A Method for the Numerical Evaluation of Finite Integrals of Oscillatory Functions

By I. M. Longman

1. Introduction. In two previous publications [1, 2] the author has demonstrated a method, based on Euler's transformation of slowly convergent alternating series, for the numerical evaluation of infinite integrals of oscillatory functions; this can be used in many cases by a double application for the evaluation of finite integrals of oscillatory functions. For example the integral

$$\int_b^a f(x)\,dx \tag{1}$$

where $f(x)$ may have a very large number of oscillations in the range of integration can conveniently be evaluated as

$$\int_a^\infty f(x)\,dx - \int_b^\infty f(x)\,dx. \tag{2}$$

However in physical problems the finiteness of the range of integration is often associated with a kind of natural boundary of $f(x)$, such that it is impossible to extend $f(x)$ to values of x beyond the upper limit b while preserving the general character of $f(x)$. Analytically speaking, $x = b$ may be a branch point of $f(x)$. Alternatively, it may be possible to extend the range of integration to infinity as in equation (2), but the infinite integrals may not converge. As an example of the branch point difficulty we can consider the integral

$$I = \int_0^a (a^2 - x^2)^{\frac{1}{2}} \sin x\,dx, \tag{3}$$

which, if a is large, would be very difficult to compute by straightforward numerical integration owing to the large number of oscillations, and we clearly cannot apply the method of equation (2). An instance of this kind of difficulty in a physical problem is given in Pekeris [3] where the solution of a problem of sound propagation in a layered liquid is given in the form of infinite integrals where the character of the oscillatory integrand changes abruptly at a certain point in the interval of integration.

The present paper gives an extension of Euler's transformation and applies it to the numerical computation of integrals such as (3).

2. Description of the Method. For simplicity let us start by considering alternating series, and suppose that we wish to calculate the sum of a series of the form

$$S = v_0 - v_1 + v_2 - \cdots + (-1)^n v_n \tag{4}$$

Received August 13, 1959. Institute of Geophysics Publication No. 139.

where the v_i's are all positive and slowly decrease in numerical value as i increases from 0 to n. For example such a series might be

$$S = 9999^{\frac{1}{2}} - 9998^{\frac{1}{2}} + 9997^{\frac{1}{2}} - \cdots + 1^{\frac{1}{2}}. \tag{5}$$

In order to transform (4) into a form convenient for computation let us consider the associated power series,

$$S(x) = v_0 - v_1 x + v_2 x^2 - \cdots + (-1)^n v_n x^n, \tag{6}$$

which reduces to (4) when $x = 1$. If we multiply by x and add we obtain

$$\begin{aligned}(1 + x)S(x) = v_0 - (v_1 - v_0)x \\ + (v_2 - v_1)x^2 - \cdots + (-1)^n(v_n - v_{n-1})x^n + (-1)^n v_n x^{n+1},\end{aligned}$$

or, with the usual notation for differences,

$$\begin{aligned}\Delta v_i &= v_{i+1} - v_i \\ \Delta^{r+1} v_i &= \Delta^r v_{i+1} - \Delta^r v_i ,\end{aligned}$$

we have

$$(1 + x)S(x) = v_0 - (\Delta v_0)x + (\Delta v_1)x^2 - \cdots + (-1)^n(\Delta v_{n-1})x^n + (-1)^n v_n x^{n+1}.$$

