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Noise rejection for two time-based multi-output modal analysis techniques

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ABSTRACT

Modal analysis is a well-developed field with many applications. In particular, forced response multi-output approaches are well suited for system identification and online damage detection because they use the natural excitations the system undergoes during its normal operation. In this work, two of these approaches, smooth orthogonal decomposition (SOD) and direct system parameter identification (DSPI), are analyzed, compared, and improved upon. SOD was originally developed as a tool for detecting features of chaotic dynamical systems. Recently, it has been used as a time-based multi-output modal analysis approach. SOD has been demonstrated to work for the free vibration case and for random excitations. DSPI was developed as a time-based multi-input multi-output approach. When the inputs are not measured, DSPI is very similar to SOD and can handle both free vibrations and random excitations. However, if the inputs are measured or known DSPI can also handle arbitrary excitations. To improve the performance of these two methods when used with noisy data, novel noise filtering algorithms are proposed. Numerical simulations are performed to compare the two methods and to show the effectiveness of the filtering algorithms in improving frequency and mode shape extraction.

1. Introduction

Experimental modal analysis can be applied to a variety of fields including system identification, model validation and analysis, sensing, and structural health monitoring. The use of modal analysis for these applications requires robust noise rejection because data extracted experimentally is corrupted by noise. This paper proposes novel noise rejection algorithms for two modal analysis techniques—smooth orthogonal decomposition (SOD), and direct system parameter identification (DSPI)—to improve their performance when dealing with practical noisy data. The novel noise filtering algorithms can be extended to other experimental modal analysis techniques. In addition, the paper reveals certain relationships between SOD and DSPI. For example, the manuscript shows (through a theoretical analysis) that these methods (under usual settings) are both second-order accurate, and despite that, DSPI outperforms SOD.

Modal analysis techniques can be separated into frequency and time domain approaches. These approaches can be broken further into single-input single-output approaches, single-input multiple-output and multiple-input multiple-output (MIMO) approaches. A survey of the time-based MIMO methods was conducted by Yang et al. [1]. One category of methods are free and impulse response methods, such as the poly-reference complex exponential [2], eigensystem realization algorithm [3] and Ibrahim time domain [4] methods. The rest of the time-based MIMO methods are forced response methods

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which use the forcing from natural excitations to determine modal properties. These methods include methods that can handle random excitations (which do not need to be measured), such as the random decrement approach [5] and SOD [6,7]. Additionally, time-based MIMO force response methods have been developed to handle general excitations (that do not need to be random, but need to be measured if they are not random). These methods include auto-regressive moving average vector techniques [8], vector backward auto-regressive with exogenous inputs method [9], and DSPI [10]. For online damage detection, forced response modal analysis techniques are better suited because they use the natural excitations the system is subjected to.

DSPI was developed by Leuridan [10] and is a modal analysis technique that actually works for free vibrations (impulse responses) and random excitations. For both these cases, no input measurements are needed. Additionally, DSPI can handle other types of excitations as long as these excitations are measured. DSPI has recently been used for extracting modal properties of augmented (nonlinear) systems that are not symmetric [11–13] and asymmetric systems designed with nonlinear controllers [14,15].

SOD was originally proposed by Chatterjee et al. [16] to track a slowly drifting parameter in chaotic dynamical systems. It was then developed further as a multidimensional damage identification tool [17,18]. Recently, SOD was modified by Chelidze and Zhou [6] to extract modes and frequencies of a system for the free vibration case. Farooq and Feeny [7] extended the method for random excitations (unmeasured). Currently, the method is restricted to symmetric systems.

The paper is organized as follows. The next section begins with background information about SOD and DSPI, and then introduces novel filtering algorithms for both approaches. The following section analyzes the relationship between SOD, DSPI, and the newly developed filtering algorithms. First, the SOD and DSPI methods are compared analytically to show their similarities in accuracy. Then, both methods are compared by using various simulations of a stiff mass-spring system, and the performance of the noise rejection techniques is discussed.

2. SOD, DSPI and noise rejection

This section provides background information for both SOD and DSPI, and introduces noise rejection approaches for each method.

