On the Utility of Newton's Method for Computing Complex Roots of Equations

By I. M. Longman

1. Introduction. With the advent of modern high speed computers, methods of computation which were once regarded as being too laborious to be of practical use can very profitably be resuscitated. This applies particularly to iterative methods which are very simple to program for a machine, but which might be very tedious for a human operator to carry out.

For example it has been noted by Whittaker and Robinson [1] that Newton's method of numerical solution of equations is theoretically applicable to complex as well as real roots, but would be extremely laborious to apply.

Despite the simplicity of the method it does not appear to have had the application it deserves, other authors preferring more complicated methods which involve less cumbersome numerical work. An interesting method is described by Ward [2].

2. Examples. Consider the equation

$$y = f(x) \equiv x^2 + 1 = 0. \tag{1}$$

The roots of this equation are known, of course, to be $x = \pm i$, but it is instructive to apply Newton's method and obtain successive approximations to the solution.

We have

$$f'(x) = 2x, \tag{2}$$

and so our recursion formula is

$$x_{n+1} = (x_n - 1/x_n)/2. \tag{3}$$

Now it is clear that the real axis separates those points of the complex plane nearer to one root from those nearer to the other. Also if we start from a first approximation x_0 which is real, all the successive x_n's must be real, in accordance with equation (3). Thus starting with a real x_0 does not lead to a convergence of the iteration. However if we start with any x_0 in the upper half-plane, the sequence (3) converges to the root $x = +i$ and if we start with any x_0 in the lower half-plane the sequence converges to the root $x = -i$. That this is true may be seen from the following:

If we start with $x_0 = a + ib$,

$$x_1 = (a/2)[1 - 1/(a^2 + b^2)] + i(b/2)[1 + 1/(a^2 + b^2)], \tag{4}$$

so that the imaginary part remains positive if it is initially so, and likewise remains negative if it is initially so.

For example, starting with $x_0 = 1 + i$, we have the sequence

$$x_1 = 0.25 + 0.75i$$

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$$x_2 = -0.075 + 0.975i$$

$$x_3 = +0.001715686 + 0.99730392i \text{ etc.},$$

which is clearly converging rapidly to the root $+i$.

The second example is an equation which arose in the course of the author's work on the reflection of wave-pulses in elastic solids (to be published elsewhere), and while it was solved easily on an electronic computer (WEIZAC)* by Newton's method, it would seem difficult to solve it by other means. The equation is

$$(5) \qquad f(x) \equiv (x^2 + a^2)^{1/2} + \alpha(x^2 + b^2)^{1/2} - i\rho x = \tau,$$

where $b > a > 0$, $\alpha > 0$, $\rho > 0$, $\tau > 0$, and the x-plane is supposed cut along the imaginary axis between $\pm ib$ so as to make $f(x)$ single valued, and so that the radicals reduce to positive arithmetic square roots on the positive real axis. Then it can be shown [3] that if $\tau > \tau_0$ where

$$(6) \qquad \tau_0 = (a^2 - v_0^2)^{1/2} + \alpha(b^2 - v_0^2)^{1/2} + \rho v_0$$

and v_0 is the (unique) root of

$$(7) \qquad v(a^2 - v^2)^{-1/2} + \alpha v(b^2 - v^2)^{-1/2} = \rho \qquad (0 < v_0 < a)$$

then (5) has a unique (complex) root $x = X$. Furthermore X lies in the positive quadrant of the x-plane.

The problem was to compute X on an electronic computer for given values of the (real) parameters a, α, b, ρ, τ. By applying Newton's method a recursion formula was found in the usual way. This method was applied to the equation

$$(x^2 + 1)^{1/2} + (x^2 + \tfrac{1}{3})^{1/2} - ix = 2,$$

and starting from $x_0 = 1 + i$ the following iterates were obtained.

$$x_1 = .49278\,43132 + .44404\,59775i$$

$$x_2 = .37934\,25961 + .38501\,14705i$$

$$x_3 = .36899\,47789 + .38109\,03135i$$

$$x_4 = .36889\,46156 + .38106\,80636i$$

$$x_5 = .36889\,46067 + .38106\,80642i$$

$$x_6 = .36889\,46067 + .38106\,80642i$$

It is seen that the iteration converges extremely rapidly, and with many different values of the parameters the same rapid convergence was found.

For a complete discussion of the conditions for convergence of Newton's method, and uniqueness of the root, reference may be made to Householder [4].

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1. E. T. Whittaker & G. Robinson, *The Calculus of Observations*, Blackie & Son, Glasgow, 1944, p. 88.

* WEIZAC is the name of the electronic computer at the Weizmann Institute of Science.

2. J. A. WARD, "The downhill method of solving $f(z) = 0$," *Jn.*, Assn. for Comp. Machinery., v. 4, 1957, p. 148.
3. L. CAGNIARD, *Réflexion et Réfraction des Ondes Seismiques Progressives*, Gauthier-Villars, Paris, 1939, p. 55–58.
4. A. S. HOUSEHOLDER, *Principles of Numerical Analysis*, McGraw-Hill, New York, 1953, p. 118–121.

A Problem in Abelian Groups, with Application to the Transposition of a Matrix on an Electronic Computer

By Gordon Pall and Esther Seiden

1. Introduction. Mr. G. A. Westlund of "Mura" (Midwestern Universities Research Association) was asked to formulate a code to transpose a matrix stored in the memory of IBM 704, using very little additional space. It appeared that such a code, by Mr. William Shooman of General Electric in Evendale, was already available and had been distributed on June 15, 1957, to SHARE members of IBM 704. Mr. Westlund asked whether there exists a more efficient method for this purpose.

A matrix of m rows and n columns is stored in the computer with the elements of each column listed in order and followed by those of the next column. The positions can be numbered as $mj + i$ $(i = 0, 1, \cdots, m - 1; j = 0, 1, \cdots, n - 1)$. To transpose the matrix, the element in position $mj + i$ must be moved to position $ni + j$. The key remark is that the new position number is obtained from the old by multiplication by n, and reducing mod $N(= mn - 1)$. Starting with any element (which we will call a *leader*), we multiply its position number by n and replace mod N, repeat this operation with the new position number again and again, and thus have a cycle of elements which are permuted cyclically in the process of transposing a matrix. If v is the g.c.d. of any of the position numbers and N, then the number of elements in the cycle is equal to the least positive integer r for which $n^r \equiv 1 (\text{mod } N/v)$.

The questions then arise: (a) can a method be devised of choosing one and only one leader in every cycle; and (b) if this is done, will the new method of transposing a matrix compare favorably in machine time with the existing method? Both questions are answered affirmatively in this note.

A program for transposing a matrix, once a set of leaders is given, was constructed at our request by G. A. Westlund, under the direction of M. R. Storm, head of the computing section at Mura. The suggestion was made that our method for forming a set of leaders (see §2) might also be programmed by building a table of indices and primitive roots into the computer. We felt that it would be more economical to carry out the construction of a set of leaders by hand. The construction of a table of leaders for all pairs m, n with certain properties is under investigation.

In transposing a matrix by the Mura program, the machine time is much smaller

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