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Pure odd-order oscillators with constant excitation

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Dedicated to the 80th birthday of
Prof. Bozidar D. Vujanovic

ABSTRACT

In this paper the excited vibrations of a truly nonlinear oscillator are analyzed. The excitation is assumed to be constant and the nonlinearity is pure (without a linear term). The mathematical model is a second-order nonhomogeneous differential equation with strong nonlinear term. Using the first integral, the exact value of period of vibration i.e., angular frequency of oscillator described with a pure nonlinear differential equation with constant excitation is analytically obtained. The closed form solution has the form of gamma function. The period of vibration depends on the value of excitation and of the order and coefficient of the nonlinear term. For the case of pure odd-order-oscillators the approximate solution of differential equation is obtained in the form of trigonometric function. The solution is based on the exact value of period of vibration. For the case when additional small perturbation of the pure oscillator acts, the so called 'Cveticanin's averaging method' for a truly nonlinear oscillator is applied. Two special cases are considered: one, when the additional term is a function of distance, and the second, when damping acts. To prove the correctness of the method the obtained results are compared with those for the linear oscillator. Example of pure cubic oscillator with constant excitation and linear damping is widely discussed. Comparing the analytically obtained results with exact numerical ones it is concluded that they are in a good agreement. The investigations reported in the paper are of special interest for those who are dealing with the problem of vibration reduction in the oscillator with constant excitation and pure nonlinear restoring force the examples of which can be found in various scientific and engineering systems. For example, such mechanical systems are seats in vehicles, supports for machines, cutting machines with periodical motion of the cutting tools, presses, etc. The examples can be find in electronics (electromechanical devices like micro-actuators and micro oscillators), in music instruments (hammers in piano), in human voice producing folds (voice cords), etc.

1. Introduction

In the most of textbooks the problem of a linear oscillator with constant excitation F_0 is widely discussed and the closed form analytical solution is given. Based on that solution for the linear oscillator, the approximate analytical solving procedures are developed for solving systems perturbed with small nonlinearities: the Kryloff–Bogoliuboff method [1] and the method of slowly varying amplitude and phase [2], the Bogoliuboff–Mitropolski method [3], the multiple scales method [4], the straightforward expansion method and the Lindstedt–Poincare method [4], combined equivalent linearization and averaging perturbation method [5] and [6], the series expansion method, the iteration procedure for calculating approximations to periodic solutions [7] and [8], the homotopy perturbation technique [9–11], the homotopy

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analysis method [12], etc. There are a lot of models where the nonlinearity is much stronger than the linearity and even where the oscillator is purely nonlinear. The nonlinearity is caused by the geometry of the system but most often by material properties. Experimentally is proved that the stress–strain law is pure nonlinear for a significant number of materials: many aircraft materials, e.g., aluminium, titanium, etc., [13], copper and copper alloys [14], aluminium alloys and annealed copper [15], wood [16], hydrophilic polymers [17], composites [18], polyurethane foam [19], ceramic materials [20], etc. For such systems the application of the aforementioned methods is not possible. Namely, the differential equations are without linear terms and also the linearization of the equation is not possible due to the properties of the system. These oscillators are not the perturbed versions of the linear ones and their behavior is far of those obtained for linear ones.

The aim of the paper is to combine several known methods, namely, the harmonic balance and averaging procedure, for solving of the oscillators with strong pure nonlinear term and a strong constant excitation F_0. The mathematical model of the oscillator is

$$\ddot{x}+c_1^2 x|x|^{\alpha-1}=F_0, \tag{1}$$

where $\alpha\geq 1$ is a real number and c_1^2 is the coefficient of nonlinear term. Using the first integral of (1) the exact period and angular frequency of vibration is analytically obtained. Based on the exact value of the angular frequency of the system the approximate analytical solution of (1), when the nonlinearity is of odd-order ($\alpha=2n-1$, where n=1,2,...), in the form of circular function for initial conditions

$$x(0)=0, \quad \dot{x}(0)=0, \tag{2}$$

is introduced. The approximate analytical solution is compared with an accurate numerically integrated solution for $\alpha=1$,3 and 5.

For the case when the pure odd-order nonlinear oscillator with constant excitation is perturbed with some small functions, the mathematical model is

$$\ddot{x}+c_1^2 x|x|^{\alpha-1}=F_0+\varepsilon f(x,\dot{x}), \tag{3}$$

where $\varepsilon \ll 1$ is a small parameter and $\varepsilon f(x,\dot{x})$ is the small perturbation function. In this paper the procedure for solving truly nonlinear oscillations, which has been termed 'Cveticanin's averaging method' (CAM) [21] by Mickens [22], is adopted for solving the differential Eq. (3). It is assumed that the amplitude and the phase of vibration are time variable. Two problems are considered: the linear and the pure cubic oscillators with constant excitation and with weak linear damping. The approximate analytical solutions are compared with those numerically obtained.