2.1. SOD

The basic idea of SOD [6,7] used for modal analysis is that the modal properties of a lightly damped symmetric system can be extracted from the following eigenvalue problem:

$$\mathbf{R}\boldsymbol{\Psi}\boldsymbol{\Lambda}' = \mathbf{S}\boldsymbol{\Psi}, \tag{1}$$

where $\mathbf{R}=\mathbf{X}\mathbf{X}^T$, $\mathbf{S}=\mathbf{V}\mathbf{V}^T$, $\boldsymbol{\Psi}$ is an $n \times n$ matrix of eigenvectors, and $\boldsymbol{\Lambda}'$ is an $n \times n$ diagonal matrix of eigenvalues. The $\mathbf{X}$ matrix is an $n \times N$ ensemble matrix where the n rows correspond to the degrees of freedom of the system, and the N columns correspond to samples in time. The $\mathbf{V}$ matrix can be found from $\mathbf{V}=\mathbf{X}\mathbf{D}^T$, where $\mathbf{D}$ is a finite difference coefficient matrix. When applied to $\mathbf{X}$, matrix $\mathbf{D}$ provides an approximation for the time derivative of $\mathbf{X}$. It has been shown [6,7] that the solution of the eigenvalue problem in Eq. (1) is similar to the eigenvalue problem given by $\mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Lambda} = \mathbf{K}\boldsymbol{\Phi}$, where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrix of the $n \times n$ system, $\boldsymbol{\Phi}$ contains the right eigenvectors, and $\boldsymbol{\Lambda}$ is a diagonal matrix with entries on the diagonal equaling a square of the natural frequencies. For free vibrations or random excitations, and using the approximation $\mathbf{D}^T\mathbf{D}\mathbf{X}^T \approx -\ddot{\mathbf{X}}^T$, it can be shown [6,7] that $\boldsymbol{\Lambda}' \approx \boldsymbol{\Lambda}$ and $\boldsymbol{\Psi}^{-1} \approx \boldsymbol{\Phi}$.

Next, we propose a seemingly minor modification to the formulation of SOD, which drastically diminishes the effects of random noise on the accuracy of the method. Consider the following eigenvalue problem $\mathbf{R}'\boldsymbol{\Psi}\boldsymbol{\Lambda}' = \mathbf{S}'\boldsymbol{\Psi}$, where $\mathbf{R}'=\mathbf{X}\mathbf{X}'^T$ and $\mathbf{S}'=\mathbf{V}\mathbf{V}'^T$. The $\mathbf{X}$ matrix is the same $n \times N$ ensemble matrix as in Eq. (1), while $\mathbf{X}'$ is an $n \times N$ ensemble matrix collected for the same system at different times (under different random excitations and different noise realizations). That means that $\mathbf{X}$ and $\mathbf{X}'$ are similar measurements collected at distinct times. Hence, the noise present in $\mathbf{X}$ is distinct from and uncorrelated with the noise present in $\mathbf{X}'$. The $\mathbf{V}$ matrix is still defined as $\mathbf{V}=\mathbf{X}\mathbf{D}^T$, while $\mathbf{V}'=\mathbf{X}'\mathbf{D}^T$. The fact that the noise realizations are distinct for $\mathbf{X}$ and $\mathbf{X}'$ greatly diminishes the effects of the random noise on the eigenvalue problem because the noise is filtered out in $\mathbf{R}'$ and $\mathbf{S}'$ (viewed as the autocorrelation of $\mathbf{X}$ with $\mathbf{X}'$, and that of $\mathbf{V}$ with $\mathbf{V}'$).

2.2. DSPI

DSPI [10] uses a discrete-time equation expressed as $\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathbf{y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{y}_{t-p} + \mathbf{e}_t$, when solving the free vibration or random forcing case. In this equation, $\mathbf{y}_t$ corresponds to the $n_0 \times 1$ measured degrees of freedom at a given time sample, $\mathbf{e}_t$ are errors in measurements or random forcing, p is chosen such that $pn_0 \geq 2n$, and $\mathbf{A}_1, \mathbf{A}_2 \ldots \mathbf{A}_p$ are discrete time correlation matrices. The basic idea behind DSPI is that the temporal correlations of an $n \times n$ system in continuous time (obtained from $\mathbf{M}$ and $\mathbf{K}$) relate to their correlations in discrete time (obtained from $\mathbf{A}_1, \mathbf{A}_2 \ldots \mathbf{A}_p$) such that the modal properties can be related. Note that DSPI works for deterministic forcing if this forcing is measured (or known). Also, note that DSPI does not require all degrees of freedom to be measured, i.e. the vector $\mathbf{y}$ is a subset of the states of the system. Grouping

