419. *Remarks on the Use of the Logarithmic Head Correction in Viscometry.*

By Guy Barr.

Many authors* have directed attention to the error that may arise in the calculation of viscosities from the rates of flow through a capillary tube, when the flow occurs under a gradually diminishing head, and the arithmetic mean of the initial and the final head is taken for insertion into Poiseuille's formula. The simplest expression for the correct mean value H is obtained when the discharge is from a vessel of constant cross-section either into the air or into a second vessel of constant cross-section : in this case, if the arithmetic mean head be h and the difference between the initial and the final head be $2x$,

$$H = 2x/[\log_e(h + x)/(h - x)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This value for H is often called the " Meissner head," since it was deduced by Meissner for the Engler viscometer, or the " logarithmic head," from the form of the denominator. The " logarithmic head correction " is $(h - H)$.

1. *Approximate Equations for the Logarithmic Head.*—Even when the construction of a viscometer is such that equation (1) is applicable, the evaluation of H requires the use of a table of logarithms, and when x/h is small a 5-figure table will be quite inadequate for any reasonable accuracy. Glass instruments very rarely have cylindrical vessels at each end of the capillary, the bulbs being approximately spherical, biconical (Bingham), fusiform, or else cylindrical with conical ends (British Standards Institution). An expression for the case of a spherical bulb with discharge into air has been given by Barr, and results for certain other forms are listed below. In practice, however, the bulbs of viscometers are not exactly of any simple geometrical shape; if an accurate value for H is required, it is thus necessary to make h so large compared with x that the correction $(h - H)$ is either negligibly small or roughly calculable. In these circumstances, it is desirable that the correction be expressed in a form which allows it to be estimated with less trouble than is necessary for evaluating the formulæ resulting from integration. Approximate values may be obtained by expanding the logarithm that occurs in each of the exact solutions : for certain conditions it has proved more convenient to make the approximation before the integration, but in either case the result is in the form of a series of terms involving ascending powers of x/h. We put $x/h = f$, and assume that f is small compared with unity.

For the conditions under which equation (1) holds, we find

$$H\left(1 + \frac{1}{3}f^2 + \frac{1}{5}f^4 + \ldots\right) = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

For a sphere of radius x, with discharge into air or into a recipient of infinite area,

$$H = 4f^3h/[6f - 3(1 - f^2)\log_e(1 + f)/(1 - f)] \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence, approximately,

$$H\left(1 + \frac{1}{5}f^2 + \frac{3}{35}f^4 + \ldots\right) = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

For a bulb that has the form of a pair of opposed cones, each of a height equal to the radius of the common base, discharging into air, or for such a bulb discharging symmetrically into a similar one, as in Bingham's viscometer :

$$H = \frac{2}{3}f^3h/[(1 + f^2)\log_e(1 + f)/(1 - f) + 2f\log_e(1 - f^2) - 2f] \quad . \quad . \quad (3)$$

whence, approximately,

$$H\left(1 + \frac{1}{10}f^2 + \frac{1}{35}f^4 + \ldots\right) = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

* References to these and to other authors cited below are given in the author's " Monograph of Viscometry " (Oxford University Press, **1931**).

For a biconical bulb, as in (3), discharging into a cylinder of radius equal to the maximum radius of the cones

$$H(1 + 0{\cdot}144f^2 + \ldots) = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

When the bulb consists of a cylinder with a cone at each end, the radius and height of the cylinder being equal to the height of each cone, and the discharge is into a cylinder of the same radius,

$$H(1 + 0{\cdot}190f^2 + \ldots) = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

For a bulb of similar shape, except that the height of the cylinder is twice the height and maximum radius of the cones, discharging into a cylinder of the same radius,

$$H(1 + 0{\cdot}222f^2 + \ldots) = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4c)$$

Bulbs approximating to those of (4*b*) and (4*c*) are specified for the viscometers standardised by the British Standards Institution.

The approximations indicate, very much more clearly than the exact solutions, the influence of the shape of the bulb on the magnitude of the correction. The difference $(h - H)$ is naturally a maximum when the " bulb " from which the discharge occurs is a cylinder, and decreases as the influence of the upper and the lower end is reduced by making their volume a smaller fraction of the total. For a given ratio of x to h, the correction increases, except in case (1), as the diameter of the recipient cylinder is reduced (cf. 3*a* and 4*a*).

The logarithmic head correction is important (i) when external pressures are applied and H has to be estimated from the values of the constant external pressure and the varying hydrostatic pressure in the viscometer, and (ii) when it is necessary to derive H from the dimensions of the apparatus, *e.g.*, in the use of " consistometers " of the burette type (Cooke, Auerbach, *et al.*). Comparison of equations (1*a*) and (3*a*) shows that the external pressure need not be so large a multiple of the hydrostatic pressure as was suggested by Bingham, Schlesinger, and Coleman in order to make the correction negligible in the use of Bingham's viscometer. The approximation (1*a*) will serve to indicate at what stage during the discharge of a consistometer it is desirable to make use of the exact equation (1).

