A Note on Rational Approximation

By Robert W. Floyd

It is suggested by plausible reasoning and confirmed by experience that the error of an nth degree polynomial approximation, in the Chebyshev sense of least maximum error, to an analytic function, is roughly a multiple of the $n + 1$st Chebyshev polynomial, $T_{n+1}(x)$, on the interval of approximation. Therefore if the nth degree polynomial $f^*(x)$ is equal to the function, $f(x)$, on the roots of $T_{n+1}(x)$, we expect that $f^*(x)$ will be a satisfactory approach to a Chebyshev approximation of $f(x)$.

Because $f(x)$ is analytic, it may be represented with negligible error in the interval of approximation by a polynomial $p(x)$ of sufficiently high degree; e.g., a truncated Taylor's or Maclaurin's series. Applying the division algorithm for polynomials,

$$\begin{aligned} p(x) &= q_0(x)\cdot T_{n+1}(x) + r_0(x) \\ T_{n+1}(x) &= q_1(x)\cdot r_0(x) + r_1(x) \\ r_0(x) &= q_2(x)\cdot r_1(x) + r_2(x) \\ r_1(x) &= q_3(x)\cdot r_2(x) + r_3(x), \text{ etc.,} \end{aligned}$$

where the degrees of the r_i form a strictly decreasing sequence. From these equations we may write $r_i(x) = a_i(x)\cdot p(x) + b_i(x)\cdot T_{n+1}(x)$, where a_i and b_i are defined recursively by

$$\begin{aligned} a_i &= a_{i-2} - q_i\cdot a_{i-1}, \qquad a_{-1} = 0, \qquad a_{-2} = 1 \\ b_i &= b_{i-2} - q_i\cdot b_{i-1}, \qquad b_{-1} = 1, \qquad b_{-2} = 0. \end{aligned}$$

It may be proven that the sum of the degrees of $a_i(x)$ and $r_i(x)$ is at most n. The first set of equations may be written $p(x) = [r_i(x)/a_i(x)] - [b_i(x)/a_i(x)]\cdot T_{n+1}(x)$, so that $r_i(x)/a_i(x)$ is a rational approximation to $p(x)$, exact wherever $T_{n+1}(x)$ vanishes. Since $T_{n+1}(x) \leqq 1$ in the interval of approximation, $b_i(x)/a_i(x)$ provides a bound for the error of the approximation. If $b_i(x)/a_i(x)$ is nearly constant on the interval of approximation, the error oscillates between $n + 2$ extrema of nearly equal magnitude, and the method of approximation is justified, for Chebyshev approximation is characterized by an error which oscillates at least $n + 1$ times between positive and negative extrema of equal magnitude. For the particular case $i = 0$, $a_i = 1$, and $r_0(x)$ is a polynomial approximation to $f(x)$ of degree at most n.

For example; $f(x) = e^x = 1 + x + (x^2/2!) + (x^3/3!) + \cdots$;

$$\begin{aligned} p(x) = 1 + x + .5x^2 + .1666\,6667x^3 + .0416\,6667x^4 + .0083\,3333x^5 \\ + .0013\,8889x^6 + .0001\,9841x^7 + .0000\,2480x^8 + .0000\,0276x^9. \end{aligned}$$

For $-1 \leqq x \leqq 1$, $|\,p(x) - f(x)\,| \leqq 3.0 \times 10^{-7}$. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$. Then $q_0 = (317.5625 + 38.75x + 4.3125x^2) \times 10^{-8}$;

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$$r_0 = 1 + 1.0000\,2223x + .5000\,0271x^2 + .1664\,8913x^3 + .04164497x^4 + .00868659x^5 + .0014\,3229x^6;$$

$$|p(x) - r_0| = |q_0| \cdot |T_7(x)| \leqq 3.61 \times 10^{-6} \quad (-1 \leqq x \leqq 1).$$

Therefore $|f(x) - r_0| \leqq 3.91 \times 10^{-6}$ $(-1 \leqq x \leqq 1)$. Dividing $T_7(x)$ by r_0,

$q_1 = -270{,}998.81 + 44{,}683.688x.$

$$r_1 = 270{,}998.81 + 226{,}314.15x + 90{,}815{,}458x^2 + 22{,}832.391x^3 + 3{,}846.3890x^4 + 381.2048x^5.$$

$a_0 = 1;\ b_0 = -q_0$

$a_1 = -q_1;\ b_1 = 1 + q_1 q_0$

Therefore

$$p(x) = \frac{r_1}{a_1} - \frac{b_1}{a_1} T_7 = -\frac{r_1}{q_1} + \frac{1 + q_1 q_0}{q_1} T_7(x).$$

The second term on the right is

$$\frac{.1394\,0940 + .0368\,86598x + .0056\,281054x^2 + .0019\,269840x^3}{-270{,}998.81 + 44{,}683.688x} T_7(x)$$

whose absolute value is bounded by 8.121×10^{-7} for $-1 \leqq x \leqq 1$. Thus e^x may be approximated on this interval by

$$-\frac{r_1}{q_1} = \frac{1 + .8351\,1123x + .3351\,1386x^2 + .0842\,5274x^3 + .0141\,9338x^4 + .0014\,0667x^5}{1 - .1648\,8518x},$$

where the error is bounded by $\pm(3 \times 10^{-7} + 8.1 \times 10^{-7}) = \pm 1.1 \times 10^{-6}$.

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The Complete Factorization of $2^{132} + 1$

By K. R. Isemanger

The integer $2^{132} + 1$ is divisible by $2^{44} + 1 = 17 \cdot 353 \cdot 2931542417$ and the quotient, $2^{88} - 2^{44} + 1$, is divisible by $241 \cdot 7393$. There remains the formidable problem of factoring the resultant quotient N, where N is the integer

1 73700 82040 22350 83057.

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