$p+N$ equations one can write

$$[\mathbf{y}_{p+1}\ldots\mathbf{y}_{p+N}]=[\mathbf{A}_1\ldots\mathbf{A}_p]\begin{bmatrix}\mathbf{y}_p & \cdots & \mathbf{y}_{p+N-1}\\ \vdots & & \vdots\\ \mathbf{y}_1 & \cdots & \mathbf{y}_N\end{bmatrix}+[\mathbf{e}_{p+1}\cdots\mathbf{e}_{p+N}]. \tag{2}$$

The matrices $\mathbf{A}_1\ldots\mathbf{A}_p$ can be obtained from Eq. (2) via a pseudoinverse $[\mathbf{A}_1\ldots\mathbf{A}_p]=[\mathbf{y}_{p+1}\ldots\mathbf{y}_{p+N}][\mathbf{\Gamma}]^T([\mathbf{\Gamma}][\mathbf{\Gamma}]^T)^{-1}$, where

$$[\mathbf{\Gamma}]=\begin{bmatrix}\mathbf{y}_p & \cdots & \mathbf{y}_{p+N-1}\\ \vdots & & \vdots\\ \mathbf{y}_1 & \cdots & \mathbf{y}_N\end{bmatrix}.$$

The modal properties of the system can then be extracted [10] using $\mathbf{A}_1\ldots\mathbf{A}_p$.

In a manner similar to SOD, a significant amount of random noise can be eliminated with a minor modification. Consider two sets of measurements one grouped in $\mathbf{\Gamma}$ and the other in $\mathbf{\Gamma}'$. Since, in general noise is temporally uncorrelated, the realizations of the noise in $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ are distinct (and uncorrelated). Next, substituting $[\mathbf{\Gamma}]^T$ with $[\mathbf{\Gamma}']^T$ in Eq. (2) one obtains

$$[\mathbf{A}_1\ldots\mathbf{A}_p]=[\mathbf{y}_{p+1}\ldots\mathbf{y}_{p+N}][\mathbf{\Gamma}']^T([\mathbf{\Gamma}][\mathbf{\Gamma}']^T)^{-1}. \tag{3}$$

3. Analysis

This section provides a comparison of SOD and DSPI, and shows the effectiveness of the filtering algorithms. First, an analysis is done showing that both methods can be related as second-order accurate methods. Next, the methods are compared for a stiff mass-spring system, and the filtering algorithms are examined.

3.1. SOD vs DSPI: analytical

To compare SOD and DSPI, consider a symmetric undamped n degree of freedom system given by $\mathbf{M}\ddot{\mathbf{x}}+\mathbf{K}\mathbf{x}=0$. To simplify the analysis, a modal transformation can be carried out to diagonalize the mass and stiffness matrices by the modal transformation $\mathbf{x}=\mathbf{\Phi}\mathbf{q}$, where $\mathbf{\Phi}$ is a matrix of the right eigenvectors, and $\mathbf{q}$ are the modal coordinates. One obtains

$$\mathbf{M}\mathbf{\Phi}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{\Phi}\mathbf{q}=0. \tag{4}$$

As is very well known, premultiplying Eq. (4) by $\mathbf{\Phi}^T$ transforms the system to

$$\mathbf{I}\ddot{\mathbf{q}}+\mathbf{\Lambda}\mathbf{q}=\mathbf{0}, \tag{5}$$

where $\mathbf{I}$ is an $n\times n$ identity matrix. Since both $\mathbf{I}$ and $\mathbf{\Lambda}$ are diagonal matrices, Eq. (5) is a set of n independent, one degree of freedom systems given by $\ddot{q}_i+\omega_i^2 q_i=0,\ i=1\ldots n$.

This modal transformation does not affect the order of the errors introduced by the approximations in SOD. To show that, consider a hth order approximation of the time derivative in SOD given in modal coordinates by

$$\mathbf{q}\mathbf{D}^T\cong\dot{\mathbf{q}}+\mathbf{O}(\Delta t^h). \tag{6}$$

Premultiplying Eq. (6) by $\mathbf{\Phi}$ yields

$$\mathbf{\Phi}\mathbf{q}\mathbf{D}^T\cong\mathbf{\Phi}\dot{\mathbf{q}}+\mathbf{\Phi}\mathbf{O}(\Delta t^h),$$

$$\mathbf{x}\mathbf{D}^T\cong\dot{\mathbf{x}}+\mathbf{O}(\Delta t^h), \tag{7}$$

which is also a hth order approximation in the physical coordinates.