This gives

$$\begin{aligned}S(x) = \frac{v_0 + (-1)^n v_n x^{n+1}}{1 + x} \\ - y[\Delta v_0 - (\Delta v_1)x + (\Delta v_2)x^2 - \cdots + (-1)^{n-1}(\Delta v_{n-1})x^{n-1}],\end{aligned} \tag{7}$$

where $y = x/(1 + x)$. A second application of this transformation to the series in square brackets in (7) yields

$$\begin{aligned}S = \frac{v_0 + (-1)^n v_n x^{n+1}}{1 + x} - \frac{\Delta v_0 + (-1)^{n-1}(\Delta v_{n-1})x^n}{1 + x} y \\ + y^2[\Delta^2 v_0 - (\Delta^2 v_1)x + (\Delta^2 v_2)x^2 - \cdots + (-1)^{n-2}(\Delta^2 v_{n-2})x^{n-2}],\end{aligned}$$

and p applications give

$$\begin{aligned}S(x) = (v_0 - y\Delta v_0 + y^2\Delta^2 v_0 - \cdots + (-1)^{p-1} y^{p-1}\Delta^{p-1} v_0)/(1 + x) \\ + (-1)^n[v_n x^{n+1} + (\Delta v_{n-1})x^n y + (\Delta^2 v_{n-2})x^{n-1}y^2 + \cdots \\ + (\Delta^{p-1} v_{n-p+1})x^{n-p+2}y^{p-1}]/(1 + x) \\ + (-1)^p y^p[\Delta^p v_0 - (\Delta^p v_1)x + (\Delta^p v_2)x^2 - \cdots \\ + (-1)^{n-p}(\Delta^p v_{n-p})x^{n-p}], \qquad p \leqq n.\end{aligned}$$

Putting $x = 1$ we have as a transformed form of (4)

$$\begin{aligned}S = [(1/2)v_0 - (1/4)\Delta v_0 + (1/8)\Delta^2 v_0 - \cdots + (-1)^{p-1}2^{-p}(\Delta^{p-1} v_0)] \\ + (-1)^n[(1/2)v_n + (1/4)\Delta v_{n-1} + (1/8)\Delta^2 v_{n-2} + \cdots + 2^{-p}\Delta^{p-1} v_{n-p+1}] \\ + 2^{-p}(-1)^p[\Delta^p v_0 - \Delta^p v_1 + \Delta^p v_2 - \cdots + (-1)^{n-p}\Delta^p v_{n-p}], \qquad p \leqq n.\end{aligned} \tag{8}$$

This result is of course exact, but for large values of p (assuming that the high order differences are small) the later terms in the first two square brackets and the whole of the third square bracket in equation (8) can be neglected since 2^{-p} will be negligible.

Assuming then, that n is large, we have for (4) the excellent approximation

$$\begin{aligned} S = (1/2)v_0 - (1/4)\Delta v_0 + (1/8)\Delta^2 v_0 - \cdots \\ + (-1)^n[(1/2)v_n + (1/4)\Delta v_{n-1} + (1/8)\Delta^2 v_{n-2} + \cdots] \end{aligned} \tag{9}$$

which represents a kind of double application of Euler's transformation.

3. Examples. The utility of the series (9) will be demonstrated by a number of examples.

1. Consider the series

$$S = 1 - 1/2 + 1/3 - 1/4 + \cdots - 1/1000. \tag{10}$$

To evaluate (10) by means of equation (9) we split off the first eight terms (whose differences do not decrease very rapidly) and evaluate

$$S' = 1/9 - 1/10 + 1/11 - \cdots - 1/1000. \tag{11}$$

The contribution at the first eight terms of S is 0.634524, and so

$$S = 0.634524 + S'.$$

The differencing of the first few and the last few terms of S' is shown in Table 1. From this, by means of equation (9), we obtain

$$\begin{aligned} S' = (1/2) \times 0.111111 + (1/4) \times 0.011111 + (1/8) \times 0.002020 \\ + (1/16) \times 0.000505 + (1/32) \times 0.000156 + (1/64) \times 0.000057 \\ - (1/2) \times 0.001000 + (1/4) \times 0.000001 = 0.058123. \end{aligned}$$

Thus $S = 0.692647$. As a check we can calculate (10) as the difference between two infinite series

$$\begin{aligned} S &= (1 - 1/2 + 1/3 - 1/4 + \cdots) \\ &\quad - (1/1001 - 1/1002 + 1/1003 - \cdots) \\ &= \ln 2 - (1/1001 - 1/1002 + 1/1003 - \cdots). \end{aligned} \tag{12}$$

The series in parentheses in (12) can be evaluated by applying Euler's transformation, and its sum is easily shown to be

$$0.000499.$$

Thus working with equation (12) we obtain

$$S = 0.693147 - 0.000499 = 0.692648,$$

agreeing with our previous result.

