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Reflection and transmission of waves in pyroelectric and piezoelectric materials

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ABSTRACT

The reflection and transmission theories of waves in pyroelectric and piezoelectric medium are studied in this paper. In general in an infinite homogeneous pyroelectric medium there are four bulk wave modes: quasi-longitudinal, two quasi-transversal and temperature waves. In an infinite homogeneous piezoelectric medium there are three bulk wave modes: quasi-longitudinal and two quasi-transversal waves. In the reflection and transmission problem there are five complex boundary conditions in the pyroelectric medium and four complex boundary conditions for the piezoelectric medium. In this paper, we find that the surface waves will be revealed in the reflection and transmission wave problem. The surface waves have the same wave vector component with the incident waves on the interface plane. The two dimensional reflection problem of waves at the interface between the semi-infinite pyroelectric medium and vacuum is researched in greater detail and a numerical example is given.

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1. Introduction

The propagation theory of waves in elasticity and thermoelasticity is completely solved many years ago [1], but the literatures of the reflection and transmission problem in piezoelectric and pyroelectric materials are few. In appendix of a paper, Wang [2] discussed the reflection and transmission problem on the interface in piezoelectric materials under the dynamic antiplane mechanical loading. Burkov et al. [3] discussed the reflection and transmission of bulk acoustic waves in piezoelectric materials under the action of an external electric field. In their studies they got the wave vector and their corresponding wave amplitudes from the Christoffel's equation at first, then they added the amplitudes of all wave modes of the electric potential together and multiplied it by a new amplitude constant. So they could get all the amplitudes of different wave modes through the interface continuum conditions. However, the solution obtained by this method did not satisfy the momentum equation, because the momentum equation gives a certain relation between the amplitudes of displacements and electric potential. Sharma et al. [4] discussed the reflection of piezothermoelastic waves. In their theory, the amplitude coefficients of waves are related to the positions on the interface. The main puzzle in the reflection and transmission theory of waves in piezoelectric and pyroelectric materials is that the electric potential has not its own independent wave mode under the postulation of quasi-static electric field [5–7]. Kyame [8] discussed a special case of piezoelectric wave. In his discussion he abandoned the postulation of the quasi-static electric field. He let the displacements and electric potential satisfy all the piezoelectric equations and Maxwell electro-dynamic equations.

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However, the velocities of the elastic waves are much less than the velocities of the electromagnetic waves, so these two physical phenomena are difficult to couple each other.

In the geophysical field the inhomogeneous wave theory has been extensively researched for the viscoelastic media [9–12]. As in the previous paper [6,7] in this paper we introduce this theory into the pyroelectric and piezoelectric materials.

To solve the reflection and transmission problem of inhomogeneous waves in piezoelectric and pyroelectric materials, we [16] found that an extra independent surface wave is revealed on each side of the boundary surface except the bulk reflection waves and the bulk transmission waves. The surface wave has the same wave vector component with the component of bulk wave along the interface (along the x_1 direction). In this paper the reflection problem from the interface of the pyroelectric material and vacuum is studied in detail and some numerical results are given. In two dimensional reflection problem there is only one quasi-transversal wave, so there are only three wave modes in pyroelectric materials and two wave modes in piezoelectric materials. Our numerical researches show that the surface wave exists certainly in the wave reflection and transmission problem.

2. General inhomogeneous wave theory

2.1. Governing equations of pyroelectric media

The governing equations of the pyroelectric media can be found in many literatures [4,6,13–16]. The constitutive and heat conductive and geometric equations are

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta, \quad D_i = \in_{ij}E_j + e_{ikl}\varepsilon_{kl} + \tau_i\vartheta,$$
$$s = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + C\vartheta/T_0, \quad q_i = -\lambda_{ij}\vartheta_{,j}, \quad \vartheta = T - T_0,$$
$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad E_k = -\varphi_{,k} \tag{1}$$

where $\mathbf{u}$, $\boldsymbol{\varepsilon}$, φ, $\mathbf{E}$, T, T_0, $\mathbf{q}$ are the displacement, strain, electric potential, electric field, temperature of the medium, temperature of the environment and heat flow; C_{ijkl}, e_{kij}, $\in_{ij}$, α_{ji}, τ_j, C, λ_{ij} are the elastic, piezoelectric, permittivity, thermo-mechanical, thermo-electric, heat capacity and heat conductive coefficients.

In literatures there are several generalized dynamical theories of piezothermoelasticity. In this paper, only two theories are used. In the inertial entropy theory [15,16] the entropy and conductive equations are

$$T\dot{s} + T\dot{s}^{(a)} = -q_{i,i}, \quad \dot{s}^{(a)} = C\varpi\ddot{T}/T_0, \quad q_i = -\lambda_{ij}T_{,j} \tag{2}$$

where $s^{(a)}$is the inertial entropy, ϖ is the inertial entropy coefficient or the relaxation time. In Lord–Shulman theory [17] the entropy equation and heat conductive equation are

$$T\dot{s} = -q_{i,i}, \quad q_i + \varpi\delta_{ij}\dot{q}_i = -\lambda_{ij}T_{,j} \tag{3}$$

Substituting Eq. (2) or Eq. (3) into the momentum equation and after some manipulations the governing equations of the pyroelectric media in displacements, electric potential and temperature are obtained as

$$C_{ijkl}u_{k,lj} + e_{kij}\varphi_{,kj} - \alpha_{ji}\vartheta_{,j} = \rho\ddot{u}_i, \quad e_{ikl}u_{k,li} - \in_{ij}\varphi_{,ji} + \tau_j\vartheta_{,j} = 0$$
$$T_0\alpha_{ij}(\dot{\varepsilon}_{ij} + \varpi_1\ddot{\varepsilon}_{ij}) + T_0\tau_i(\dot{E}_i + \varpi_2\ddot{E}_i) + \rho C(\dot{\vartheta} + \varpi\ddot{\vartheta}) = \lambda_{ij}\vartheta_{,ji} \tag{4}$$

In the inertial entropy theory $\varpi_1 = \varpi_2 = 0$ and in Lord-Shulman theory $\varpi_1 = \varpi_2 = \varpi$. The relaxation time only affects the wave attenuation and its effects on the wave velocity and the wave mode are very limited in the stationary reflection and transmission problems [6]. The numerical results show that the difference in reflection and transmission problem between these two theories is small for small relaxation time. So in the following we discuss these two theories together and do not distinguish them.