Since the order of the error stays the same when diagonalizing an n degree of freedom system, in the following we explore the error of the one degree of freedom system given by

$$\ddot{q}+\omega^2 q=0. \tag{8}$$

For a one degree of freedom system, SOD is only calculating the frequency ω of the system. For the noise free case, the dominant source of error in the estimate comes from the approximation of $\ddot{q}$ by finite differencing. Consider the central differencing scheme given by

$$\ddot{q}_t\cong\frac{q_{t+1}-2q_t+q_{t-1}}{\Delta t^2}+O(\Delta t^2). \tag{9}$$

For the one degree of freedom system DSPI can be formulated in a similar manner using the minimum value of p for the one degree of freedom case (p=2), which gives

$$y_{t+1}=a_1y_t+a_2y_{t-1}+e_{t+1}. \tag{10}$$

For the no noise case, the dominant error in DSPI comes from the linear approximation of y_{t+1} by the previous two time steps. In order to show the minimum order of error in e_{t+1}, consider the following Taylor series approximations:

$$y_{t-1} = y_t - \frac{\dot{y}}{1!}\Delta t + \frac{\ddot{y}}{2!}\Delta t^2 - O(\Delta t^3), \tag{11}$$

$$y_{t+1} = y_t + \frac{\dot{y}}{1!}\Delta t + \frac{\ddot{y}}{2!}\Delta t^2 + O(\Delta t^3). \tag{12}$$

Adding Eqs. (11) and (12) and rearranging the terms one obtains

$$y_{t+1} = 2y_t - y_{t-1} + O(\Delta t^2). \tag{13}$$

Comparing Eqs. (13) and (10) one concludes that if $a_1 = 2 + O(\Delta t^2)$ and $a_2 = -1 + O(\Delta t^2)$, then e_{t+1} is $O(\Delta t^2)$.

To show the order of error in the estimates of y_{t+1} from y_t and y_{t-1}, consider the one degree of freedom system given by Eq. (8). If a value of $\omega = 20\pi$ is used, then the system can be excited by a random forcing and the response can be calculated using an integration technique with an accuracy higher than second order. The fourth-order MATLAB function *ode*45 was used for the following results. For time steps varying between 5×10^{-8} and 5×10^{-3} the order of the error can be calculated from the slope of the dependence of $\log(e') = \log\left(\sum_{i=2}^{N-1} |y_{i+1} - (a_1 y_i + a_2 y_{i-1})|/(N-2)\right)$ upon $\log(\Delta t)$, where N is the number of time samples at a given Δt. A plot of these results is shown in Fig. 1. These results are for the case where $a_1 = 2$ and $a_2 = -1$ (as identified from Eqs. (10 and 13), and for the case where a_1 and a_2 are computed using Eq. (2) (DSPI). A line of slope two is plotted for comparison, and it shows a very good agreement for $5 \times 10^{-8} \leq \Delta t \leq 2 \times 10^{-5}$.

The results in Fig. 1 illustrate computationally for a one degree of freedom system that the order of the error in the approximations in the parameters of DSPI (a_1, a_2) is the same as the order of the error in the approximation of the derivative (D) used in SOD. Note that it is possible to use higher order approximations of the derivative for SOD and larger p values for DSPI, which in turn lead to higher order approaches. However, this can create additional issues when noise is present.

3.2. SOD vs DSPI: stiff system

An investigation of a stiff mass-spring system, shown in Fig. 2, is undertaken to further explore SOD, DSPI and the proposed filtering algorithms. The stiff system is excited randomly at all five degrees of freedom. The applied forces are distributed randomly about zero with a maximum amplitude of 1000 N. The time response data are used by SOD and DSPI to extract frequency and mode shape information. For both methods 100,000 time steps were computed. For SOD a time step of 0.002 s was used for a total sample time of 200 s. For DSPI a time step of 0.01 s was used for a total sample time of 1000 s.

The first scenario explored is one where a low level of 0.1 percent relative noise is introduced into the dataset for 100 separate cases. The relative noise added to the time series was a non-Gaussian white noise generated by the Matlab function

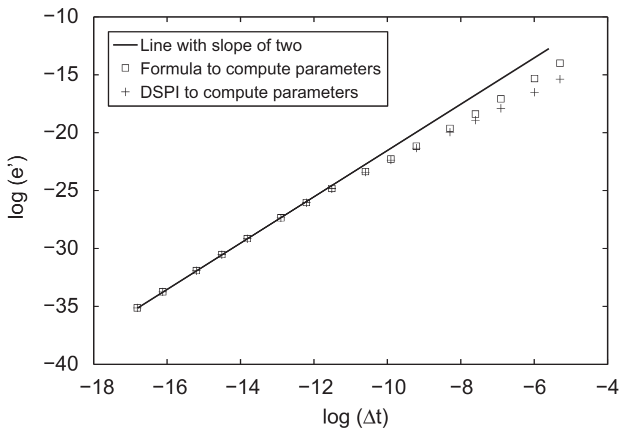

Fig. 1. Order of error in DSPI shown by the slope of the error line on a log scale.

