side, the yellow result light will flash on, indicating the truth value of $KNpNq$ for p = wrong and q = wrong according to the left position of the switches for the variables p and q on the right side of the keyboard.

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Evaluation at Half Periods of Weierstrass' Elliptic Function with Rectangular Primitive Period-Parallelogram

By Chih-Bing Ling

The purpose of this paper is to evaluate the following Weierstrass' elliptic function at half periods [1],

$$(1)\qquad e_1 = \wp(\omega_1), \qquad e_2 = \wp(\omega_2), \qquad e_3 = \wp(\omega_3),$$

where $2\omega_1$ and $2\omega_2$ are double periods of the function and ω_3 is defined by

$$(2)\qquad \omega_1 + \omega_2 + \omega_3 = 0.$$

This paper tabulates only the values of the function whose primitive period-parallelogram is a rectangle with $2\omega_1 = 1$ and $2\omega_2 = ai$, where $a \geqq 1$.

The three functions in (1) form a set of distinct roots of the cubic [1]

$$(3)\qquad x^3 - px - q = 0,$$

where

$$(4)\qquad p = 15\sigma_4, \qquad q = 35\sigma_6,$$

and

$$(5)\qquad \begin{aligned} \sigma_{2k} &= \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(2m\omega_1 + 2n\omega_2)^{2k}} \\ &= 2\sum_{m=1}^{\infty} \frac{1}{m^{2k}} + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{(m + nai)^{2k}}. \end{aligned}$$

The accent on the summation sign denotes the omission of simultaneous zero values of m and n from the double summation.

The cubic (3) indicates that

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$$e_1 + e_2 + e_3 = 0. \tag{6}$$

Also, since e_1, e_2 and e_3 are distinct, the discriminant $(4p^3 - 27q^2)$ of the cubic does not vanish. As will be seen later, in the present case both σ_4 and σ_6 are real and $(4p^3 - 27q^2)$ is positive. This implies that all the roots of the cubic are real.

The evaluation of σ_4 and σ_6 is facilitated by using the known relation [2]

$$\cot x = \frac{1}{x} + \sum_{m=-\infty}^{\infty}{}' \left(\frac{1}{m\pi + x} - \frac{1}{m\pi}\right) \tag{7}$$

where the accent on the summation sign denotes the omission of the zero value of m from the summation. By repeated differentiation of Equation (7) and substitution of ix for x, it is found that

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \frac{1}{(m\pi + ix)^4} &= \frac{2}{3\sinh^2 x} + \frac{1}{\sinh^4 x} \\ \sum_{m=-\infty}^{\infty} \frac{1}{(m\pi + ix)^6} &= -\frac{2}{15\sinh^2 x} - \frac{1}{\sinh^4 x} - \frac{1}{\sinh^6 x}. \end{aligned} \tag{8}$$

Hence we have

$$\begin{aligned} \sigma_4 &= \frac{\pi^4}{45} + \frac{4\pi^4 K_1}{3\sinh^2 \pi a} \\ \sigma_6 &= \frac{2\pi^6}{945} - \frac{4\pi^6 K_2}{15\sinh^2 \pi a} \end{aligned} \tag{9}$$

where

$$\begin{aligned} K_1 &= \sinh^2 \pi a \sum_{n=1}^{\infty} \left(\frac{1}{\sinh^2 n\pi a} + \frac{3}{2\sinh^4 n\pi a}\right) \\ K_2 &= \sinh^2 \pi a \sum_{n=1}^{\infty} \left(\frac{1}{\sinh^2 n\pi a} + \frac{15}{2\sinh^4 n\pi a} + \frac{15}{2\sinh^6 n\pi a}\right). \end{aligned} \tag{10}$$

Consequently, we find

$$\frac{4p^3 - 27q^2}{16\pi^{12}} = \frac{5K_1 + 7K_2}{3\sinh^2 \pi a} + \frac{100K_1^2 - 147K_2^2}{\sinh^4 \pi a} + \frac{2000K_1^3}{\sinh^6 \pi a}. \tag{11}$$

With the aid of known tables [3, 4], values of K_1, K_2, and then σ_4, σ_6 and $(4p^3 - 27q^2)^{\frac{1}{2}}$ are computed to 16D for $a = 1(0.25)2(1)6$ and ∞ as shown in Table 1.