In piezoelectric materials the governing equations can be obtained from that of pyroelectric materials by making the terms containing temperature equal to zero. So the problem discussed here for pyroelectric materials can also be used directly to the piezoelectric materials. In piezoelectric materials Eq. (4) becomes

$$C_{ijkl}u_{k,lj} + e_{kij}\varphi_{,kj} = \rho\ddot{u}_i, \quad e_{ikl}u_{k,li} - \in_{ij}\varphi_{,ji} = 0 \tag{5}$$

2.2. Homogeneous and inhomogeneous waves

In general a plane attenuation wave f can be expressed as

$$f = f_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} = f_0 e^{i(k_m x_m - \omega t)}, \quad \mathbf{k} = \mathbf{P} + i\mathbf{A}, \quad \mathbf{P} = P\mathbf{n}, \quad \mathbf{A} = A\mathbf{m}$$
$$k_j = Pn_j + iAm_j, \quad P = \sqrt{(Pn_1)^2 + (Pn_2)^2}, \quad 4A = \sqrt{(Am_1)^2 + (Am_2)^2}$$
$$k^2 = \mathbf{k}\cdot\mathbf{k} = P^2 - A^2 + 2i\mathbf{P}\cdot\mathbf{A}, \quad c = \omega/P \tag{6}$$

where f_0 is the amplitude of f, ω is the circular frequency, c is the phase velocity of the wave, $\mathbf{k}$, with components k_1, k_2, is a complex wave vector for a attenuation wave. In general the real part $\mathbf{P}$ and imaginary part $\mathbf{A}$ are all vectors. $\mathbf{P}$=$P\mathbf{n}$, where

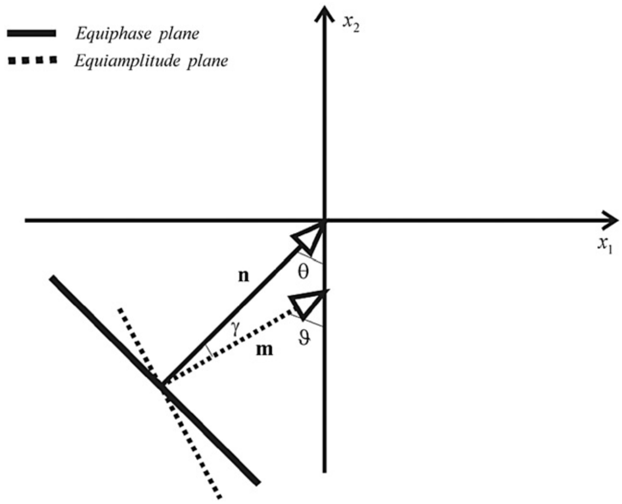

Fig. 1. Wave propagation direction **n**, attenuation direction **m** and the attenuation angle γ.

$\mathbf{n}=[\sin\theta,\ \cos\theta]^T$ is the wave propagation direction, θ is the angle between **n** and the coordinate axis x_2. The plane perpendicular to **n** is the equiphase plane; $\mathbf{A}=A\mathbf{m}$, $\mathbf{m}=[\sin(\theta+\gamma),\cos(\theta+\gamma)]^T$ is the maximum attenuation direction, γ is the attenuation angle and $\cos\gamma=\mathbf{n}\cdot\mathbf{m}$ (Fig. 1). The plane perpendicular to **m** is the equiamplitude plane. If $\mathbf{n}\neq\mathbf{m}$, we call the wave the inhomogeneous wave and **k** is expressed by (P,A,θ,γ). If $\mathbf{n}=\mathbf{m}$,$\mathbf{k}=(P+\mathrm{i}A)\mathbf{n}$, we call the wave the homogeneous wave and $\mathbf{k}\cdot\mathbf{x}=kn_ix_i$, $k=P+\mathrm{i}A$, so **k** can be expressed by (P,A,θ).

2.3. Inhomogeneous plane wave in pyroelectric materials

For the inhomogeneous plane wave propagated in an infinite space we can assume

$$u_k=U_k\,\mathrm{e}^{\mathrm{i}(k_mx_m-\omega t)},\quad \varphi=\Phi\,\mathrm{e}^{\mathrm{i}(k_mx_m-\omega t)},\quad \vartheta=\Theta\,\mathrm{e}^{\mathrm{i}(k_mx_m-\omega t)},\quad k=1\text{-}3 \tag{7}$$

where U_k, Φ, Θ are amplitudes of u_k, φ, ϑ, respectively. Substituting Eq. (7) into Eq. (4) we get the Christoffel's equation

$$\mathbf{\Lambda}(\mathbf{k},\omega)\mathbf{U}=\mathbf{0},\quad \mathbf{U}=[U_1,U_2,U_3,U_4,U_5]^T,\quad U_4=\Phi,\quad U_5=\Theta$$

$$\mathbf{\Lambda}=\begin{bmatrix}\Gamma_{11}(\mathbf{k})-\rho\omega^2 & \Gamma_{12}(\mathbf{k}) & \Gamma_{13}(\mathbf{k}) & e_1^*(\mathbf{k}) & \mathrm{i}\alpha_1^*(\mathbf{k})\\ \Gamma_{21}(\mathbf{k}) & \Gamma_{22}(\mathbf{k})-\rho\omega^2 & \Gamma_{23}(\mathbf{k}) & e_2^*(\mathbf{k}) & \mathrm{i}\alpha_2^*(\mathbf{k})\\ \Gamma_{31}(\mathbf{k}) & \Gamma_{32}(\mathbf{k}) & \Gamma_{33}(\mathbf{k})-\rho\omega^2 & e_3^*(\mathbf{k}) & \mathrm{i}\alpha_3^*(\mathbf{k})\\ e_1^*(\mathbf{k}) & e_2^*(\mathbf{k}) & e_3^*(\mathbf{k}) & -\in^*(\mathbf{k}) & -\mathrm{i}\tau^*(\mathbf{k})\\ T_0\alpha_1^*(\mathbf{k})\xi_1 & T_0\alpha_2^*(\mathbf{k})\xi_1 & T_0\alpha_3^*(\mathbf{k})\xi_1 & -T_0\tau^*(\mathbf{k})\xi_2 & \lambda^*-\mathrm{i}\rho C\xi\end{bmatrix} \tag{8}$$

where

$$\begin{gathered}\Gamma_{ik}(\mathbf{k})=C_{ijkl}k_jk_l,\quad e_i^*(\mathbf{k})=e_{kij}k_kk_j,\quad \alpha_i^*(\mathbf{k})=\alpha_{ij}k_j\\ \tau^*(\mathbf{k})=\tau_jk_j,\quad \in^*(\mathbf{k})=\in_{jk}k_kk_j,\quad \lambda^*(\mathbf{k})=\lambda_{ij}k_ik_j\\ \xi_1=\omega-\mathrm{i}\omega^2\varpi_1,\quad \xi_2=\omega-\mathrm{i}\omega^2\varpi_2,\quad \xi=\omega-\mathrm{i}\omega^2\varpi\end{gathered} \tag{9}$$