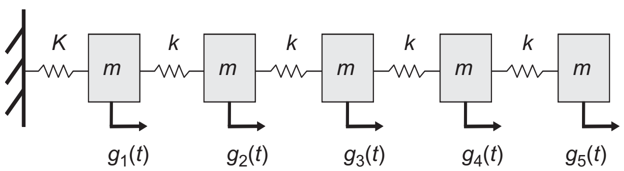

Fig. 2. Stiff five degree of freedom system forced at each mass by random excitations $g_i(t)$ $i = 1 \ldots 5$, where m=1 kg, K=10,000 N/m and k=1 N/m.

rand. A summary of the frequency information obtained is summarized for the unfiltered case in Table 1 and for the filtered case in Table 2. These results highlight the effectiveness of the filtering algorithm for both SOD and DSPI. The error for both methods is reduced significantly with filtering. Additionally, DSPI outperforms SOD for both the filtered and unfiltered cases. A comparison of the extracted eigenvectors is summarized in Fig. 3 for the unfiltered case and in Fig. 4 for the filtered case. For this low level of noise the modes are extracted quite accurately and the filtering has little effect.

The next scenario explored is where a 1 percent relative noise was introduced into the data for 100 separate cases. A summary of the extracted frequency information is included in Table 3 for the filtered case. For the unfiltered case, the frequency extraction performed very poorly for both SOD and DSPI. After filtering, the frequencies extracted are fairly

Table 1
Frequencies of a stiff system obtained *without* filtering applied for SOD and DSPI for 0.1 percent random noise.

Exact frequencies (rad/s)	Average SOD frequency estimate (rad/s)	Error in average SOD estimate (%)	Standard deviation of SOD estimate (rad/s)	Average DSPI frequency estimate (rad/s)	Error in average DSPI estimate (%)	Standard deviation of DSPI estimate (rad/s)
1.098	1.152	4.861	0.00036	1.100	0.169	0.00015
3.162	3.187	0.792	0.00017	3.163	0.009	0.00003
4.845	4.918	1.507	0.00069	4.850	0.107	0.00013
5.943	6.019	1.276	0.00082	5.947	0.072	0.00007
316.244	311.043	1.644	0.00000	316.279	0.011	0.00001

Table 2
Frequencies of a stiff system obtained *with* filtering applied for SOD and DSPI for 0.1 percent random noise.

Exact frequencies (rad/s)	Average SOD frequency estimate (rad/s)	Error in average SOD estimate (%)	Standard deviation of SOD estimate (rad/s)	Average DSPI frequency estimate (rad/s)	Error in average DSPI estimate (%)	Standard deviation of DSPI estimate (rad/s)
1.098	1.099	0.085	0.00062	1.098	0.051	0.00019
3.162	3.168	0.182	0.00044	3.162	0.008	0.00003
4.845	4.833	0.330	0.00355	4.845	0.006	0.00134
5.943	5.948	0.077	0.00138	5.944	0.010	0.00019
316.244	311.150	1.610	0.00023	316.290	0.015	0.00006

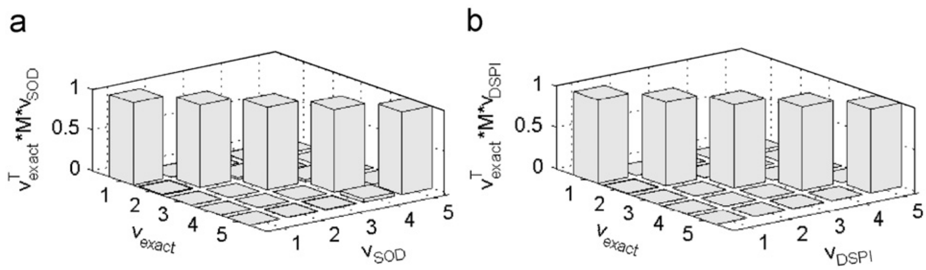

Fig. 3. Modal assurance criterion for modes extracted using (a) SOD and (b) DSPI *without* filtering algorithms for the stiff system and for 0.1 percent random noise.