Suggested solution procedure requires to calculate the exact period of vibration for the pure nonlinear oscillator with constant excitation described with (1) with initial conditions (2). Based on the 'exact' frequency of vibration Ω the approximate solution is assumed as

$$x=F(1-\cos\Omega t), \tag{4}$$

where F is the amplitude of oscillation, while F_0 is the amplitude of forcing term. In spite of the fact that Eq. (3) is a strong nonlinear one and the method of superposition fails, Burton [23] showed that as long as the waveform is reasonably close to simple harmonic, it is possible to use the approximative solution for nonlinear oscillations with the form which corresponds to the linear oscillator. It is the reason that the solution of the strong nonlinear differential Eq. (3) is assumed in the form (4). Substituting (4) into (1) and using the harmonic balance method the unknown value for F is obtained. Now, it is at this point where the adopted 'Cveticanin's method' can be applied. The assumption of time variable amplitude and phase in (4) is introduced. The averaging over the exact period of vibration gives an accurate approximate solution.

2. The exact period of vibration

In general, the first integral of (1) for the initial conditions (2) has the following form

$$\frac{\dot{x}^2}{2}+x\left(c_1^2\frac{|x|^\alpha}{\alpha+1}-F_0\right)=0. \tag{5}$$

Assuming that the direction of excitation is constant and does not depend on motion direction, the curves in the $x-\dot{x}$ plane are determined

$$\frac{\dot{x}^2}{2}\pm|x|\left(c_1^2\frac{|x|^\alpha}{\alpha+1}-F_0\right)=0, \tag{6}$$

where the upper sign is for $x\geq 0$ and the other for $x\leq 0$. Due to equality of the curves, the analyses is done only for one of them. The curves in $x-\dot{x}$ which correspond to (6), when $x\geq 0$ and $x\leq 0$, are closed ones and it gives the conclusion that the solution of (1) is periodic.

For calculating the period of vibration let us rewrite the relation (6) into

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2F_0|x| - 2c_1^2 \frac{|x|^{\alpha+1}}{\alpha+1}}. \tag{7}$$

i.e.,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2F_0|x|}\sqrt{1 - \frac{c_1^2}{F_0}\frac{|x|^{\alpha}}{\alpha+1}}. \tag{8}$$

For mathematical expediency, we introduce a new variable

$$u = \frac{c_1^2}{F_0}\frac{|x|^{\alpha}}{\alpha+1}, \tag{9}$$

with time derivative

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{c_1^2}{F_0}\frac{\alpha}{\alpha+1}|x|^{\alpha-1}\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{10}$$

Substituting (9) and (10) into (7) and separating the variables the following expression is obtained

$$\mathrm{d}t = \frac{1}{\alpha u\sqrt{2F_0(1-u)}}\left(\frac{F_0(\alpha+1)u}{c_1^2}\right)^{1/2\alpha}\mathrm{d}u. \tag{11}$$

After suitable transformation and integration half the period of vibration is shown to be

$$\frac{T}{2} = \frac{1}{\alpha\sqrt{2F_0}}\left(\frac{F_0(\alpha+1)}{c_1^2}\right)^{1/2\alpha}\int_0^1 u^{(1-2\alpha)/2\alpha}(1-u)^{-1/2}\,\mathrm{d}u. \tag{12}$$

According to the definition of the beta function given in the appendix

$$\int_0^1 u^{(1/2\alpha)-1}(1-u)^{(1/2)-1}\,\mathrm{d}u = B\left(\frac{1}{2\alpha},\frac{1}{2}\right), \tag{13}$$

the period of vibration is

$$T = \frac{2}{\alpha\sqrt{2F_0}}\left(\frac{F_0(\alpha+1)}{c_1^2}\right)^{1/2\alpha} B\left(\frac{1}{2\alpha},\frac{1}{2}\right). \tag{14}$$

The practical application of beta function in integral form is connected with many difficulties and is not convenient for use by engineers and technicians. It is recommended to rewrite the expression (14) by using the gamma function, which is much more appropriate for practice. Introducing the transformation (A5) the period of vibration is

$$T = \frac{\sqrt{2}}{\alpha F_0^{(\alpha-1)/2\alpha}}\left(\frac{(\alpha+1)}{c_1^2}\right)^{1/2\alpha}\frac{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2\alpha}\right)}. \tag{15}$$

Analyzing the relation (15) it can be concluded:

(a) For higher values of parameter F_0, the period of vibrations is shorter. If F_0 tends to infinity, the period tends to zero. If F_0 is zero the period of vibration is infinitely large and no oscillatory motion exists.