If **U** has nontrivial solution, then the determinant of the characteristic matrix $\mathbf{\Lambda}$ must be vanishing, i.e.

$$|\mathbf{\Lambda}|=0. \tag{10a}$$

Substituting $k_j=Pn_j+\mathrm{i}Am_j$ from Eq. (6) into (10a) and decomposing $|\mathbf{\Lambda}|=0$ into the real and imaginary parts, we get the following coupling real equations in (P,A,θ,γ)

$$\mathrm{Re}|\mathbf{\Lambda}|=0,\quad \mathrm{Im}|\mathbf{\Lambda}|=0 \tag{10b}$$

For given (θ,γ), we have enough equations to solve the unknowns (P,A). Because the expressions of Eq. (10b) are very tedious, so it is not given here, but it is easily generated by numerical computation. The range of γ is determined by that (P,A) should be nonnegative real value. It is found that γ should be in the range $(-\pi/2,\pi/2)$ [7,9,18]. However, how to determine γ is still an open problem [11,12].

Eq. (10) is a 5×5 complex matrix, or two coupling 5×5 real matrix. But the electric potential has not its own independent wave velocity. It is to say that along one propagation direction (the opposite propagation direction is not considered here) Eq. (10) only has four independent complex eigenvalues, and the eigenvalue corresponding to φ is zero, which means that φ has the independent mode with infinite propagation velocity [5,6]. It is emphasized that φ can still propagate with mechanical and thermal wave velocities through constitutive equations.

As shown above and numerical results, Eq. (10) has four independent complex eigenvalues $\mathbf{k}_i=P_i\mathbf{n}+\mathrm{i}A_i\mathbf{m}$ (i=1–4). The four phase velocities c_i corresponding to $\mathbf{k}_i$ are

$$c_i=\omega/P_i, \quad P_i=\sqrt{(P_i n_1)^2+(P_i n_2)^2} \tag{11}$$

The quasi-longitudinal wave has the fastest phase velocity, the temperature wave has the slowest phase velocity, two quasi-transversal wave have the middle phase velocity [5,6].

Corresponding to each complex eigenvalue $\mathbf{k}_i$ the amplitudes (or the eigenvectors) $\mathbf{U}_i$ can also be determined through Eq. (8) and the ratio U_{1i}:U_{2i}:U_{3i}:Φ_i:Θ_i is definite, so only one of the five components is undetermined, say U_{1i}, and all other components can be expressed by it. So the general solutions of the wave propagation problem in an infinite space can be written as

$$u_k=\sum\nolimits_{j=1}^4 \beta_j U_{kj}\,\mathrm{e}^{\mathrm{i}(k_{jm}x_m-\omega t)}, \quad \varphi=\sum\nolimits_{j=1}^4 \beta_j \Phi_j\,\mathrm{e}^{\mathrm{i}(k_{jm}x_m-\omega t)}, \quad \vartheta=\sum\nolimits_{j=1}^4 \beta_j \Theta_j\,\mathrm{e}^{\mathrm{i}(k_{jm}x_m-\omega t)}$$

$$\mathrm{e}^{\mathrm{i}(k_{jm}x_m-\omega t)}=\mathrm{e}^{\mathrm{i}\left[(P_j\mathbf{n}+\mathrm{i}A_j\mathbf{m})\mathbf{x}-\omega t\right]}=\mathrm{e}^{-A_j\mathbf{m}\cdot\mathbf{x}}\,\mathrm{e}^{\mathrm{i}(P_j\mathbf{n}\cdot\mathbf{x}-\omega t)} \tag{12}$$

where β_j(j=1–4) is the undetermined amplitude coefficient and $\mathbf{U}_i$ is completely determined by making one, say U_{1i}, of the five components equal to 1.

2.4. Governing equations in piezoelectric materials

In piezoelectric problem Eq. (8) becomes

$$\mathbf{\Lambda}(\mathbf{k},\omega)\mathbf{U}=\mathbf{0}, \quad \mathbf{U}=[U_1,U_2,U_3,U_4]^T, \quad U_4=\Phi,$$

$$\mathbf{\Lambda}=\begin{bmatrix} \Gamma_{11}(\mathbf{k})-\rho\omega^2 & \Gamma_{12}(\mathbf{k}) & \Gamma_{13}(\mathbf{k}) & e_1^*(\mathbf{k}) \\ \Gamma_{21}(\mathbf{k}) & \Gamma_{22}(\mathbf{k})-\rho\omega^2 & \Gamma_{23}(\mathbf{k}) & e_2^*(\mathbf{k}) \\ \Gamma_{31}(\mathbf{k}) & \Gamma_{32}(\mathbf{k}) & \Gamma_{33}(\mathbf{k})-\rho\omega^2 & e_3^*(\mathbf{k}) \\ e_1^*(\mathbf{k}) & e_2^*(\mathbf{k}) & e_3^*(\mathbf{k}) & -\in^*(\mathbf{k}) \end{bmatrix} \tag{13}$$

where

$$\Gamma_{ik}(\mathbf{k})=C_{ijkl}k_jk_l, \quad e_i^*(\mathbf{k})=e_{kij}k_kk_j, \quad \in^*(\mathbf{k})=\in_{jk}k_kk_j \tag{14}$$

Eq. (12) becomes

$$u_k=\sum\nolimits_{j=1}^3 \beta_j U_{kj}\,\mathrm{e}^{\mathrm{i}(k_{jm}x_m-\omega t)}, \quad \varphi=\sum\nolimits_{j=1}^3 \beta_j \Phi_j\,\mathrm{e}^{\mathrm{i}(k_{jm}x_m-\omega t)} \tag{15}$$

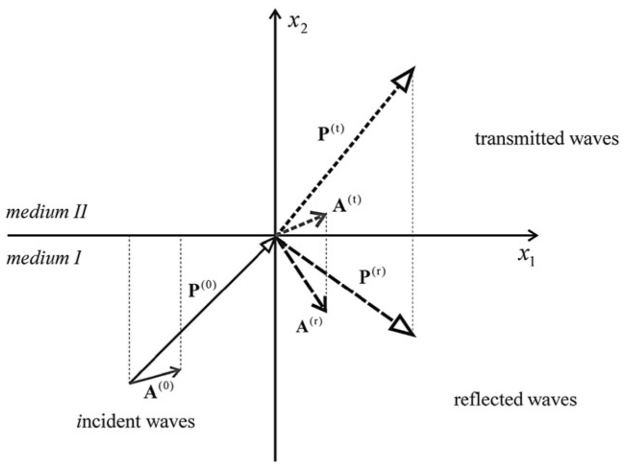

Fig. 2. General sketch of incident, reflection and transmission waves.

