathbf{C}_s + \sum_{j=1}^m (i\lambda)^{\alpha_j} \mathbf{e}_j c_j \mathbf{e}_j^T + \mathbf{K}_s + \mathbf{K}_d \right]^{-1}. \tag{47}$$

In the case of a frame with Maxwell dampers, after substituting relationships (42) and (43) into Eqs. (19) and (22) the following relationships are obtained:

$$\mathbf{D}_{ss}(\lambda)\mathbf{Q}_s(\lambda) + \mathbf{D}_{sr}(\lambda)\mathbf{Q}_r(\lambda) = \tilde{\mathbf{P}}, \quad \mathbf{D}_{rs}(\lambda)\mathbf{Q}_s(\lambda) + \mathbf{D}_{rr}(\lambda)\mathbf{Q}_r(\lambda) = \mathbf{0}, \tag{48}$$

where

$$\begin{aligned} \mathbf{D}_{ss}(\lambda) &= -\lambda^2 \mathbf{M}_s + i\lambda \mathbf{C}_s + \sum_{j=1}^m (i\lambda)^{\alpha_j} \tilde{\mathbf{e}}_j c_j \tilde{\mathbf{e}}_j^T + \mathbf{K}_s + \mathbf{K}_{ss}^d, \\ \mathbf{D}_{sr}(\lambda) &= \sum_{j=1}^m (i\lambda)^{\alpha_j} \tilde{\mathbf{e}}_j c_j \mathbf{h}_j^T - \sum_{j=1}^m \tilde{\mathbf{e}}_j k_j \mathbf{h}_j^T, \\ \mathbf{D}_{rs}(\lambda) &= \sum_{j=1}^m (i\lambda)^{\alpha_j} \mathbf{h}_j c_j \tilde{\mathbf{e}}_j^T - \sum_{j=1}^m \mathbf{h}_j k_j \tilde{\mathbf{e}}_j^T, \\ \mathbf{D}_{rr}(\lambda) &= \sum_{j=1}^m (i\lambda)^{\alpha_j} \mathbf{h}_j c_j \mathbf{h}_j^T + \sum_{j=1}^m \mathbf{h}_j k_j \mathbf{h}_j^T. \end{aligned} \tag{49}$$

Finally, it is possible to write relationships

$$\mathbf{Q}_s(\lambda) = \mathbf{H}_{ss}(\lambda)\mathbf{P}, \quad \mathbf{Q}_r(\lambda) = \mathbf{H}_{rs}(\lambda)\mathbf{P}, \tag{50}$$

where the frequency response functions $\mathbf{H}_{ss}(\lambda)$ and $\mathbf{H}_{rs}(\lambda)$ could be written in the following form:

$$\begin{aligned} \mathbf{H}_{ss}(\lambda) &= [\mathbf{D}_{ss}(\lambda) - \mathbf{D}_{sr}(\lambda)\mathbf{D}_{rr}^{-1}(\lambda)\mathbf{D}_{rs}(\lambda)]^{-1}, \\ \mathbf{H}_{rs}(\lambda) &= -\mathbf{D}_{rr}^{-1}(\lambda)\mathbf{D}_{rs}(\lambda)\mathbf{H}_{ss}(\lambda) = -\mathbf{D}_{rr}^{-1}(\lambda)\mathbf{D}_{rs}(\lambda)[\mathbf{D}_{ss}(\lambda) - \mathbf{D}_{sr}(\lambda)\mathbf{D}_{rr}^{-1}(\lambda)\mathbf{D}_{rs}(\lambda)]^{-1}. \end{aligned} \tag{51}$$

Element $H_{ij}(\lambda)$ of the matrix frequency response function is the displacement of the i th degree of freedom of the structure subjected to the unit harmonically varying force at the j th degree of freedom.

5. Results of calculation

5.1. Example 1—Two-storey frame

A typical calculation was made for a two-storey frame with a damper mounted on the second storey (see Fig. 3).

The following data were chosen: the masses of the first and second storeys are $m_1 = 21.6$ Mg and $m_2 = 17.28$ Mg, respectively; the height and rigidity of the columns are 3 m and $El_c = 11685$ kNm², the span and rigidity of the beam are 6 m and $El_b = 47416$ kNm², respectively. The dampers data are: $k_d = k_{11}$ where k_{11} is the element of \mathbf{K}_s matrix, $c_d = 376.456$ kNs/m, $\tau_d = c_d/k_d = 0.02$. The static condensation is used to eliminate the rotational nodal parameters.