(b) For the linear oscillator ($\alpha = 1$) the period of vibration does not depend on the excitation frequency F_0 and has the well known value

$$T = \frac{2}{c_1}B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{2\pi}{c_1}, \tag{16}$$

as $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$. The frequency of vibration is

$$\Omega = \frac{2\pi}{T} = c_1. \tag{17}$$

(c) Coefficient of the nonlinearity c_1 has also a significant influence on the period of vibration. In a certain oscillator higher the value of the coefficient, the shorter the period of vibration. The influence of the parameter c_1 is not of the same order for various types of oscillators. Namely, $T = f(F_0,\alpha)(c_1)^{-1/\alpha}$: for the linear oscillator ($\alpha = 1$) the angular frequency is directly dependent on c_1, for higher order of nonlinearity the influence of c_1 on the period of vibration is smaller and for α tends to infinity the influence of the parameter c_1 disappears.

3. The approximate harmonic solution for odd-order nonlinear oscillators

Based on the exact period of vibration (15) the exact angular frequency is

$$\Omega=\frac{2\pi}{T}=\sqrt{2}\pi\alpha F_0^{(\alpha-1)/2\alpha}\left(\frac{c_1^2}{\alpha+1}\right)^{1/2\alpha}\frac{\Gamma\left(\frac{1+\alpha}{2\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{18}$$

For the case when α is an odd number the approximate solution for (1) is assumed in the form (4), where F is an unknown value. The assumed solution is the corrected version of the previously used ones, as it includes the exact angular frequency Ω to describe the oscillatory motion of the system. Substituting (4) into (1) we have

$$F\Omega^2\cos\Omega t+c_1^2F^\alpha(1-\cos\Omega t)^\alpha=F_0. \tag{19}$$

The determination of F using (19) is not an easy task. Namely, the harmonic balance method is not directly applicable due to the form of the relation (19). For mathematical reasons, let us introduce the series expansion of the power order function (see [24]) into (19)

$$F_0=F\Omega^2\cos\Omega t+F^\alpha c_1^2\left[1+\sum_{n=1}\frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(2n)!}\cos^{2n}\Omega t.\right.$$
$$\left.-\sum_{n=1}\frac{\alpha(\alpha-1)\dots(\alpha-(2n-1)+1)}{(2n-1)!}\cos^{2n-1}\Omega t\right], \tag{20}$$

and the series expansion for function $\cos^{2n-1}\Omega t$ [24]

$$\cos^{2n-1}(\Omega t)=\frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos(2n-2k-1)(\Omega t). \tag{21}$$

Using (20) with (21), and separating only the terms with $\cos\Omega t$ the following algebraic equation is obtained

$$F\Omega^2-F^\alpha c_1^2\sum_{n=1}\frac{\alpha(\alpha-1)\dots(\alpha-(2n-1)+1)}{(2n-1)!}\frac{1}{2^{2n-2}}\binom{2n-1}{n-1}=0. \tag{22}$$

It follows that

$$F=\left(\frac{\Omega^2}{qc_1^2}\right)^{1/(\alpha-1)}, \tag{23}$$

where

$$q=\sum_{n=1}\frac{\alpha(\alpha-1)\dots(\alpha-(2n-1)+1)}{(2n-1)!}\frac{1}{2^{2n-2}}\binom{2n-1}{n-1}. \tag{24}$$

Substituting (18) into (23) we have

$$F=\left(\frac{1}{q}\left(\frac{1}{\alpha+1}\right)^{1/\alpha}\left(\frac{\Gamma\left(\frac{1+\alpha}{2\alpha}\right)}{\sqrt{2\pi}}\alpha\Gamma\left(\frac{1}{2\alpha}\right)\right)^2\right)^{1/(\alpha-1)}\left(\frac{F_0}{c_1^2}\right)^{1/\alpha}. \tag{25}$$

For a certain oscillator the amplitude of vibration depends on the excitation value and coefficient of the nonlinear term: The higher the excitation F_0, the higher is the amplitude of vibration F. The influence of the coefficient of nonlinearity c_1 on the amplitude of vibration F is the same as on the frequency of vibration (18).