3. Reflection/transmission problem in inhomogeneous materials

3.1. Continuous conditions on the interface of pyroelctric materials

Consider the problem of two bounded semi-infinite pyroelectric materials with the interface x_2=0 subjected to a harmonic incident wave of frequency ω with an incident angle θ, as shown in Fig. 2. The continuous conditions on the interface are

$$u_i^{\mathrm{I}} = u_i^{\mathrm{II}}, \quad \varphi^{\mathrm{I}} = \varphi^{\mathrm{II}}, \quad \vartheta^{\mathrm{I}} = \vartheta^{\mathrm{II}}$$
$$\sigma_{ij}^{\mathrm{I}} n_i^{\mathrm{I}} + \sigma_{ij}^{\mathrm{II}} n_i^{\mathrm{II}} = 0, \quad D_i^{\mathrm{I}} n_i^{\mathrm{I}} + D_i^{\mathrm{II}} n_i^{\mathrm{II}} = 0, \quad \lambda_{ij}^{\mathrm{I}} \vartheta_{,j}^{\mathrm{I}} n_i^{\mathrm{I}} + \lambda_{ij}^{\mathrm{II}} \vartheta_{,j}^{\mathrm{II}} n_i^{\mathrm{II}} = 0 \tag{16}$$

where $\lambda_{ij}^{\mathrm{I}} \vartheta_{,j}^{\mathrm{I}}$ and $\lambda_{ij}^{\mathrm{II}} \vartheta_{,j}^{\mathrm{II}}$ are equivalent to q_i^{I} and q_i^{II}, respectively, where $n_i^{\mathrm{II}} = -n_i^{\mathrm{I}}$.

3.2. Reflection and transmission waves in pyroelectric materials

Let an incident wave with a wave vector $\mathbf{k}^{(0)}$is in the semi-infinite plane I,$x_2 \le 0$ and expressed by

$$u_k^{(o)} = U_k^{(o)} \mathrm{e}^{\mathrm{i}(k_m^{(0)} x_m - \omega t)}, \quad \varphi^{(0)} = \Phi^{(0)} \mathrm{e}^{\mathrm{i}(k_m^{(0)} x_m - \omega t)}, \quad \vartheta^{(0)} = \Theta^{(0)} \mathrm{e}^{\mathrm{i}(k_m^{(0)} x_m - \omega t)} \tag{17}$$

where $U_k^{(o)}$,$\Phi^{(0)}$,$\Theta^{(0)}$ and $k_m^{(0)}$ are all known. The reflection wave in the semi-infinite plane I, $x_2 \le 0$ can be expressed by

$$u_k^{(r)} = \sum\nolimits_{j=1}^{N} \beta_j^{(r)} U_{kj}^{(r)} \mathrm{e}^{\mathrm{i}(k_{jm}^{(r)} x_m - \omega t)}, \quad \varphi^{(r)} = \sum\nolimits_{j=1}^{N} \beta_j^{(r)} \Phi_j^{(r)} \mathrm{e}^{\mathrm{i}(k_{jm}^{(r)} x_m - \omega t)}, \quad \vartheta^{(r)} = \sum\nolimits_{j=1}^{N} \beta_j^{(r)} \Theta_j^{(r)} \mathrm{e}^{\mathrm{i}(k_{jm}^{(r)} x_m - \omega t)} \tag{18}$$

And the transmission wave in the semi-infinite plane II, $x_2 \ge 0$ can be expressed by

$$u_k^{(t)} = \sum\nolimits_{j=1}^{N} \beta_j^{(t)} U_{kj}^{(t)} \mathrm{e}^{\mathrm{i}(k_{jm}^{(t)} x_m - \omega t)}, \quad \varphi^{(t)} = \sum\nolimits_{j=1}^{N} \beta_j^{(t)} \Phi_j^{(t)} \mathrm{e}^{\mathrm{i}(k_{jm}^{(t)} x_m - \omega t)}, \quad \vartheta^{(t)} = \sum\nolimits_{j=1}^{N} \beta_j^{(t)} \Theta_j^{(t)} \mathrm{e}^{\mathrm{i}(k_{jm}^{(t)} x_m - \omega t)} \tag{19}$$

In Eqs. (18) and (19) N is the number of the independent waves which will contain four bulk waves and one surface wave, which is revealed in the reflection and transmission problem of waves. The behavior of the surface wave can be seen in [19,20]. It is obvious that

$$u_k^{\mathrm{I}} = u_k^{(o)} + u_k^{(r)}, \quad u_k^{\mathrm{II}} = u_k^{(t)}, \quad \varphi^{\mathrm{I}} = \varphi^{(0)} + \varphi^{(r)}, \quad \varphi^{\mathrm{II}} = \varphi^{(t)}, \quad \vartheta^{\mathrm{I}} = \vartheta^{(0)} + \vartheta^{(r)}, \vartheta^{\mathrm{II}} = \vartheta^{(t)}$$
$$\sigma_{ij}^{\mathrm{I}} = \sigma_{ij}^{(o)} + \sigma_{ij}^{(r)}, \quad \sigma_{ij}^{\mathrm{II}} = \sigma_{ij}^{(t)}, \quad D_i^{\mathrm{I}} = D_i^{(o)} + D_i^{(r)}, \quad D_i^{\mathrm{II}} = D_i^{(t)} \tag{20}$$

Substituting Eqs. (17)–(20) into (16) one can get

$$k_1^{(0)} = k_{j1}^{(r)} = k_{j1}^{(t)}, \quad k_{j1}^{(\alpha)} = P_{j1}^{(\alpha)} + \mathrm{i} A_{j1}^{(\alpha)} = P_j^{(\alpha)} n_1 + \mathrm{i} A_j^{(\alpha)} m_1, \quad (\alpha = r,t;\ j = 1{-}N) \tag{21}$$

which is also shown in Fig. 2. Decomposing Eq. (21) into real and imaginary parts, we get

$$P^{(0)} \sin\theta^{(0)} = P_j^{(r)} \sin\theta_j^{(r)} = P_j^{(t)} \sin\theta_j^{(t)}$$
$$A^{(0)} \sin(\theta^{(0)} + \gamma^{(0)}) = A_j^{(r)} \sin(\theta_j^{(r)} + \gamma_j^{(r)}) = A_j^{(t)} \sin(\theta_j^{(t)} + \gamma_j^{(t)}) \quad (j = 1{-}N) \tag{22}$$

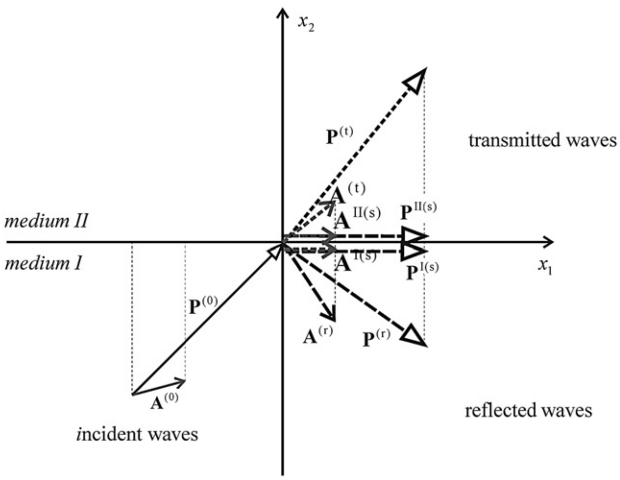

Fig. 3. General sketch of incident, reflection, transmission and surface waves.