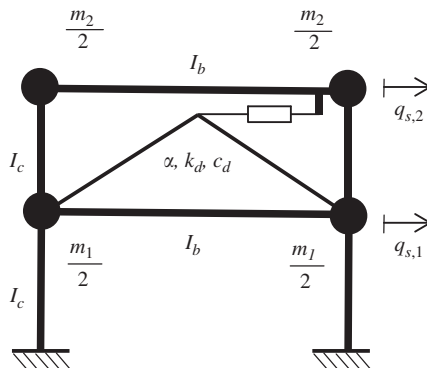


Fig. 3. Diagram of a frame with a single damper.

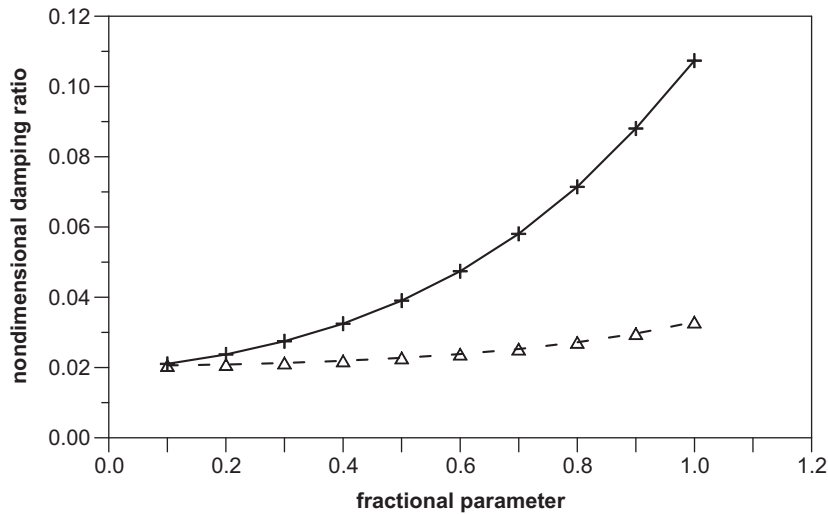


Fig. 4. Non-dimensional damping ratio γ_1 versus fractional parameter α for frame with Kelvin damper (the dashed line with triangles) and for frame with Maxwell damper (the solid line with crosses).

Changes of the first natural frequency of vibration due to changes of the fractional parameter α are minor. These changes are less than 0.15% for the Kelvin model and less than 5.7% for the Maxwell model.

In the case where the frame with the Maxwell damper is considered, the solution to the nonlinear eigenproblem provides five eigenvalues. For $\alpha = 1$ we obtain one real eigenvalue and two pairs of complex, conjugate ones. The real eigenvalue is associated with rheological properties of damper. Below, the natural frequency and the non-dimensional damping ratio conjugate with complex eigenvalues are presented. The plot of the non-dimensional damping ratio versus fractional parameter α is shown in Fig. 4 for the Kelvin model (the dashed line with triangles) and the Maxwell model (the solid line with crosses), respectively. It is easy to observe that both models provide the non-dimensional damping ratio rises when the fractional parameter increases.

5.2. Example 2—A ten-storey frame

In the second numerical test, the ten storey frame shown in Fig. 5 is investigated.

We assume the same value of mass $m_s = 18$ Mg on each storey and the same rigidity of columns on each storey $k_s = 51.60$ MN/m. The viscous damping matrix of structure is assumed to be proportional to the mass and stiffness matrices of frame structure as follows:

$$\mathbf{C}_s = \beta_1 \mathbf{K}_s + \beta_2 \mathbf{M}_s, \quad (52)$$

where $\beta_1 = 0.0093$, $\beta_2 = 0.1006$. Two groups of dampers are installed between the selected storeys. Two dampers, characterized by parameters $k_1 = 40$ MN/m and $c_1 = 800$ kNs/m are mounted on storeys 2 and 3. The second group of dampers, which parameters are $k_2 = 30$ MN/m and $c_2 = 600$ kNs/m, occurs on three storeys from 6 to 8.

Four rheological models are applied to describe the dynamic behaviour of dampers which differ in the value of fractional order α . The following models are chosen: (i) the Kelvin model ($\alpha_1 = \alpha_2 = 1$), (ii) the Maxwell model ($\alpha_1 = \alpha_2 = 1$), (iii) the fractional Kelvin model ($\alpha_1 = 0.8$, $\alpha_2 = 0.6$), (iv) and the fractional Maxwell model ($\alpha_1 = 0.8$, $\alpha_2 = 0.6$).

Using the procedure developed in this work the eigenvalues are computed (Table 1) for various damper models, the natural frequencies of structure (Table 2), and the values of non-dimensional damping ratio (Table 3). In each case, what is obtained is ten pairs of conjugate complex eigenvalues representing the dynamic behaviour of frames and, additionally, for the frame with classic Maxwell dampers, five negative real eigenvalues reflecting the creeping behaviour in the dynamics.