1. For the linear oscillator ($\alpha=1$) the amplitude of vibration is, according to (25), $F=F_0/c_1^2$ and for $\Omega=c_1^2$, the relation (4) gives the exact solution

$$x=\frac{F_0}{c_1^2}[1-\cos(c_1t)]. \tag{26}$$

2. For the pure cubic oscillator ($\alpha=3$) the exact period of vibration is

$$T=\frac{\sqrt{2}}{3F_0^{1/3}}\left(\frac{(4)}{c_1^2}\right)^{1/6}\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{3}\right)}=\frac{4.3274}{\sqrt[3]{F_0c_1}}, \tag{27}$$

and the corresponding angular frequency is

$$\Omega=1.452\sqrt[3]{F_0c_1}. \tag{28}$$

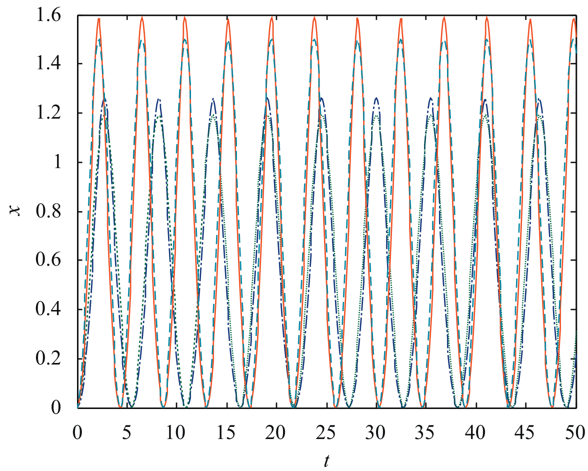

Fig. 1. Analytical x_A and numerical x_N solutions of pure cubic oscillator for: in solid line $F_0=1$, x_N; dashed line $F_0=1$, x_A; dashed–dotted line $F_0=0.5$, x_N; dotted line $F_0=0.5$, x_A.

The amplitude of vibration is according to (25)

$$F = 0.74981\left(\frac{F_0}{c_1^2}\right)^{1/3}. \tag{29}$$

In general the approximate solution of the oscillator with a pure cubic term is

$$x = 0.74981\left(\frac{F_0}{c_1^2}\right)^{1/3}(1-\cos(1.452t\sqrt[3]{F_0 c_1})). \tag{30}$$

Some numerical examples are considered and the analytical solutions are compared with numerical ones:

(a) For F_0=1 and c_1^2=1 and the differential equation

$$\ddot{x}+x^3 = 1, \tag{31}$$

the approximate solution is

$$x_A = 0.74981(1-\cos(1.452t)). \tag{32}$$

(b) For F_0=0.5 and c_1^2=1 the mathematical model of the oscillator is

$$\ddot{x}+x^3 = 0.5, \tag{33}$$

the approximate analytical solution is

$$x_A = 0.59512(1-\cos(1.1525t)). \tag{34}$$

In Fig. 1 the analytical (x_A) and numerical (x_N) solutions for F_0=0.5 and F_0=1 are plotted. It can be concluded that the approximate solution is on the top of the numerical one. The maximal difference is for the amplitudes, but even this difference is smaller than 5percent and is negligible.

2. For the pure fifth-order oscillator

$$\ddot{x}+x^5 = 0.5, \tag{35}$$

the approximate analytical solution is

$$x_A = 0.62119(1-\cos(1.2429t)). \tag{36}$$

In Fig. 2 the analytical solution x_A, (36), is compared with numerical solution x_N to (35). The difference between the solutions is seen to be minimal.

4. The oscillator with additional small nonlinearity

The model of the oscillator with additional small nonlinearity is described with (3). The additional nonlinearity represent the perturbation of the oscillatory system (1). It is known that in the real oscillatory systems the additional nonlinearity may cause amplitude and phase variations. Such a phenomena should be included into the assumed solution

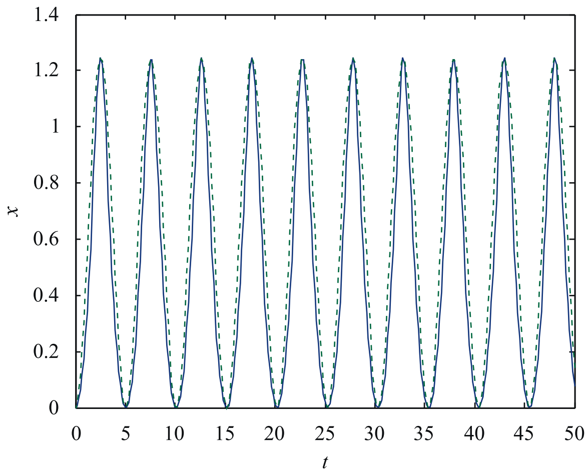

Fig. 2. Analytical x_A (dashed line) and numerical x_N (solid line) solutions of pure fifth order oscillator with constant excitation $F_0=1$.