The subsequent evaluation of e_1, e_2, and e_3 requires the solution of the cubic (3). It appears that one of the roots, e_1, can be easily evaluated to 16D as shown in Table 2 by using Newton's method or otherwise, but difficulty arises in evaluating the other two roots for in most cases they are almost equal. However, they can be separated by forming a new cubic

$$x^3 + p'x - q' = 0 \tag{12}$$

whose roots are the differences of the roots of the cubic (3). Let $(e_1 - e_2)$, $(e_2 - e_3)$ and $(e_3 - e_1)$ be the roots of the new cubic. We have

$$\begin{aligned} p' &= (e_1 - e_2)(e_2 - e_3) + (e_2 - e_3)(e_3 - e_1) + (e_3 - e_1)(e_1 - e_2) = -3p, \\ q'^2 &= (e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = 4p^3 - 27q^2. \end{aligned} \tag{13}$$

TABLE 1

a	K_1	K_2	σ_4	σ_6	$(4p^3 - 27q^2)^{\frac{1}{2}}$
1	1.01311,06293,09539	1.05851,88947,76779	3.15121,20021,53898	0	6.49955,04200,35218 × 10^2
1.25	1.00271,90816,67748	1.01206,13101,78251	2.36702,93923,35617	1.63147,62559,94511	3.01664,67968,60964 × 10^2
1.5	1.00056,49682,73190	1.00250,28510,82885	2.20660,15468,91272	1.95170,97194,76020	1.38049,25551,66708 × 10^2
1.75	1.00011,74335,66488	1.00052,00996,05499	2.17336,30560,32070	2.01747,33403,07279	6,29902,26479,45028 × 10
2	1.00002,44115,30272	1.00010,81097,89987	2.16645,82514,80805	2.03110,95062,61006	2.87242,26104,19235 × 10
3	1.00000,00455,86885	1.00000,02018,84784	2.16464,98507,19257	2.03467,94456,07301	1.24133,82088,92023
4	1.00000,00000,85131	1.00000,00003,77008	2.16464,64737,40389	2.03468,61114,97443	5.36430,92081,07968 × 10^{-2}
5	1.00000,00000,00159	1.00000,00000,00704	2.16464,64674,34075	2.03468,61239,45609	2.31812,81969,45459 × 10^{-3}
6	1.00000,00000,00000	1.00000,00000,00001	2.16464,64674,22298	2.03468,61239,68855	1.00175,40242,77741 × 10^{-4}
∞	1.00000,00000,00000	1.00000,00000,00000	2.16464,64674,22276	2.03468,61239,68898	0

TABLE 2

a	e_1	$-(e_2 - e_3)$	$-e_2$	$-e_3$
1	6.87518,58180,20373	6.87518,58180,20373	6.87518,58180,20373	0
1.25	6.64106,26950,12943	3.11618,71546,58608	4.87862,49248,35775	1.76243,77701,77167
1.5	6.59248,08531,44224	1.41904,24293,36329	4.00576,16412,40277	2.58671,92119,03947
1.75	6.58238,54370,71645	6.46830,14416,89006 × 10^{-1}	3.61460,77906,20273	2.96777,76464,51372
2	6.58028,69683,44880	2.94898,84967,54931 × 10^{-1}	3.43759,29090,10186	3.14269,40593,34693
3	6.57973,72957,91816	1.27435,57352,40785 × 10^{-2}	3.29624,04265,72112	3.28349,68692,19704
4	6.57973,62693,13382	5.50699,03149,79195 × 10^{-4}	3.29014,34841,72440	3.28959,27851,40942
5	6.57973,62673,96492	2.37978,62933,93412 × 10^{-5}	3.28988,00326,29713	3.28985,62347,66779
6	6.57973,62673,92912	1.02839,89036,79391 × 10^{-6}	3.28986,86478,95908	3.28986,76194,97004
∞	6.57973,62673,92906	0	3.28986,81336,96453	3.28986,81336,96453

Consequently, by taking a positive sign for q', the new cubic is in the form

$$(14) \qquad x^3 - 3px - (4p^3 - 27q^2)^{\frac{1}{2}} = 0.$$

From this cubic, values of $(e_2 - e_3)$ and then e_2 and e_3 are computed to $16D$ as shown in Table 2.

It is mentioned that the values of the function, for $0 \leqq a < 1$ or in general for ω_2/ω_1 purely imaginary, can be computed from the tabulated values with the aid of the following relation [1]

$$(15) \qquad \wp(\lambda z \mid \lambda\omega_1, \lambda\omega_2) = \lambda^{-2}\wp(z \mid \omega_1, \omega_2)$$

where λ is a constant, real or complex.

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A Note on the Nonexistence of Certain Projective Planes of Order Nine

By Raymond B. Killgrove

1. Introduction. Every finite projective plane may be coordinatized in at least one way [1]. In this process some line is chosen to be the line at infinity, and the points not on this line are represented by an ordered pair of elements. The elements x and y for any point (x, y) on a given line of the plane satisfy the equation $y = x \cdot m \circ b$, where m and b are specific elements for the given line. This ternary operation on x, m, and b includes an additive loop in a special case.

A sequence of SWAC computer routines has been written to search for all planes having a specific additive loop in an appropriate ternary ring. Using these routines, a complete search had been made previously using the elementary Abelian group for the additive loop [2]. Now a complete search has been made using the

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