One may see in Table 3 that the value of the non-dimensional damping ratio γ_i rises when the value of parameter α_i increases. The incremental-iteration method used here enables the nonlinear eigenproblem to be solved very fast. It is noted that the presented method is very efficient for any value of parameter α increment. The iteration process converges very quickly, around three iteration steps, which enable the solution to the nonlinear eigenproblem to be found very fast. Other existing methods [41–43] which can be used to solve the considered nonlinear eigenvalue problem also require the iterative procedure. However, the detailed comparison of numerical effectiveness of these methods is beyond the scope of this paper.

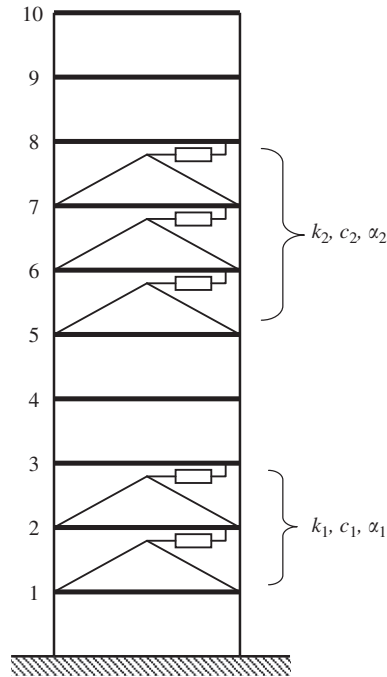


Fig. 5. Diagram of a 10-storey frame supplemented with two groups of dampers.

Table 1
Eigenvalues for various damper models.

Modal number	Kelvin model		Maxwell model	
	$\alpha_1 = \alpha_2 = 1$	$\alpha_1 = 0.8, \alpha_2 = 0.6$	$\alpha_1 = \alpha_2 = 1$	$\alpha_1 = 0.8, \alpha_2 = 0.6$
1	-0.2150 ± i9.05373	-0.15050 ± i9.06947	-0.31341 ± i8.04753	-0.199819 ± i8.06423
2	-1.6618 ± i27.3432	-0.69798 ± i27.4453	-2.0351 ± i24.8743	-0.866345 ± i24.2741
3	-1.5845 ± i41.0483	-0.98350 ± i40.9677	-1.63258 ± i40.2153	-1.13531 ± i39.4970
4	-7.26030 ± i59.5285	-2.65052 ± i60.1429	-4.3333 ± i57.4251	-2.63126 ± i54.8554
5	-12.1042 ± i78.9621	-4.53308 ± i76.5824	-5.92816 ± i73.6432	-4.51219 ± i69.2738
6	-8.3160 ± i91.3798	-5.49980 ± i89.9669	-6.56773 ± i87.3558	-5.96771 ± i81.7537
7	-19.7512 ± i90.5160	-9.40134 ± i97.1531	-6.85865 ± i94.8283	-5.99637 ± i91.3018
8	-32.5646 ± i97.5756	-7.83897 ± i110.907	-9.18200 ± i106.718	-6.60849 ± i100.102
9	-60.9843 ± i110.201	-11.6385 ± i131.633	-13.2786 ± i124.874	-9.64612 ± i109.115
10	-70.3351 ± i105.891	-29.4717 ± i134.166	-13.9426 ± i126.266	-15.6599 ± i114.493
11	-	-	-29.3319	-
12	-	-	-32.5576	-
13	-	-	-33.9397	-
14	-	-	-36.4305	-
15	-	-	-41.3625	-

The non-dimensional damping ratios are also determined using the modal strain method [46,47] to make a comparison of the results obtained. Basing on the modal strain method the non-dimensional damping ratio γ_i for the considered system can be defined as follows:

$$\gamma = \frac{\mathbf{a}_s^T \mathbf{K}_i \mathbf{a}_s}{2 \mathbf{a}_s^T \mathbf{K}_r \mathbf{a}_s}, \tag{53}$$

where $\mathbf{K}_i, \mathbf{K}_r, \mathbf{a}_s$ denote the imaginary and real part of the stiffness matrix and the modal shape, respectively. For a structure equipped with dampers which differ in the value of fractional parameters α_i , the above matrices are:

$$\mathbf{K}_r = \mathbf{K}_s + \sum_{j=1}^m \left(\mathbf{K}_{d,j} + \mathbf{C}_{d,j} \omega^{\alpha_j} \cos \frac{\alpha_j \pi}{2} \right), \quad \mathbf{K}_i = \omega \mathbf{C}_s + \sum_{j=1}^m \mathbf{C}_{d,j} \omega^{\alpha_j} \sin \frac{\alpha_j \pi}{2}. \tag{54}$$

Table 2
Natural frequencies of structure ω_i [rad/s].