of (3). Let us introduce the approximate solution as

$$x = F - F^*(t)\cos\Psi(t) \equiv F - F^*\cos\Psi, \tag{37}$$

where

$$\Psi = \Omega t + \theta(t) \equiv \Omega t + \theta, \tag{38}$$

$F^*(t)$ and $\theta(t)$ are unknown time t functions. The amplitude and phase of vibration, $F^*(t)$ and $\theta(t)$, are the perturbed values of the previously calculated one, F and θ. The solution (37) with (38) has the same form as (4) for (1) but with time variable amplitude and phase.

In this section the CAM averaging method suggested for truly nonlinear oscillators [21,22] is adopted for solving of the differential Eq. (3). The developed method is based on the solution of the generating solution (4) of (1) and its first time derivative. For the trial solution (37) of (3) the first time derivative has the same form as that for the generating solution

$$\dot{x} = F^*\Omega\sin\Psi, \tag{39}$$

for

$$F^*\dot{\theta}\sin\Psi - \dot{F}^*\cos\Psi = 0. \tag{40}$$

Substituting the time derivative of (39) and the function (37) into (3) we obtain

$$\begin{aligned}\dot{F}^*\Omega\sin\Psi + F^*\Omega^2\cos\Psi + F^*\dot{\theta}\Omega\cos\Psi + c_1^2(F - F^*\cos\Psi)^\alpha \\ = F_0 + \varepsilon f((F - F^*\cos\Psi), F^*\Omega\sin\Psi).\end{aligned} \tag{41}$$

The two first-order differential Eqs. (40) and (41) represent the rewritten version of the second-order differential Eq. (3) in the new variables F^* and θ. Eqs. (40) and (41) are coupled and linear for $\dot{F}^*$ and $\dot{\theta}$. Separating the variables $\dot{F}^*$ and $\dot{\theta}$ in (40) and (41) the two uncoupled equations for $\dot{F}^*$ and $\dot{\theta}$ are obtained

$$\begin{aligned}F^*\dot{\theta}\Omega = -c_1^2(F - F^*\cos\Psi)^\alpha\cos\Psi - F^*\Omega^2\cos^2\Psi \\ + F_0\cos\Psi + \varepsilon f((F - F^*\cos\Psi), F^*\Omega\sin\Psi)\cos\Psi,\end{aligned} \tag{42}$$

$$\begin{aligned}\dot{F}^*\Omega = -c_1^2(F - F^*\cos\Psi)^\alpha\sin\Psi - F^*\Omega^2\cos\Psi\sin\Psi \\ + F_0\sin\Psi + \varepsilon f(F - F^*\cos\Psi, F^*\Omega\sin\Psi)\sin\Psi.\end{aligned} \tag{43}$$

In general, the two coupled Eqs. (42) and (43) cannot be solved analytically for F^* and θ. Observing that the right-hand sides of (42) and (43) are periodic in Ψ with period 2π, the averaging procedure over this period is introduced. It gives the two averaged expressions in the new variables F^* and θ

$$\dot{F}^* = \frac{\varepsilon}{\Omega}\int_0^{2\pi} f((F - F^*\cos\Psi), F^*\Omega\sin\Psi)\sin\Psi\,d\Psi, \tag{44}$$

$$F^*\dot{\theta}=-\frac{c_1^2}{\Omega}\int_0^{2\pi}(F-F^*\cos\Psi)^{\alpha}\cos\Psi\,\mathrm{d}\Psi-\frac{1}{2}F^*\Omega$$
$$+\frac{\varepsilon}{\Omega}\int_0^{2\pi}f((F-F^*\cos\Psi),F^*\Omega\sin\Psi)\cos\Psi\,\mathrm{d}\Psi. \quad (45)$$

Solving the averaged differential Eqs. (44) and (45) for the initial conditions (2), i.e.,

$$F^*(0)=F,\quad \theta(0)=0, \quad (46)$$

the approximate amplitude and phase of vibration are obtained.

Some special cases of additional terms are considered.