2. In example 1 we were able to extend our series to infinity so that we really had no need to use the transformation (9). We now consider, however, an example

TABLE 1

		Δ^2		Δ^4		Δ^6
$9^{-1} = 0.111111$						
	-11111					
$10^{-1} = 0.100000$		2020				
	-9091		-505			
$11^{-1} = 0.090909$		1515		156		
	-7576		-349		-57	
$12^{-1} = 0.083333$		1166		99		24
	-6410		-250		-33	
$13^{-1} = 0.076923$		916		66		
	-5454		-184			
$14^{-1} = 0.071429$		732				
	-4762					
$15^{-1} = 0.066667$						
- - -						
$994^{-1} = 0.001006$						
	-1					
$995^{-1} = 0.001005$						
	-1					
$996^{-1} = 0.001004$						
	-1					
$997^{-1} = 0.001003$						
	-1					
$998^{-1} = 0.001002$						
	-1					
$999^{-1} = 0.001001$						
	-1					
$1000^{-1} = 0.001000$						

TABLE 2

		Δ^2		Δ^4		
$9999^{1/2} = 99.995000$						
	-5000					
$9998^{1/2} = 99.990000$		-2				
	-5002		$+2$			
$9997^{1/2} = 99.984998$		0		1		
	-5002		$+3$			
$9996^{1/2} = 99.979996$		3				
	-4999		-6			
$9995^{1/2} - 99.974997$		-3				
	-5002					
$9994^{1/2} = 99.969995$						
- - -						
$15^{1/2} = 3.872983$						
	-131326					
$14^{1/2} = 3.741657$		-4780				
	-136106		-563			
$13^{1/2} = 3.605551$		-5343		-122		
	-141449		-685		-35	
$12^{1/2} = 3.464102$		-6028		-157		-27
	-147477		-842		-62	
$11^{1/2} = 3.316625$		-6870		-219		
	-154347		-1061			
$10^{1/2} = 3.162278$		-7931				
	-162279					
$9^{1/2} = 3.000000$						

of a series which cannot be so extended. Such a series is given in equation (5). This series has 9,999 terms and its direct evaluation would be a rather lengthy computation. However it is easily computed by an application of our transformation in equation (9). Differencing the first few terms, and a few terms near the end of the series we obtain Table 2. We split off the last eight terms of the series (which do not difference so well) and obtain

$$\begin{aligned} S = {} & (1/2) \times 99.995000 + (1/4) \times 0.005000 - (1/8) \times 0.000002 \\ & + (1/2) \times 3.000000 - (1/4) \times 0.162278 - (1/8) \times 0.007931 \\ & - (1/16) \times 0.001061 - (1/32) \times 0.000219 - (1/64) \times 0.000062 \\ & - 8^{\frac{1}{2}} + 7^{\frac{1}{2}} - 6^{\frac{1}{2}} + 5^{\frac{1}{2}} - 4^{\frac{1}{2}} + 3^{\frac{1}{2}} - 2^{\frac{1}{2}} + 1^{\frac{1}{2}} = 50.378853. \end{aligned}$$