Modal number	Kelvin model		Maxwell model	
	$\alpha_1 = \alpha_2 = 1$	$\alpha_1 = 0.8, \alpha_2 = 0.6$	$\alpha_1 = \alpha_2 = 1$	$\alpha_1 = 0.8, \alpha_2 = 0.6$
1	9.05628	9.08026	8.05363	8.06670
2	27.3937	27.4542	24.9574	24.2896
3	41.0788	40.9795	40.2485	39.5133
4	59.9696	60.2013	57.5884	54.9184
5	79.8844	76.7164	73.8814	69.4206
6	91.7575	90.1348	87.6024	81.9712
7	92.6459	97.6069	95.0760	91.4985
8	102.866	111.184	107.112	100.320
9	125.950	132.147	125.578	109.540
10	127.122	137.365	127.033	115.559

Table 3
The values of non-dimensional damping ratio γ_i .

Modal number	Kelvin model				Maxwell model	
	The incremental-iteration method		Modal strain method		$\alpha_1 = \alpha_2 = 1$	$\alpha_1 = 0.8, \alpha_2 = 0.6$
	$\alpha_1 = \alpha_2 = 1$	$\alpha_1 = 0.8, \alpha_2 = 0.6$	$\alpha_1 = \alpha_2 = 1$	$\alpha_1 = 0.8, \alpha_2 = 0.6$		
1	0.023740	0.016575	0.023802	0.016630	0.038915	0.024770
2	0.060663	0.025423	0.061125	0.025676	0.081542	0.035667
3	0.038572	0.024000	0.040230	0.024297	0.040562	0.028732
4	0.121066	0.044027	0.122104	0.044416	0.075246	0.047912
5	0.151521	0.059088	0.157011	0.061873	0.080238	0.064997
6	0.090630	0.061017	0.162223	0.066233	0.074972	0.072802
7	0.213190	0.096318	0.190022	0.085053	0.072138	0.065535
8	0.316573	0.070504	0.277835	0.068962	0.085723	0.065874
9	0.484195	0.088072	0.483859	0.095964	0.080238	0.088060
10	0.553288	0.214550	0.526706	0.183351	0.109756	0.135514

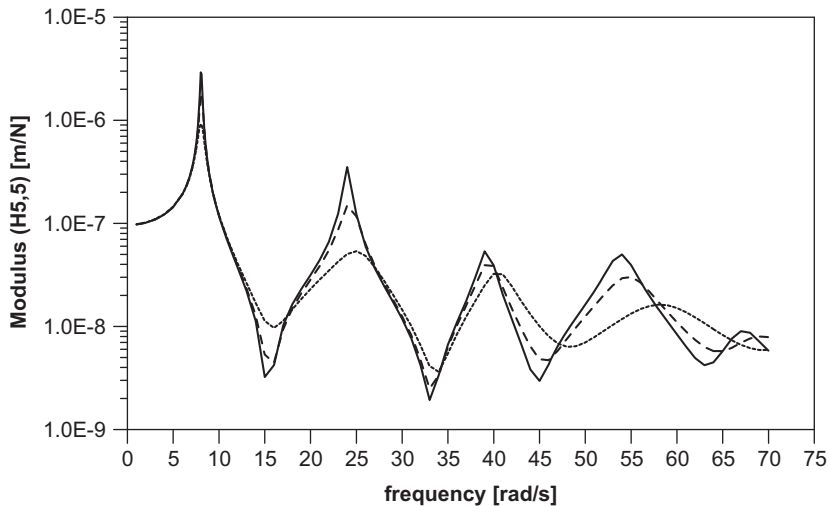


Fig. 6. The modulus of frequency response function $H(5,5)$. The solid line – results for $\alpha_1 = \alpha_2 = 0.2$; the dashed line – results for $\alpha_1 = \alpha_2 = 0.6$; the dotted line – results for $\alpha_1 = \alpha_2 = 1.0$.