From Eqs. (11) and (22) we can get the generalized Snell's law (Fig. 3):

$$\frac{\sin\theta^{(0)}}{c^{(0)}}=\frac{\sin\theta_j^{(r)}}{c_j^{(r)}}=\frac{\sin\theta_j^{(t)}}{c_j^{(t)}},\quad c^{(0)}=\frac{\omega}{P^{(0)}},\quad c_j^{(r)}=\frac{\omega}{P_j^{(r)}},\quad c_j^{(t)}=\frac{\omega}{P_j^{(t)}}\quad (j=1\text{–}N) \tag{23}$$

In general $\theta^{(0)}\neq\theta_j^{(r)}$ due to the anisotropic behavior of materials. So that the 4N real unknowns, $\theta_j^{(r)},\theta_j^{(t)},\gamma_j^{(r)},\gamma_j^{(t)}$, for each j can be determined by Eqs. (22) and (23).

It is emphasized that in two cases unknowns are different. Case (1): The problem of the bulk waves propagating in an infinite space: In this case Eq. (10) is for given (θ,γ) to solve unknowns (P,A). According to Eq. (6) in this case though k_{j1} and k_{j2} are complex numbers, but they are only related to two real constant P_j and A_j, or complex constants k_{j1} and k_{j2} are related each other. There only four pair (P_j,A_j) can be obtained, so we have $\mathbf{k}_j^{(\alpha)}=P_j^{(\alpha)}\mathbf{n}+\mathrm{i}A_j^{(\alpha)}\mathbf{m}$, where $j=1-4$; $\alpha=r,t$. Case (2): The problem of the reflection and transmission problem: From Eq. (21) it is known that k_{j1} is given, i.e. $k_1^{(0)}=k_{j1}^{(r)}=k_{j1}^{(t)}$, the components $k_{j2}^{(r)}$ and $k_{j2}^{(t)}$ are unknown complex constants and not related to k_{j1}. It is fortunate that in this case five $k_{j2}^{(r)}$ and $k_{j2}^{(t)}$ $(j=1-5)$ can be obtained and an independent surface wave is revealed in them, which can supplement the lack of the independent bulk electric wave to satisfy boundary conditions. It is also noted that in case (1) if k_{j1} is given , then k_{j2} is determined only by one real number; however, in case (2) given k_{j1}, k_{j2} is still determined by two real number or one complex number. So in case (2) we can get an extra independent surface wave, but in case (1) we cannot.

All the physical variables in the incident wave are known. In the reflection and transmission waves there are five unknowns $\beta_j^{(r)}$ and $\beta_j^{(t)}$ corresponding to four bulk waves and one surface wave, so that we have total 10 complex unknowns to satisfy 10 complex interface continuous conditions (see Eq. (16)). Therefore, the number of the boundary equations is equal to the number of unknowns. This situation shows that the reflection and transmission waves are complete.

3.3. Reflection and surface waves in vacuum/semi-infinite pyroelectric materials

If the medium II is a vacuum and on the boundary (interface) the stress, electric displacement and heat flow are all free, the wave cannot transmit to vacuum from the medium I, so that all variables in vacuum can be neglected. In this case there is no transmission waves, i.e.

$$\sigma_{ij}^{(t)}n_i^{II}=\sigma_{ij}^{II}n_i^{II}=D_i^{(t)}n_i^{II}=D_i^{II}n_i^{II}=\lambda_{ij}^{II}\vartheta_j^{(t)}n_i^{II}=\lambda_{ij}^{II}\vartheta_j^{II}n_i^{II}=0 \tag{24}$$

and the boundary conditions on the boundary x_2=0 of the pyroelectric medium become

$$\begin{gathered}\sigma_{2j}n_2=\left(\sigma_{2j}^{(o)}+\sigma_{2j}^{(r)}\right)n_2=0,\quad D_2n_2=\left(D_2^{(o)}+D_2^{(r)}\right)n_2=0\\ \lambda_{2j}\vartheta_{,j}n_2=\lambda_{2j}\left(\vartheta_{,j}^{(o)}+\vartheta_{,j}^{(r)}\right)n_2=0,\quad j=1\text{-}3,\quad n_2=1\end{gathered} \tag{25}$$

3.4. The reflection/transmission waves in piezoelectric materials

In piezoelectric materials Eqs. (17)–(19) become

$$\begin{gathered}u_k^{(o)}=U_k^{(o)}\,\mathrm{e}^{\mathrm{i}(k_m^{(0)}x_m-\omega t)},\quad \varphi^{(0)}=\Phi^{(0)}\,\mathrm{e}^{\mathrm{i}\left(k_m^{(0)}x_m-\omega t\right)}\\ u_k^{(r)}=\sum\nolimits_{j=1}^{4}\beta_j^{(r)}U_{kj}^{(r)}\,\mathrm{e}^{\mathrm{i}(k_{jm}^{(r)}x_m-\omega t)},\quad \varphi^{(r)}=\sum\nolimits_{j=1}^{4}\beta_j^{(r)}\Phi_j^{(r)}\,\mathrm{e}^{\mathrm{i}(k_{jm}^{(r)}x_m-\omega t)}\\ u_k^{(t)}=\sum\nolimits_{j=1}^{4}\beta_j^{(t)}U_{kj}^{(t)}\,\mathrm{e}^{\mathrm{i}(k_{jm}^{(t)}x_m-\omega t)},\quad \varphi^{(t)}=\sum\nolimits_{j=1}^{4}\beta_j^{(t)}\Phi_j^{(t)}\,\mathrm{e}^{\mathrm{i}(k_{jm}^{(t)}x_m-\omega t)}\end{gathered} \tag{26}$$

The continuous conditions on the interface Eqs. (16) and (20) become

$$u_k^{(o)}+u_k^{(r)}=u_k^{(t)},\quad \left(\sigma_{ij}^{(o)}+\sigma_{ij}^{(r)}\right)n_i^I=\sigma_{ij}^{(t)}n_i^{II},\quad \varphi^{(0)}+\varphi^{(r)}=\varphi^{(t)},\quad \left(D_i^{(o)}+D_i^{(r)}\right)n_i^I=D_i^{(t)}n_i^{II} \tag{27}$$

It is emphasized that there are three elastic bulk waves propagating in the homogeneous infinite piezoelectric materials and the elastic bulk waves do not attenuate. So in Eqs. (13) and (26) we can get three real wave vectors $k_i(i=1\text{–}3)$. But in the reflection/transmission problem of wave propagation in piezoelectric materials, except three bulk waves with real wave vector k_i, a surface wave with complex wave vector k_s will appear. So in the reflection/transmission problem of wave propagation in piezoelectric materials we should use $u_k^{(r)},\varphi^{(r)},u_k^{(t)},\varphi^{(t)}$ with complex wave vector in Eqs. (26) and (27).