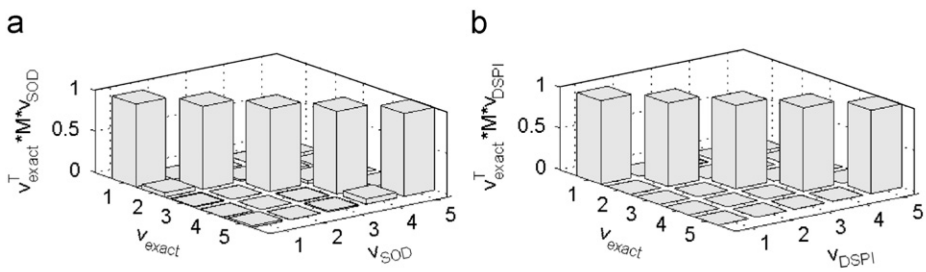

Fig. 4. Modal assurance criterion for modes extracted using (a) SOD and (b) DSPI *with* filtering algorithms for the stiff system and for 0.1 percent random noise.

Table 3
Frequencies of a stiff system obtained *with* filtering applied for SOD and DSPI for 1 percent random noise.

Exact frequencies (rad/s)	Average SOD frequency estimate (rad/s)	Error in average SOD estimate (%)	Standard deviation of SOD estimate (rad/s)	Average DSPI frequency estimate (rad/s)	Error in average DSPI estimate (%)	Standard deviation of DSPI estimate (rad/s)
1.098	1.094	0.347	0.03415	1.102	0.317	0.02403
3.162	3.168	0.184	0.02920	3.160	0.062	0.00544
4.845	4.809	0.732	0.18225	4.859	0.302	0.17812
5.943	5.954	0.186	0.06819	5.944	0.010	0.02718
316.244	311.151	1.610	0.00342	316.290	0.015	0.00082

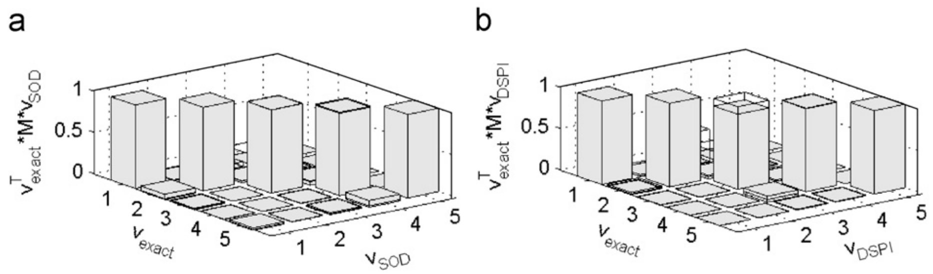

Fig. 5. Modal assurance criterion for modes extracted using (a) SOD and (b) DSPI *with* filtering algorithms for the stiff system and for 1 percent random noise.

accurate and DSPI again outperforms SOD. The results of the eigenvector extraction are summarized in Fig. 5 for the filtered case.

Results similar to those above have been obtained for other systems, including less stiff (and thus less challenging) systems, but they are omitted here for the sake of brevity.

4. Conclusions

A comparison of SOD and DSPI has been presented, and filtering algorithms to greatly reduce the effects of random noise on the performance of these two methods have been proposed.

SOD was originally developed for damage detection in chaotic systems. Recently, it has been extended to a time-based multi-output modal analysis approach that works for the case of free vibrations and that of random excitations. DSPI was developed as a time-based multi-input multi-output approach. DSPI is a more general approach than SOD since it can handle (not only the case of free vibrations and that of random excitations, but also) the cases where harmonic and other nonrandom excitations are present (and as long as these excitations are known/measured).

The presence of noise was shown to significantly affect the performance and accuracy of both SOD and DSPI. To address that challenge, novel filtering algorithms were proposed. These algorithms were shown to considerably reduce the impact of random noise, and drastically improve the frequency and mode shape extraction. Additionally, although DSPI and SOD are both second-order methods, DSPI was shown to outperform SOD both for the case of unfiltered data and that of filtered data.

The filtering algorithms proposed are not limited to DSPI and SOD. These filtering algorithms can be extended to a variety of other modal analysis techniques. For example, system identification and modeling methods based on auto-regressive moving average methods can significantly benefit from the proposed filtering techniques. The key requirement in the extension of the filtering algorithms to other modal analysis techniques is that the modal analysis method must use an autocorrelation of an ensemble matrix with itself. In such cases, the filtering algorithm can be applied by using two different realizations of that ensemble matrix to filter out the effects of uncorrelated noise.

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