when the fractional Kelvin model of dampers is used. The vector of modal shape \mathbf{a}_s and the corresponding natural frequency ω can be obtained from the following eigenproblem:

$$(\tilde{\mathbf{K}}_r - \omega^2 \mathbf{M}_s) \mathbf{a}_s = 0. \tag{55}$$

where $\tilde{\mathbf{K}}_r = \mathbf{K}_s + \sum_{j=1}^m \mathbf{K}_{d,j}$

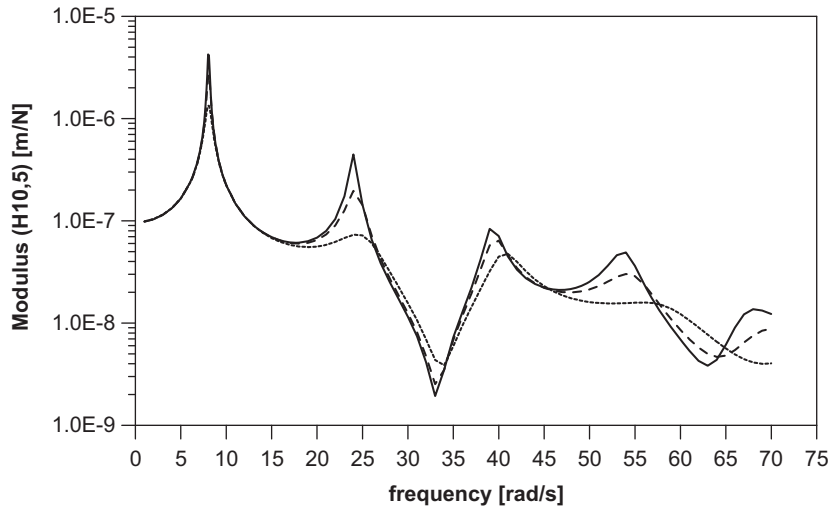


Fig. 7. The modulus of frequency response function $H(10,5)$. The solid line – results for $\alpha_1 = \alpha_2 = 0.2$; the dashed line – results for $\alpha_1 = \alpha_2 = 0.6$; the dotted line – results for $\alpha_1 = \alpha_2 = 1.0$.

The calculated non-dimensional damping ratios are shown in Table 3. The results obtained by both methods agree very well. For a few first modes of vibration, the maximal differences are of the order of 1%. However, the differences were observed to grow for higher modes of vibration and reached 14% for the mode of number ten, for which the non-dimensional damping ratio has a very high value. This observation is in agreement with the remarks written in paper [46].

Finally, for the considered structure we investigate the frequency response functions. Using relationships (50) the function $H_{5,5}(\lambda)$ was calculated taking into account various values of parameter α_i , which describes the fractional Maxwell damper (see Fig. 6).

The results obtained for $\alpha_1 = \alpha_2 = 0.2$ are presented in Fig. 6 by the solid line, for $\alpha_1 = \alpha_2 = 0.6$ by the dashed line, and for $\alpha_1 = \alpha_2 = 1.0$ by the dotted line. Fig. 7 shows the frequency response function $H_{10,5}(\lambda)$ which expresses the last storey displacements caused by force acting on the fifth storey. One may observe the increase of the non-dimensional damping ratio when parameter α rises.

6. Concluding remarks

In this paper, the equations of motion for planar frames with VE dampers are derived. Two fractional, three-parameter rheological models, i.e., the fractional Kelvin model and the fractional Maxwell model are used to describe the dynamic behaviour of the considered systems. The equations of motion of the structure with dampers are written in terms of both physical and state-space variables. The proposed approach in the state space formulation is new. This is the main advantage of the proposed formulation, where matrices with huge dimensions are not required. The resulting matrix equation of motion is a fractional differential equation.

Moreover, the paper is dedicated to the determination of the dynamics characteristics of the considered structures. The nonlinear eigenvalue problem is formulated from which the dynamics characteristics of a system can be determined. The continuation method is used to solve the above-mentioned nonlinear eigenvalue problem. In contrast to the method presented previously, the dimension of the eigenvalue problem arising here is much smaller. Numerical results demonstrate the effectiveness and applicability of the proposed approach. The influence of the key parameter, which describes the order of the fractional derivative, on the dynamic parameters of frames with VE dampers, is also shown.

Acknowledgments

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Appendix A. The equation of motion in physical coordinates

When all the fractional parameters are equal, i.e., $\alpha_i = \alpha = const.$, the equation of motion for frame with the Kelvin dampers (10) takes the form:

$$\mathbf{M}_s D_t^2 \mathbf{q}_s(t) + \mathbf{C}_s D_t^1 \mathbf{q}_s(t) + \mathbf{C}_d D_t^\alpha \mathbf{q}_s(t) + (\mathbf{K}_s + \mathbf{K}_d) \mathbf{q}_s(t) = \mathbf{p}(t), \quad (\text{A1})$$

where the damping matrix of the dampers and the matrix of dampers stiffness are defined as

$$\mathbf{C}_d = \sum_{i=1}^m \mathbf{e}_i c_i \mathbf{e}_i^T, \quad \mathbf{K}_d = \sum_{i=1}^m \mathbf{e}_i k_i \mathbf{e}_i^T. \tag{A2}$$