4. Two dimensional reflection problem in vacuum/semi-infinite pyroelectric materials

4.1. Fundamental formula in two dimensional reflection problem

In two-dimensional case all the variables are independent to x_3 and u_3 is neglected, so there is only one quasi-transversal wave. In this case the Christoffel's equation (8) becomes

$$\mathbf{\Lambda}(\mathbf{k},\omega)\mathbf{U}=\mathbf{0},\quad \mathbf{U}=[U_1,U_2,U_4,U_5]^T,\quad U_4=\Phi,\quad U_5=\Theta$$

$$\mathbf{\Lambda}=\begin{bmatrix} \Gamma_{11}(\mathbf{k})-\rho\omega^2 & \Gamma_{12}(\mathbf{k}) & e_1^*(\mathbf{k}) & i\alpha_1^*(\mathbf{k}) \\ \Gamma_{21}(\mathbf{k}) & \Gamma_{22}(\mathbf{k})-\rho\omega^2 & e_2^*(\mathbf{k}) & i\alpha_2^*(\mathbf{k}) \\ e_1^*(\mathbf{k}) & e_2^*(\mathbf{k}) & -\in^*(\mathbf{k}) & -i\tau^*(\mathbf{k}) \\ T_0\alpha_1^*(\mathbf{k})\xi_1 & T_0\alpha_2^*(\mathbf{k})\xi_1 & -T_0\tau^*(\mathbf{k})\xi_2 & \lambda^*-i\rho C\xi \end{bmatrix} \tag{28}$$

It is noted that in Eq. (28) $k_1^{(0)}=k_{j1}^{(r)}\quad (j=1-4)$.

The boundary conditions (25) on the boundary of the pyroelectric materials become

$$\sigma_{i2}^{(o)}+\sigma_{i2}^{(r)}=0,\quad D_2^{(o)}+D_2^{(r)}=0,\quad \lambda_{i2}\left(\vartheta_{,i}^{(o)}+\vartheta_{,i}^{(r)}\right)=0,\quad i=1,2 \tag{29}$$

From Eqs. (1), (17), (18) and (29) we get

$$C_{i2kl}k_l^{(0)}U_l^{(o)}+e_{ki2}k_k^{(0)}\Phi^{(0)}+\mathrm{i}\alpha_{i2}\Theta^{(0)}+\sum_{j=1}^{4}\beta_j^{(r)}\left(C_{i2kl}k_{jl}^{(r)}U_{kj}^{(r)}+e_{ki2}k_{jk}^{(r)}\Phi_j^{(r)}+\mathrm{i}\alpha_{i2}\Theta_j^{(r)}\right)=0,\quad i=1,2$$

$$e_{2kl}k_l^{(0)}U_k^{(o)}-\in_{2j}k_j^{(0)}\Phi^{(0)}-\mathrm{i}\tau_i\Theta^{(0)}+\sum_{j=1}^{4}\beta_j^{(r)}\left(e_{2kl}k_{jl}^{(r)}U_{kj}^{(r)}-\in_{2l}k_{jl}^{(r)}\Phi_j^{(r)}-\mathrm{i}\tau_2\Theta_j^{(r)}\right)=0$$

$$\lambda_{i2}k_i^{(0)}\Theta^{(0)}+\sum_{j=1}^{4}\beta_j^{(r)}\lambda_{i2}k_{ji}^{(r)}\Theta_j^{(r)}=0 \tag{30}$$

Eq. (30) contains 4 complex equations with 4 complex unknowns $\beta_1^{(\tau)},\beta_2^{(r)},\beta_3^{(r)},\beta_4^{(r)}\equiv\beta^{(s)}$, so the problem is solved.

4.2. Example

As an example we discuss the two-dimensional reflection problem from the interface of $BiTiO_3$/ vacuum. When ox_2 is the pole axis, the material constants of $BiTiO_3$ in Voigt compact form for two-dimensional problem are (the usual three subscript piezoelectric coefficients in tensor form are changed to two subscript piezoelectric coefficients in Voigt vector form as: $e_{211}\Rightarrow e_{21}$, $e_{222}\Rightarrow e_{22}$, $e_{112}\Rightarrow e_{16}$)

$$C_{11}=15.0\times10^{10}\,\mathrm{Pa},\quad C_{12}=6.6\times10^{10}\,\mathrm{Pa},\quad C_{22}=14.6\times10^{10}\,\mathrm{Pa},\quad C_{66}=4.3\times10^{10}\,\mathrm{Pa},$$
$$e_{21}=-4.35\,C/m^2,\quad e_{22}=17.5\,C/m^2,\quad e_{16}=11.4\,C/m^2,\quad \in_{11}=9.87\times10^{-9}\,F/m,$$
$$\in_{22}=11.15\times10^{-9}\,F/m,\quad \overline{\alpha}_{11}=8.53\times10^{-6}\,1/\mathrm{K},\quad \overline{\alpha}_{22}=1.99\times10^{-6}\,1/\mathrm{K},$$
$$\lambda_{11}=1.1J/m\,s\mathrm{K},\quad \lambda_{22}=3.5J/m\,s\mathrm{K},\quad \tau_2=5.53\times10^{-3}\,C/m^2\,\mathrm{K},\quad C=500J/\mathrm{kg\,K}$$
$$\rho=5700\,\mathrm{kg}/m^3,\quad \varpi=10^{-10}\,\mathrm{s},\quad \omega=2\pi\times10^6\,1/\mathrm{s} \tag{31}$$

The thermo-mechanical coupling coefficients α_{ij} can be calculated as follows:

$$\alpha_{11}=(C_{11}+C_{12})\overline{\alpha}_{11}+(C_{12}+e_{21})\overline{\alpha}_{22},\quad \alpha_{22}=2C_{12}\overline{\alpha}_{11}+(C_{22}+e_{22})\overline{\alpha}_{22} \tag{32}$$

Three bulk wave vectors, phase velocities and the ratios of the quasi-longitudinal, quasi-transversal, temperature wave amplitudes for θ=20°, γ=0° propagating in an infinite space are

$$\begin{aligned} &\mathbf{k}_1=(390.37, && 1072.537); && c_1=5504.97\\ &\mathbf{k}_2=(774.57+1.14\times10^{-7}\mathrm{i}, && 2128.11+3.12\times10^{-7}\mathrm{i}); && c_2=2774.42\\ &\mathbf{k}_3=(570019.87+569661.83\mathrm{i}, && 1.57\times10^6+1.57\times10^6\mathrm{i}); && c_3=3.77 \end{aligned} \tag{33}$$

It can be seen from Eq. (33) that the attenuation of the elastic waves is very small, but the attenuation of the temperature wave is very large. Any one of the three waves corresponding to $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ can be used as the incident wave.