1. For the case when the small additional term is a function only of x the relation (44) transforms into

$$\dot{F}^*=0, \quad (47)$$

i.e., for the initial conditions (46), it is

$$F^*=F. \quad (48)$$

Substituting (48) into (45) and integrating, it follows

$$\theta=-\frac{c_1^2F^{\alpha-1}t}{\Omega}\int_0^{2\pi}(1-\cos\Psi)^{\alpha}\cos\Psi\,\mathrm{d}\Psi-\frac{1}{2}\Omega t$$
$$+\frac{\varepsilon t}{\Omega F}\int_0^{2\pi}f(F-F\cos\Psi)\cos\Psi\,\mathrm{d}\Psi. \quad (49)$$

The approximate analytical solution is

$$x=F-F\cos\left(\frac{\Omega t}{2}-\frac{c_1^2F^{\alpha-1}t}{\Omega}\int_0^{2\pi}(1-\cos\Psi)^{\alpha}\cos\Psi\,\mathrm{d}\Psi+\frac{\varepsilon t}{\Omega F}\int_0^{2\pi}f(F-F\cos\Psi)\cos\Psi\,\mathrm{d}\Psi\right). \quad (50)$$

The amplitude of vibration is independent of the value of the additional term, but the frequency of vibration is modified.

2. For the case when the small additional term is a function of damping $\dot{x}$

$$\ddot{x}+c_1^2x|x|^{\alpha-1}=F_0+\varepsilon f(\dot{x}), \quad (51)$$

the averaged differential equations are

$$\dot{F}^*=\frac{\varepsilon}{\Omega}\int_0^{2\pi}f(F^*\Omega\sin\Psi)\sin\Psi\,\mathrm{d}\Psi, \quad (52)$$

$$\dot{\theta}=-\frac{c_1^2F^{\alpha}}{\Omega F^*}\int_0^{2\pi}\left(1-\frac{F^*}{F}\cos\Psi\right)^{\alpha}\cos\Psi\,\mathrm{d}\Psi-\frac{1}{2}\Omega. \quad (53)$$

The amplitude of vibration is a function of the additional nonlinearity, but the phase angle is independent on that value.

Due to damping during the long time ($t\to\infty$) the oscillatory behavior of the system disappears and the motion stops at a fixed position which is the steady-state solution of the equation

$$c_1^2x^{\alpha}=F_0,$$

i.e.,

$$x=\left(\frac{F_0}{c_1^2}\right)^{1/\alpha}.$$

The coefficient of damping has no influence on the steady-state position of the system. The higher the damping coefficient, the shorter the time for the system to get into the fixed position.

5. Examples

(1) To prove the correctness of the suggested procedure the result of CAM is applied for solving of a linear differential equation with a constant excitation and a small linear damping

$$\ddot{x}+c_1^2x=F_0-\varepsilon b\dot{x}, \quad (54)$$

where $\varepsilon b \ll 0$ is the damping coefficient. According to (50) the approximate analytical solution is

$$x=\frac{F_0}{c_1^2}\left(1-\exp\left(-\frac{\varepsilon bt}{2}\right)\cos(c_1t)\right). \quad (55)$$

Comparing (55) with the exact analytical solution of (54) for initial conditions (2)

$$x=\frac{F_0}{c_1^2}+\frac{F_0}{2c_1^2}\sqrt{1+\frac{(\varepsilon b)^2}{c_1^2-\frac{(\varepsilon b)^2}{4}}}\exp\left(-\frac{\varepsilon bt}{2}\right)$$

$$\left[-\frac{2}{\sqrt{1+\frac{(\varepsilon b)^2}{c_1^2-\frac{(\varepsilon b)^2}{4}}}}\cos\left(\sqrt{c_1^2-\frac{(\varepsilon b)^2}{4}}t\right)-\frac{\varepsilon b}{\sqrt{c_1^2+\frac{3}{4}(\varepsilon b)^2}}\sin\left(\sqrt{c_1^2-\frac{(\varepsilon b)^2}{4}}t\right)\right], \quad (56)$$

it can be concluded that for $(\varepsilon b)^2\approx 0$ the expression (56) simplifies into (55). Due to the negligible difference between the relations (56) and (55), which describe the transient motion to the steady-state position $x_{t\to\infty}=F_0/c_1^2$, it is obvious that the CAM method, developed in the paper, gives very accurate results.

(2) The suggested CAM procedure is applied for solving a pure cubic oscillator with constant excitation and small linear damping, too

$$\ddot{x}+x^3=F_0-\varepsilon b\dot{x}, \quad (57)$$

where εb is the constant coefficient. According to (52) the approximate amplitude is

$$F^*=F\exp\left(-\frac{\varepsilon b}{2}t\right), \quad (58)$$

and the approximate solution is

$$x=F-F\exp\left(-\frac{\varepsilon b}{2}t\right)\cos\left(\left(\frac{1}{2}\Omega+\frac{3c_1^2F^2}{2\Omega}\right)t+\frac{3c_1^2F^2}{8\varepsilon b\Omega}(1-\exp(-\varepsilon bt))\right). \quad (59)$$

For the numerical case when $\varepsilon b=0.01$, F_0=1 and c_1^2=1 the analytical solution is

$$x_A=0.74981-0.74981\exp(-0.005t)\cos((1.31)t+14.233(1-\exp(-0.01t)). \quad (60)$$

The relation (60) describes the transient motion to a fixed position x_A=0.99975. In Fig. 3 the analytical solution x_A and the numerical x_N for (57) are compared.