Eqs. (21) and (22) which describe the dynamic behaviour of frame with the Maxwell dampers are much simplified when all the fractional parameters are equal, i.e., $\alpha_i = \alpha = const.$ In such case the following can be written:

$$\mathbf{M}_s D_t^\alpha \mathbf{q}_s(t) + \mathbf{C}_s D_t^1 \mathbf{q}_s(t) + \mathbf{C}_{ss}^d D_t^\alpha \mathbf{q}_s(t) + (\mathbf{K}_s + \mathbf{K}_{ss}^d) \mathbf{q}_s(t) + \mathbf{C}_{sr}^d D_t^\alpha \mathbf{q}_r(t) - \mathbf{K}_{sr}^d \mathbf{q}_r(t) = \mathbf{p}(t), \tag{A3}$$

$$\mathbf{C}_{rs}^d D_t^\alpha \mathbf{q}_s(t) + \mathbf{C}_{rr}^d D_t^\alpha \mathbf{q}_r(t) - \mathbf{K}_{rs}^d \mathbf{q}_s(t) + \mathbf{K}_{rr}^d \mathbf{q}_r(t) = \mathbf{0}, \tag{A4}$$

where

$$\mathbf{C}_{ss}^d = \sum_{i=1}^m \hat{\mathbf{e}}_i c_i \hat{\mathbf{e}}_i^T, \quad \mathbf{C}_{sr}^d = \sum_{i=1}^m \hat{\mathbf{e}}_i c_i \mathbf{h}_i^T, \quad \mathbf{C}_{rs}^d = \sum_{i=1}^m \mathbf{h}_i c_i \hat{\mathbf{e}}_i^T = (\mathbf{C}_{sr}^d)^T, \tag{A5}$$

$$\mathbf{K}_{rr}^d = \sum_{i=1}^m \mathbf{h}_i k_i \mathbf{h}_i^T, \quad \mathbf{K}_{rs}^d = \sum_{i=1}^m \mathbf{h}_i k_i \hat{\mathbf{e}}_i^T = (\mathbf{K}_{sr}^d)^T, \quad \mathbf{K}_{rr}^d = \sum_{i=1}^m \mathbf{h}_i k_i \mathbf{h}_i^T, \tag{A6}$$

$$\mathbf{K}_{sr}^d = \sum_{i=1}^m \hat{\mathbf{e}}_i k_i \mathbf{h}_i^T, \quad \mathbf{K}_{ss}^d = \sum_{i=1}^m \hat{\mathbf{e}}_i k_i \hat{\mathbf{e}}_i^T. \tag{A7}$$

Appendix B. The equation of motion in state space

When all the fractional parameters are equal, i.e., $\alpha_i = \alpha = const.$ the equation of motion in state space (24) takes the form:

$$\mathbf{A} D_t^1 \mathbf{z}(t) + \hat{\mathbf{A}}_1 D_t^\alpha \mathbf{z}(t) + \mathbf{B} \mathbf{z}(t) = \tilde{\mathbf{p}}(t). \tag{B1}$$

For the Kelvin model of dampers we have

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} \mathbf{C}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{B2}$$

For the Maxwell model of dampers the equation of motion has the form of Eq. (B1), where now:

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} \mathbf{C}_{rr}^d & \mathbf{C}_{rs}^d & \mathbf{0} \\ \mathbf{C}_{sr}^d & \mathbf{C}_{ss}^d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K}_{rr}^d & -\mathbf{K}_{rs}^d & \mathbf{0} \\ -\mathbf{K}_{sr}^d & \mathbf{K}_s + \mathbf{K}_{ss}^d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{M}_s \end{bmatrix}. \tag{B3}$$

Appendix C. The incremental equations of the Newton method

If fractional parameters α_i are identical (i.e., $\alpha_i = \alpha = const.$) for all dampers, then the formulas associated with the incremental equations of the Newton method (36) are in the following form:

$$\begin{aligned} \mathbf{g}_1 &\equiv [\mathbf{s}\mathbf{A} + s^\alpha \mathbf{A}_1 + \mathbf{B}] \mathbf{Z} = \mathbf{0}, \\ \mathbf{g}_2 &= \frac{1}{2} \mathbf{Z}^T \mathbf{G}_s - a = \frac{1}{2} \mathbf{Z}^T (\alpha s^{\alpha-1} \mathbf{A}_1 + \mathbf{A}) \mathbf{Z} - a = 0, \\ \mathbf{G}_z &\equiv \mathbf{G}_z(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}) = \frac{\partial \mathbf{g}_1}{\partial \mathbf{Z}} = s^\alpha \mathbf{A}_1 + \mathbf{s}\mathbf{A} + \mathbf{B}, \\ \mathbf{G}_s &\equiv \mathbf{G}_s(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}) = \frac{\partial \mathbf{g}_1}{\partial s} = [\alpha s^{\alpha-1} \mathbf{A}_1 + \mathbf{A}] \mathbf{Z}, \\ \mathbf{G}_s &\equiv \mathbf{G}_s(s_{r+1}^{(i)}, \mathbf{Z}_{r+1}^{(i)}, \alpha_{r+1}) = \frac{\partial \mathbf{g}_2}{\partial s} = \frac{1}{2} \mathbf{Z}^T [\alpha(\alpha-1) s^{\alpha-2} \mathbf{A}_1] \mathbf{Z}. \end{aligned} \tag{C1}$$