2. *Application of the Correction in the Standardisation of Ostwald Viscometers.*—In the normal use of viscometers of the Ostwald type for measurements of relative kinematic viscosities, the " logarithmic head " is a constant of the apparatus and does not need to be evaluated. The method introduced by Grüneisen for determining the minimum time of flow that is proportional to the kinematic viscosity (without the application of a kinetic-energy correction) consists in finding the " logarithmic head," h_0, experimentally from observations of the time-intervals corresponding with various stages during the discharge, and then measuring the times of flow of water when the mean head is modified by applying different external pressures h_e. It is assumed that, so long as the kinetic-energy correction is negligible, the product ht or $(h_0 + h_e)t$ of the time of flow and the effective head will remain constant. But it is clear from equation (2*a*) that, if the bulb be spherical and the kinetic-energy correction be negligible throughout the series, the time of flow t_0 when no external pressure is applied will appear to be unduly long, the product $(h_0 + h_e)t$ being greater by $20f^2\%$ than that obtained when the applied head has been increased until $(x/h)^2$ has become negligibly small. The practical effect is to increase the mean value derived for ht, so that the range of usefulness of the viscometer will be under-estimated.

Bury's procedure (J., 1934, 1380) avoids the experimental difficulty associated with the estimation of the " logarithmic head " by using a new graphical method to represent the results of the flow tests at different heads. He assumes that with a given filling the equations $\nu = ah_0t_0 - b/t_0 = a(h_0 + h_e)t - b/t$ will hold, whence $h_e/(t_0 - t) = h_0 + b/(at_0t)$; he therefore plots values of $h_e/(t_0 - t)$, calculated from the observations of the times of flow t corresponding with different values of h_e against $1/t$, and draws the best straight line through the points. The slope of this straight line gives $b/(at_0)$ and from this, knowing the time t_0 corresponding with no external head, it is possible to find b by making use of an approximate value of a deduced from a calibration with a liquid of known kinematic viscosity ν. The method depends, however, like that of Grüneisen, on the assumed constancy of h_0: Bury remarks that for all ordinary Ostwald viscometers the increase with h_e is very small. Equation (2*a*) shows that the increase might amount to 0·23% for the instrument he describes, a spherical measuring bulb and a large recipient bulb being assumed; introduction of the correction would have led to the deduction of a value of b some 5% greater than that given by Bury. The capillary of his viscometer was so narrow that such an error was of negligible importance—in fact, the coefficient might have been calculated from the dimensions with sufficient accuracy. If, however, the kinetic-energy correction is considerable, and particularly if the length of the bulb is not a very small fraction of the mean head, it will be essential to apply the logarithmic head correction to each of the observations. Bury's method appears to be a great improvement over that of Grüneisen, since it not only reduces the experimental work but also affords a determination of b instead of merely finding a range within which it is permissible to assume that $b = 0$. It is suggested that a few of the observations of flow time with different external heads should first be plotted as recommended by Bury to find an approximate value for h_0. Knowing the volume and, roughly, the shape of the bulb and the area of the recipient, it is then possible to estimate the logarithmic head corrections Δ_0 and Δ_n to the head h_0 and to the various values of $h_0 + h_e$. (If the bulb of Bury's instrument were made up of two right-angled cones, Δ_0 would be 0·18% instead of the 0·23% cited above for a sphere.) Introduction of the corrections indicates that

$$(h_e - \Delta_n + \Delta_0)/(t - t_0) = h_0 - \Delta_0 + b/(att_0) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that, when the values of the expression on the left are plotted against $1/t$, a straight line should be obtained intersecting the axis of ordinates at $h_0 - \Delta_0$ and having a slope of $\tan^{-1} b/(at_0)$. If the kinematic viscosity ν be known, then

$$\nu = a(h_0 - \Delta_0)t_0 - b/t_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

$$a = \nu/\{(h_0 - \Delta_0)t_0 - b/(at_0)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

Both terms of the denominator in (6*a*) are known, so the calibration constants in equation (6) may be evaluated.

3. *Use of the Correction in the Examination of "Abnormal" Liquids.*—External pressures have often been applied to viscometers of the Ostwald type when it was desired to investigate the possibility of deviations from Poiseuille's law in the flow of colloidal solutions or suspensions. Wo. Ostwald and Föhre avoided the errors that may occur in the estimation of h_0 and of the mean effective pressures acting when external pressures are added, by calibrating the instrument for each case by means of a flow test with a normal liquid. Abnormality is then shown by a decrease in the relative time of flow as the pressure is increased. The procedure advocated by Bury, modified as suggested above, may well be used to check the calibration data, which should give points lying on a straight line when this method of plotting is adopted: it will then be safe to interpolate flow times for the normal liquid at pressures other than those actually used in the calibration. When the viscosity of the liquid under examination is so high that it is inconvenient, or impossible, to make a satisfactory preliminary calibration with a liquid that is known to obey Poiseuille's law, the best use will be made of the data relating flow times to applied pressure if the mean effective pressures are corrected as in equations (2)—(4*c*).

THE NATIONAL PHYSICAL LABORATORY, TEDDINGTON. [*Received, November 5th,* 1935.]