In Fig. 4 the $x-t$ diagrams obtained numerically for $\varepsilon b=0.1$; 0.5; 0.75, F_0=1 and c_1^2=1 for large value of t are plotted. It is evident that for large t the oscillator tends to a fixed position which is independent on the coefficient of damping. The value of the damping coefficient has no influence on the steady-state position, as it was previously stated after the analytical analysis. In Fig. 5 the time-history diagrams for $\varepsilon b=0.1$, F_0=1 and various values of coefficient of nonlinearity c_1^2=0.5; 1; 1.5 are plotted. The result obtained for the steady-state solution is compared with analytically obtained one

$$x_A=\sqrt[3]{\frac{F_0}{c_1^2}}=const. \quad (61)$$

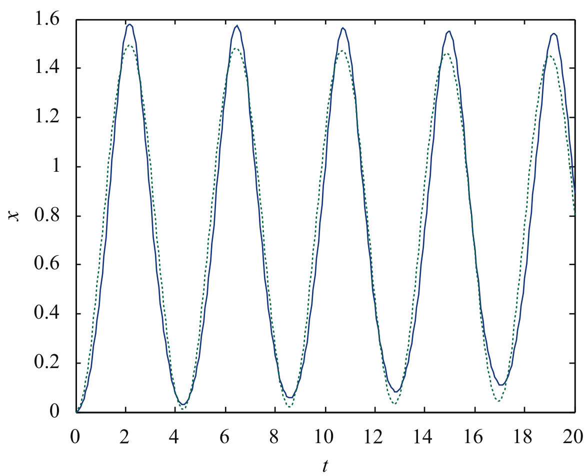

Fig. 3. $x-t$ diagrams obtained analytically (solid line) and numerically (dashed line) for the pure cubic oscillator with constant excitation and small damping.

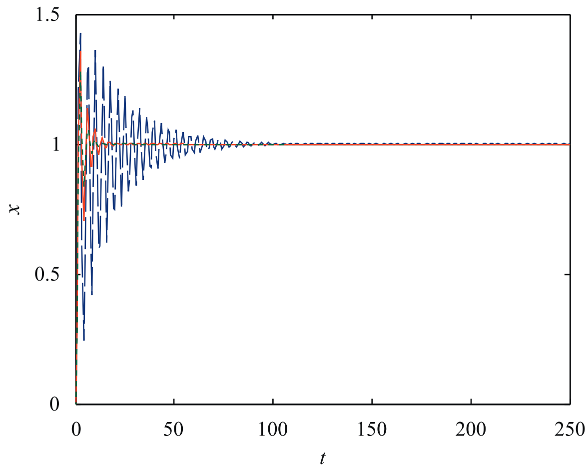

Fig. 4. $x-t$ diagrams for F_0=1, c_1^2=1. and various damping coefficients: $\varepsilon b=0.1$ (dashed–dotted line); 0.5 (solid line); 0.75 (dashed line).

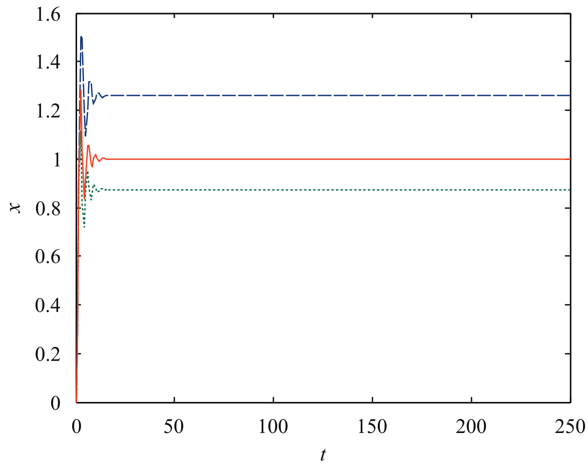

Fig. 5. $x-t$ diagrams for $\varepsilon b=0.01$, F_0=1 and c_1^2=0.5. (dashed–dotted line); c_1^2=1. (solid line); c_1^2=0.5 (dashed line).