References

- [1] T.T. Soong, B.F. Spencer, Supplemental energy dissipation: state-of-the-art and state-of-the-practice, *Engineering Structures* 24 (2002) 243–259.
- [2] C. Christopoulos, A. Filiatrault, *Principles of Passive Supplemental Damping and Seismic Isolation*, IUSS Press, Pavia, Italy, 2006.
- [3] Z. Osiński, *Tłumienie drgań mechanicznych*, Wydawnictwo Naukowo-Techniczne, Warszawa, 1979.
- [4] R.L. Bagley, P.J. Torvik, Fractional calculus – a different approach to the analysis of viscoelastically damped structures, *AIAA Journal* 27 (1989) 1412–1417.
- [5] S.W. Park, Analytical modeling of viscoelastic dampers for structural and vibration control, *International Journal of Solids and Structures* 38 (2001) 8065–8092.
- [6] N. Makris, M.C. Constantinou, Fractional-derivative Maxwell model for viscous dampers, *ASCE Journal of Structural Engineering* 117 (1991) 2708–2724.

- [7] T. Pritz, Analysis of four-parameter fractional derivative model of real solid materials, *Journal of Sound and Vibration* 195 (1996) 103–115.
- [8] T. Pritz, Five-parameter fractional derivative model for polymeric damping materials, *Journal of Sound and Vibration* 265 (2003) 935–952.
- [9] A. Aprile, J.A. Inaudi, J.M. Kelly, Evolutionary model of viscoelastic dampers for structural applications, *Journal of Engineering Mechanics* 123 (1997) 551–560.
- [10] A. Idesman, R. Niekamp, E. Stein, Finite elements in space and time for generalized viscoelastic model, *Computational Mechanics* 27 (2001) 49–60.
- [11] A. Palmeri, F. Ricciardelli, A. De Luca, G. Muscolino, State space formulation for linear viscoelastic dynamic systems with memory, *Journal of Engineering Mechanics* 129 (2003) 715–724.
- [12] A. Pálfalvi, A comparison of finite element formulations for dynamics of viscoelastic beams, *Finite Elements in Analysis and Design* 44 (2008) 814–818.
- [13] T. Hatada, T. Kabori, M.A. Ishida, N. Niwa, Dynamic analysis of structures with Maxwell model, *Earthquake Engineering and Structural Dynamics* 29 (2000) 159–176.
- [14] M.P. Singh, L.M. Moreschi, Optimal placement of dampers for passive response control, *Earthquake Engineering and Structural Dynamics* 31 (2002) 955–976.
- [15] V.A. Matsagar, R.S. Jangid, Viscoelastic damper connected to adjacent structures involving seismic isolation, *Journal of Civil Engineering and Management* 11 (2005) 309–322.
- [16] M.P. Singh, N.P. Verma, L.M. Moreschi, Seismic analysis and design with Maxwell dampers, *Journal of Engineering Mechanics* 129 (2003) 273–282.
- [17] W.S. Zhang, Y.L. Xu, Vibration analysis of two buildings linked by Maxwell model defined fluid dampers, *Journal of Sound and Vibration* 233 (2000) 775–796.
- [18] M. Enelund, P. Olsson, Damping described by fading memory analysis and application to fractional derivative models, *International Journal of Solids and Structures* 36 (1999) 939–970.
- [19] A. Fenander, Modal synthesis when modeling damping by use of fractional derivatives, *AIAA Journal* 34 (1996) 1051–1058.
- [20] T. Chang, M.P. Singh, Seismic analysis of structures with a fractional derivative model of viscoelastic dampers, *Earthquake Engineering and Engineering Vibration* 1 (2002) 251–260.
- [21] A.K. Shukla, T.K. Datta, Optimal use of viscoelastic dampers in building frames for seismic force, *Journal of Structural Engineering* 125 (1999) 401–409.
- [22] J.H. Park, J. Kim, K.W. Min, Optimal design of added viscoelastic dampers and supporting braces, *Earthquake Engineering and Structural Dynamics* 33 (2004) 465–484.
- [23] S.H. Lee, D.I. Son, J. Kim, K.W. Min, Optimal design of viscoelastic dampers using eigenvalue assignment, *Earthquake Engineering and Structural Dynamics* 33 (2004) 521–542.
- [24] M.H. Tsai, K.C. Chang, Higher-mode effect on the seismic responses of buildings with viscoelastic dampers, *Earthquake Engineering and Engineering Vibration* 1 (2002) 119–129.