(1) As an example the incident quasi-longitudinal wave is discussed. The incident waves are

$$u_1^{(o)}=U_{11}^{(o)}\mathrm{e}^{\mathrm{i}(k_m^{(0)}x_m-\omega t)},\quad u_2^{(o)}=U_{21}^{(o)}\mathrm{e}^{\mathrm{i}(k_m^{(0)}x_m-\omega t)},\quad \vartheta^{(0)}=\Theta_1^{(0)}\mathrm{e}^{\mathrm{i}(k_m^{(0)}x_m-\omega t)},\quad \varphi^{(0)}=\Phi_1^{(0)}\mathrm{e}^{\mathrm{i}(k_m^{(0)}x_m-\omega t)}$$
$$U_{11}^{(o)}:U_{21}^{(o)}:\Theta_1^{(0)}:\Phi_1^{(0)}=(2.11\times10^{-10}-2.51\times10^{-19}\mathrm{i})$$
$$:(6.28\times10^{-10}-6.14\times10^{-19}\mathrm{i}):(-1.27\times10^{-11}-0.000054\mathrm{i}):(1.0) \tag{34}$$

From Eq. (28) we get the wave vectors of the reflected wave are

$$\begin{aligned}
&\mathbf{k}_1^{(r)} = (390.37, \quad -1072.53 - 4.04\times10^{-8}\mathrm{i}); && c_1 = 5504.97\\
&\mathbf{k}_2^{(r)} = (390.37, \quad -2229.26 - 9.56\times10^{-8}\mathrm{i}); && c_2 = 2776.26\\
&\mathbf{k}_3^{(r)} = (390.37, \quad -1597811.46 - 1596807.85\mathrm{i}); && c_3 = 3.93\\
&\mathbf{k}_4^{(r)} \equiv \mathbf{k}_s^{(r)} = (390.37, \quad 8.68\times10^{-9} - 368.07\mathrm{i}); && c_s = 16095.46
\end{aligned} \tag{35}$$

where the components $\mathbf{k}_1^{(r)}$, $\mathbf{k}_2^{(r)}$ and $\mathbf{k}_3^{(r)}$ are just wave vectors of bulk waves, $\mathbf{k}_4^{(r)} \equiv \mathbf{k}_s^{(r)}$ is the wave vectors of the new surface wave which attenuates along x_2 direction. It is also noted that in Eq. (35) the relation $k_1^{(0)} = k_{j1}^{(r)} \quad (j = 1-4)$ is used.

The ratios of the bulk and surface wave amplitudes for θ=20°,γ=0° are

$$U_{11}^{(r)} : U_{12}^{(r)} : \Theta_1^{(r)} : \Phi_1^{(r)} = (-2.11\times10^{-10} + 1.50\times10^{-19}\mathrm{i}) : (6.28\times10^{-10} - 3.24\times10^{-19}\mathrm{i}) : (1.27\times10^{-11} + 5.39\times10^{-5}\mathrm{i}) : (1.0)$$

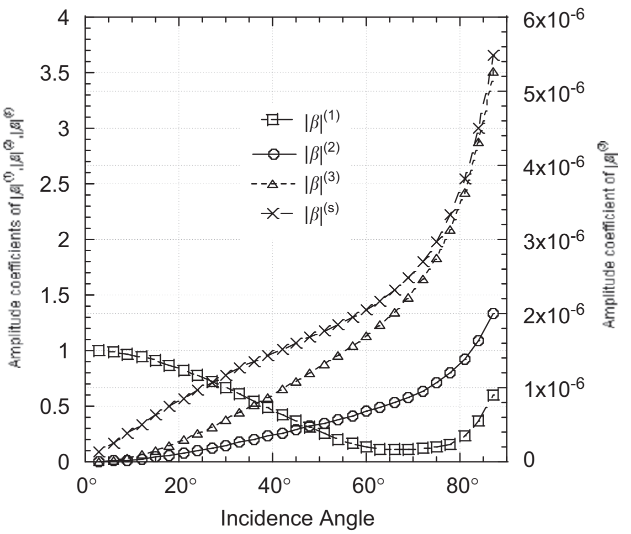

Fig. 4. Variations of modulus of the reflected and surface wave amplitude coefficients $|\beta^{(i)}|$ with θ for quasi-longitudinal incident wave with γ=0.