The following numerical values are calculated: for c_1^2=0.5 the analytical value is 1.2596 and the numerically obtained one is 1.259921; for c_1^2=1 we have x_A=0.99975 and x_N=1 and for c_1^2=1.5 we have x_A=0.87336 and x_N=0.873580. Comparing the analytical and numerical values it can be seen that the difference is negligible.

6. Conclusions

The following is concluded:

1. For the pure nonlinear oscillator with constant excitation the period and angular frequency of vibration in the closed analytical form can be calculated. The period and also the angular frequency of vibration depend on the order of nonlinearity, value of excitation and the coefficient of nonlinearity.
   For higher values of excitation the period of vibration is shorter and the frequency is higher. If the excitation tends to zero, the frequency is zero. If the excitation tends to infinity, the period of vibration tends to zero. For the linear oscillator the period of vibration is independent on the excitation.
   Increasing the coefficient of nonlinearity the period of vibration decreases. The influence of the parameter c_1 is not of the same order for various types of oscillators. Namely, for the linear oscillator the angular frequency is directly dependent on coefficient. For higher order of nonlinearity, the influence of coefficient on the period of vibration is smaller and tends to disappear for large order (α tends to infinity).
2. The amplitude of vibration of the pure nonlinear oscillator with constant excitation also depends on the excitation value, order and coefficient of nonlinearity. In a certain oscillator the amplitude of vibration increases with increasing of the excitation value. For the oscillator with fixed excitation the amplitude of vibration is higher for smaller value of coefficient of nonlinearity when the order of nonlinearity is fixed.

3. For the case when additional small perturbation to the oscillator exists, the amplitude and frequency of vibration are time dependent.
4. If the perturbation of the pure nonlinear and excited oscillator is of damping type, for the long time t, the oscillatory motion stops, and the system gets into a constant steady-state position. The distance from to zero is smaller for higher values of coefficient of nonlinearity, smaller excitation and higher order of nonlinearity. Besides, the value of the damping coefficient has no influence on the steady-state position.
5. The suggested 'Cveticanin averaging method' gives approximate analytical solutions which describe also the transient motion. The difference between analytical and 'exact' numerical solutions is negligible.
6. The investigations reported in the paper are of special interest for engineers who are dealing with the problem of vibration reduction in the oscillator with constant excitation. It can be seen that the proper choice of material of oscillator (for example, with high coefficient of nonlinearity) decreases the vibration level and when even small damping exists the oscillations disappear.
7. The Duffing problems represent the special case of (3) when $\alpha=2$. The future investigation will be directed to Van der Pol problems and also parametrically excited systems like Duffing ones and Mathie–Hill models.

Appendix A. The Euler's integrals of the first and second kind

Euler's integral of the first kind also called beta function, $B(p,q)$, is defined as (see [24])

$$B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}\,\mathrm{d}u, \tag{A1}$$

which exists for

$$\mathrm{Re}(p)>0, \quad \mathrm{Re}(q)>0. \tag{A2}$$

Introducing the new variable $x=1-u$ into (A1), the beta function is expressed as (see [25])

$$B(p,q)=-\int_1^0 x^{q-1}(1-x)^{p-1}\,\mathrm{d}x=\int_0^1 x^{q-1}(1-x)^{p-1}\,\mathrm{d}x=B(q,p). \tag{A3}$$

The beta function is symmetric in (p,q).

Euler's integral of the second kind also called gama function is (see [26])

$$\Gamma(p)=\int_0^\infty u^{p-1}e^{-u}\,\mathrm{d}u, \tag{A4}$$

where p satisfies the relation (A2). The connection between the Euler's integrals of the first and second kind is

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{A5}$$

For $(p-1)=n$, where n is a whole positive number, the relation (A4) modifies into

$$\Gamma(n+1)=\int_0^\infty u^n e^{-u}\,\mathrm{d}u=n!. \tag{A6}$$

Thus,

$$\Gamma(n)=(n-1)!, \tag{A7}$$

and the relation between (A6) and (A7) is

$$\Gamma(n+1)=n(n-1)!=n\Gamma(n). \tag{A8}$$

Generalizing (A6) for any value of p we have

$$\Gamma(p+1)=p!, \tag{A9}$$

and the corresponding relations

$$\Gamma(p)=(p-1)!, \tag{A10}$$

and

$$\Gamma(p+1)=p\Gamma(p). \tag{A11}$$

Substituting (A10) into (A3) the transformed version of the beta function is

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}=\frac{(p-1)!(q-1)!}{(p+q-1)!}, \tag{A12}$$

which is suitable for calculation.

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