- [25] R. Okada, N. Nakata, B.F. Spencer, K. Kasai, B.S. Kim, Rational polynomial approximation modeling for analysis of structures with VE dampers, *Journal of Earthquake Engineering* 10 (2006) 97–125.
- [26] L.E. Suarez, A. Shokooh, An eigenvector expansion method for the solution of motion containing fractional derivatives, *Journal of Applied Mechanics* 64 (1997) 629–635.
- [27] M. Kuroda, Formulation of a state equation including fractional-order state vectors, *Journal of Computational and nonlinear Dynamics* 3 (2008) 021202–1–8.
- [28] S. Sorrentino, A. Fassana, Finite element analysis of vibrating linear systems with fractional derivative viscoelastic models, *Journal of Sound and Vibration* 299 (2007) 839–853.
- [29] G. Catania, A. Fasana, A condensation technique for finite element dynamic analysis using fractional derivative viscoelastic models, *Journal of Vibration and Control* 14 (2008) 1573–1586.
- [30] A.C. Galucio, J.F. Deu, R. Ohayon, Finite element formulation of viscoelastic sandwich beams using fractional derivative operators, *Computational Mechanics* 33 (2004) 282–291.
- [31] F. Cortés, M.J. Elejabarrieta, Homogenised finite element for transient dynamic analysis of unconstrained layer damping beams involving fractional derivative models, *Computational Mechanics* 40 (2007) 313–324.
- [32] F. Cortés, M.J. Elejabarrieta, Finite element formulations for transient dynamic analysis in structural systems with viscoelastic treatments containing fractional derivative models, *International Journal of Numerical Methods in Engineering* 69 (2007) 2173–2195.
- [33] A. Schmidt, L. Gaul, Finite element formulation of viscoelastic constitutive equations using fractional time derivatives, *Journal of Nonlinear Dynamics* 29 (2002) 37–55.
- [34] S. Krenk, Damping mechanisms and models in structural dynamics, *Proceedings of the 4th International Conference on Structural Dynamics EURO-DYN2002*, Munich, Germany, 2–5 September 2002.
- [35] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.
- [36] E. Riks, Some computational aspects of the stability analysis of nonlinear structures, *Computer Methods in Applied Mechanics and Engineering* 47 (1984) 219–259.
- [37] Y.B. Yang, M.S. Shieh, Solution method for nonlinear problems with multiple critical points, *AIAA Journal* 28 (1999) 10–16.
- [38] R. Lewandowski, Solutions with bifurcation points for free vibrations of beams—an analytical approach, *Journal of Sound and Vibration* 177 (1994) 239–249.
- [39] R. Lewandowski, Computational formulation for periodic vibration of geometrically nonlinear structures, *International Journal of Solids and Structures* 34 (1997) 1925–1964.
- [40] A. Cardona, A. Lerusse, M. Geradin, Fast Fourier nonlinear vibration analysis, *Computational Mechanics* 22 (1998) 128–142.
- [41] F. Cortes, M.J. Elejabarrieta, Computational methods for complex eigenproblems in finite element analysis of structural systems with viscoelastic damping treatments, *Computer Methods in Applied Mechanics and Engineering* 195 (2006) 6448–6462.
- [42] F. Cortes, M.J. Elejabarrieta, An approximate numerical method for the complex eigenproblem in systems characterized by a structural damping matrix, *Journal of Sound and Vibration* 296 (2006) 166–182.
- [43] E.M. Daya, M. Potier-Ferry, A numerical method for nonlinear problems application to vibrations of viscoelastic structures, *Computers and Structures* 79 (2001) 533–541.
- [44] Q. Chen, K. Worden, A decomposition method for the analysis of viscoelastic structural dynamics with time-dependent Poisson's ratio, *Strain*, 2009, doi: 10.1111/j.1475-1305.2008.0057.x.
- [45] R. Seydel, *From Equilibrium to Chaos. Practical Bifurcation and Stability Analysis*, Elsevier, New York, 1988.
- [46] D.-G. Lee, S. Hong, J. Kim, Efficient seismic analysis of building structures with added viscoelastic dampers, *Engineering Structures* 24 (2002) 1217–1227.
- [47] Y. Fu, K. Kasai, Comparative study of frames using viscoelastic and viscous dampers, *Journal of Structural Engineering* 124 (1998) 513–522.