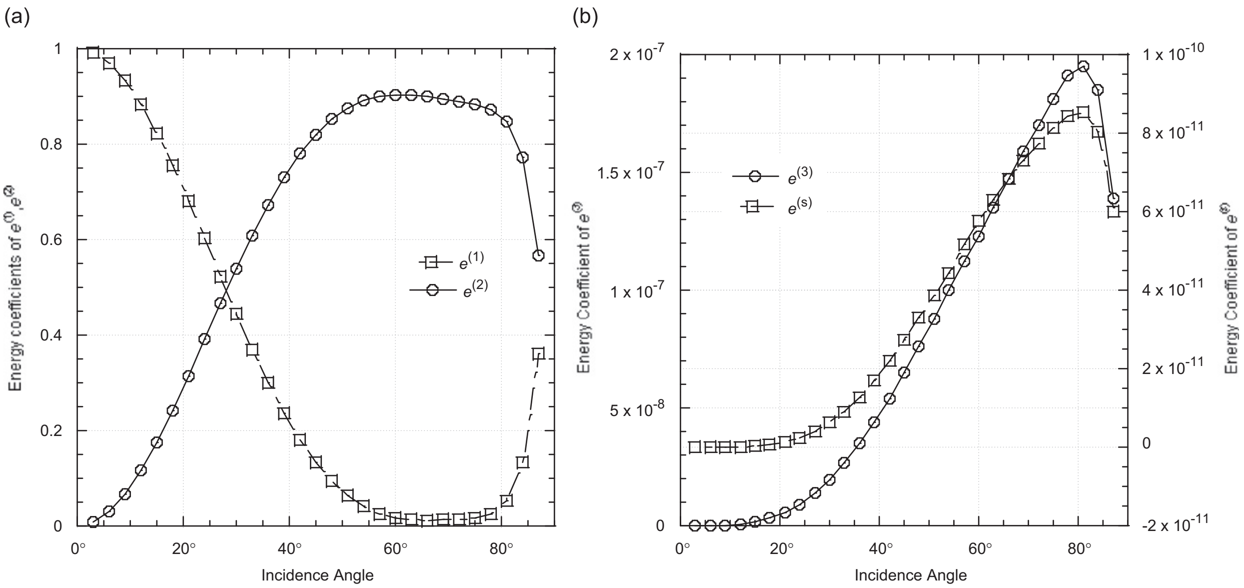

Fig. 5. The variations of the reflected elastic wave energy ratios with θ at γ=0 for quasi-longitudinal incident wave: (a) $e^{(1)}$, $e^{(2)} \sim \theta$ and (b) $e^{(3)}$, $e^{(s)} \sim \theta$.

$$U_{21}^{(r)}:U_{22}^{(r)}:\Theta_2^{(r)}:\Phi_2^{(r)}=(6.78\times10^{-9}+5.07\times10^{-17}\mathrm{i}):$$
$$(1.11\times10^{-9}+7.61\times10^{-18}\mathrm{i}):(-2.97\times10^{-10}-3.05\times10^{-4}\mathrm{i}):(1.0)$$
$$U_{31}^{(r)}:U_{32}^{(r)}:\Theta_3^{(r)}:\Phi_3^{(r)}=(-6.32\times10^{-14}+6.32\times10^{-14}\mathrm{i}):$$
$$(5.80\times10^{-11}-4.54\times10^{-18}\mathrm{i}):(29.27-29.29\mathrm{i}):(1.0)$$
$$U_{41}^{(r)}:U_{42}^{(r)}:\Theta_4^{(r)}:\Phi_4^{(r)}=(-1.18\times10^{-21}-4.57\times10^{-12}\mathrm{i})$$
$$:(2.92\times10^{-12}+3.92\times10^{-22}\mathrm{i})-:(0.000021+3.55\times10^{-13}\mathrm{i}):(1.0) \quad (36)$$

For different (θ,γ), the numerical values in Eqs. (33)–(36) are different. Their numerical values should be computed by numerical calculation for every case.

Comparing Eqs. (33) and (35) it is found that though three bulk wave velocities are almost the same in two equations, but their amplitude ratios are different. This situation shows that the amplitude ratios of the bulk reflection waves in the reflection problem are different with the waves propagating in free space.

Fig. 4 shows the variations of the modulus of ratios $(\beta^{(1)},\beta^{(2)},\beta^{(3)})$ of the reflected wave amplitude coefficients and the modulus of ratio $\beta^{(s)}$ of the surface wave amplitude coefficients with θ for γ=0. Fig. 5 shows the variations of ratios of the reflected and surface wave energy flows $e^{(1)},e^{(2)},e^{(3)},e^{(s)}$with θ for γ=0. The general expressions of the wave energy flow and its ratio of the reflected wave with the incident wave are defined as

$$\dot{W}_i=-\sigma_{ki}\dot{u}_k+\varphi\dot{D}_i-\lambda_{ik}\vartheta_{,k}\vartheta/T_0, \quad e^{(j)}=\langle\dot{W}_2^{(j)}\rangle/\langle\dot{W}_2^{(0)}\rangle \quad (37)$$

where the symbol $\langle\,\rangle$ expresses the average value over one period of a physical variable, $\dot{W}_2^{(j)}$ is the energy flow component corresponding to $\beta^{(j)}$ along x_2 direction.

The difference of the results for γ=0 and $\neq$0 is very small for small relaxation time, so we only give the results for γ=0. This means that the solution can be discussed directly by the homogeneous wave theory for the problem with small relaxation time.

From these figures it is seen that: (1) In the wave reflection problem from an interface between pyroelectric medium and vacuum, a surface wave is generated. From the generalized Snell's law Eq. (23) and numerical results it is seen that the phase velocity c_s of the surface wave is strongly dependent to the incident angle θ of the quasi-longitudinal wave: $\theta\to\pi/2,\ c_s\to c_1;\theta\to0,\ c_s\to\infty$,but the amplitudes of the surface wave are approach zero.

(2) When the incident wave is elastic wave, the component $e^{(2)}$ of the energy flow along the direction x_2 from the boundary x_2=0 is mainly contributed by the elastic wave modes, the effect of the reflected temperature and surface wave modes is very small. This is just the character of a surface wave. (3) There has only the quasi-longitudinal reflection wave for the incident wave with θ=0. (4) The attenuation angle γ almost does not play role when the incident wave is elastic wave.

When the wave vector **k**, the ratios of the bulk and surface wave amplitudes $U_{i1}^{(r)}:U_{i2}^{(r)}:\Theta_i^{(r)}:\Phi_i^{(r)}$ and the amplitude coefficient $\beta^{(j)}(i$=1–4) are solved, from Eq. (18) it is easy to get the solution of the reflection wave.

5. Conclusions

In this paper, the reflection and transmission theories of waves in pyroelectric and piezoelectric medium are studied. In this problem the puzzle is that the electric potential does not have its own independent wave mode under the postulation of quasi-static electric field. There are only four independent wave modes for the five-order Christoffel's equation of waves propagated in an infinite homogenous space. However, in the reflection and transmission problem there are five complex boundary conditions in the pyroelectric medium. It is a problem whether the reflection and transmission problem in the pyroelectric medium is solvable. In this paper we find that in the reflection and transmission wave problem a surface wave mode in each side of the boundary surface will be revealed except the four bulk wave modes propagating in an infinite homogenous space due to the general Snell's law. The surface waves have the same wave vector component with the incident waves on the interface plane. The surface wave was not found in the previous literatures. The surface wave and the bulk waves together can just satisfy the boundary conditions. The two dimensional reflection problem of waves at the interface between the semi-infinite pyroelectric medium and vacuum is researched in greater detail. The numerical example of the two-dimensional reflection problem from the interface of $BiTiO_3$/ vacuum is given. Our numerical example shows that there exists a surface wave mode certainly. It is also found that the ratios of the amplitudes of the bulk waves in the reflection problem are different with that in the propagation problem in an infinite homogenous space. The difference of the results for the attenuation angle γ=0 and $\neq$0 is very small for small relaxation time, so in many engineering problems we can use the homogeneous wave theory conveniently.

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