3. Now let us turn to the evaluation of oscillatory integrals. We will illustrate this by evaluating the integral (3) for $a = 100\pi$. We have

$$I = \int_0^{100\pi} (100^2\pi^2 - x^2)^{\frac{1}{2}} \sin x \, dx. \tag{3'}$$

By splitting up the range of integration, I can be expressed as the series

$$I = \sum_{r=0}^{99} (-1)^2 \int_0^{\pi} [100^2\pi^2 - (r\pi + x)^2]^{\frac{1}{2}} \sin x \, dx \tag{13}$$

which is of the form (4). A few integrals near the beginning and near the end of the series (13) were evaluated on an IBM 709 computer using a 16 point Gaussian integration formula. The results are tabulated and differenced in Table 3. Applying our method we obtain

$$\begin{aligned} I = {} & (1/2) \times 628.30915 + (1/4) \times 0.06285 - (1/8) \times 0.06284 \\ & + (1/16) \times 0.00004 - [(1/2) \times 325.85292 - (1/4) \times 10.14092 \\ & - (1/8) \times 0.41146 - (1/16) \times 0.03382 - (1/32) \times 0.00472 \\ & - (1/64) \times 0.00089 - (1/128) \times 0.00022] + 315.26077 \\ & - 304.17027 + 292.52472 - \cdots - 60.96022 \\ = {} & 298.43558. \end{aligned}$$

As a check we apply the method differently to Table 3, obtaining

$$\begin{aligned} I = {} & 628.30915 - 628.24630 + (1/2) \times 628.12061 - (1/4) \times 0.18857 \\ & - (1/8) \times 0.06298 - (1/16) \times 0.00002 + (1/2) \times 238.71325 \\ & - (1/4) \times 14.76162 - (1/8) \times 0.96185 - (1/16) \times 0.14230 \\ & - (1/32) \times 0.03311 - (1/64) \times 0.00997 - (1/128) \times 0.00362 \\ & - (1/256) \times 0.00150 - 222.79836 + 209.46181 - \cdots - 60.96002 \\ = {} & 298.43557 \end{aligned}$$

agreeing with our previous result.

TABLE 3

q	v_q		Δ^2		Δ^4		Δ^6	
1	628.30915							
		−6285						
2	628.24630		−6284					
		−12569		−4				
3	628.12061		−6288					
		−18857		−10				
4	627.93204		−6298					
		−25155		−2				
5	627.68049		−6300					
		−31455						
6	627.36594							

79	381.13726							
		−837885						
80	372.75841		−30116					
		−868001		−2211				
81	364.07840		−32327		−316			
		−900328		−2527		−67		
82	355.07512		−34854		−383		−22	
		−935182		−2910		−89		
83	345.72330		−37764		−472			
		−972946		−3382				
84	335.99384		−41146					
		−1014092						
85	325.85292							

85	325.85292							
		−1059215						
86	315.26077		−49835					
		−1109050		−5670				
87	304.17027		−55505		−1256			
		−1164555		−6926		−423		
88	292.52472		−62431		−1679		212	
		−1226986		−8605		−635		−150
89	280.25486		−71036		−2314		362	
		−1298022		−10919		−997		
90	267.27464		−81955		−3311			
		−1379977		−14230				
91	253.47487		−96185					
		−1476162						
92	238.71325							
93	222.79836							
94	205.46181							
95	186.30583							
96	164.30583							
97	139.47917							
98	108.12528							
99	60.96022							

4. Conclusion. An extension of Euler's transformation has been presented and its use in the computation of finite integrals of oscillatory functions demonstrated. It is believed that its application will render feasible the numerical solution of physical problems hitherto regarded as intractable.

5. Acknowledgments. The author desires to record his appreciation of the computing facilities made available to him at the Western Data Processing Center of the University of California at Los Angeles.

Institute of Geophysics
University of California
Los Angeles, California

1. I. M. LONGMAN, "Note on a method for computing infinite integrals of oscillatory functions," Cambridge Phil. Soc., *Proc.*, v. 52, 1956, p. 764.
2. I. M. LONGMAN, "Tables for the rapid and accurate numerical evaluation of certain infinite integrals involving Bessel functions," *MTAC*, v. 11, 1957, p. 166.
3. C. L. PEKERIS, Geol. Soc. Am. *Mem.*, No. 27, 1948, p. 45. (See equations A17, A18, A19).