The Fourier Transform of $[c^{2k} + (x - x^{-1})^{2k}]^{-1}$ Arising from Study of Tuned Circuit Spectra

By Charles Walmsley and Arthur S. G. Grant

This Fourier transform is given in tables [1, 2] only in the case when $k = 1$ and $c = 2 \cos \theta$ with $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. The need for it in other cases has arisen in recent work concerned with the study of a random signal obtained by applying a white noise to the input of a band pass filter.

The power transfer function of the filter, and thus the power spectrum of the signal at the filter output, is considered to be of the form

$$(1) \qquad T(f) = c^{2k}/[c^{2k} + (f/f_0 - f_0/f)^{2k}].$$

It should be noted that $T(0) = T(\infty) = 0$, $T(f/f_0) = T(f_0/f)$ and $T(f_0) = 1$. The band-width to the half-power points is obtained by setting $f/f_0 - f_0/f = \pm c$ and solving for the lower and upper cut-off frequencies f_1 and f_2, obtaining

$$(2) \qquad f_1/f_0 = [(c^2 + 4)^{1/2} - c]/2 \quad \text{and} \quad f_2/f_0 = [(c^2 + 4)^{1/2} + c]/2.$$

Thus the band-width, $f_2 - f_1$, is equal to cf_0. The shape of the filter response is determined by k, which is assumed to be restricted to positive integral values. The plot of $T(f)$ vs. f on a logarithmic frequency scale is symmetrical about the center frequency f_0 and thus, by a proper choice of c and k, good approximations to practical filter response curves may be obtained. By the Wiener-Khintchine result [3], the auto-correlation function of the filtered noise is given by the Fourier transform of $T(f)$,

$$\phi(\tau) = \int_{-\infty}^{\infty} \frac{c^{2k} \exp(-2\pi j f \tau)\, df}{c^{2k} + (f/f_0 - f_0/f)^{2k}};$$

or, with simple reduction and change of notation,

$$(3) \qquad I = \int_{-\infty}^{\infty} \exp(iyx)[c^{2k} + (x - x^{-1})^{2k}]^{-1}\, dx$$

where $c > 0$, $y > 0$, and k is a positive integer.

Contour integration, using the familiar infinite upper semi-circle and Jordan's lemma, shows that the integral I is equal to $2i\pi$ times the sum of the residues of the integrand at the poles in the upper half-plane. The relevant poles are

$$x = \left[c \operatorname{Cis} \frac{2m - 1}{2k}\pi \pm r \operatorname{Cis} \frac{\theta}{2}\right] \Big/ 2, \qquad m = 1, 2, \cdots, k$$

and

$$r^2 \operatorname{Cis} \theta = c^2 \operatorname{Cis} \frac{2m - 1}{k}\pi + 4,$$

where $\operatorname{Cis} \phi (= \exp i\phi)$ denotes $\cos \phi + i \sin \phi$. Evaluation of the residues and algebraic reduction gives the general result:

$$(4) \qquad I = (\pi c^{1-2k}/2k) \sum_{m=1}^{k} M,$$

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where

$$M = \exp\alpha\left[\cos\beta\left(\sin\frac{2m-1}{2k}\pi - cr^{-1}\sin\omega\right) + \sin\beta\left(\cos\frac{2m-1}{2k}\pi + cr^{-1}\cos\omega\right)\right] + \exp\gamma\left[\cos\delta\left(\sin\frac{2m-1}{2k}\pi + cr^{-1}\sin\omega\right) + \sin\delta\left(\cos\frac{2m-1}{2k}\pi - cr^{-1}\cos\omega\right)\right], \tag{5}$$

where α, γ denote

$$\frac{y}{2}\left(-c\sin\frac{2m-1}{2k}\pi \mp r\sin\frac{\theta}{2}\right),$$

β, δ denote

$$\frac{y}{2}\left(c\cos\frac{2m-1}{2k}\pi \pm r\cos\frac{\theta}{2}\right),$$

and

$$\omega = \frac{\theta}{2} - \frac{2m-1}{k}\pi.$$

This is applicable in all cases except when $c = 2$ and k is odd. In this exceptional case the value of M corresponding to $m = \frac{1}{2}(k+1)$ must be replaced by $2(1-y)\exp(-y)$.

In the case when $k = 1$ the result is

$$\begin{aligned} I &= \pi\exp(-\tfrac{1}{2}yc)[c^{-1}\cos\epsilon - (4-c^2)^{-1/2}\sin\epsilon], \\ &\text{or } \pi\exp(-\tfrac{1}{2}yc)[c^{-1}\cosh\zeta - (c^2-4)^{-1/2}\sinh\zeta], \\ &\text{or } \tfrac{1}{2}\pi(1-y)\exp(-y) \end{aligned} \tag{6}$$

according as $0 < c < 2$, or $c > 2$, or $c = 2$; where

$$\epsilon = \tfrac{1}{2}y(4-c^2)^{1/2}, \qquad \zeta = \tfrac{1}{2}y(c^2-4)^{1/2}.$$

This result (6), with different notation, agrees with formula 21, §1.2 (p. 9), Vol. 1 of the Bateman Project Tables [1] in so far as that formula is applicable; but it includes also the case (when $c > 2$) not covered by that formula.

For $k = 2$ the result is

$$I = \pi\exp(-yc/2\sqrt{2})[A\exp\phi + B\exp(-\phi)]/(2\sqrt{2}c^3) \tag{7}$$

with $\phi = y(r^2-4)^{1/2}/2\sqrt{2}$, $r^4 = c^4 + 16$,

$A = [1 - cr^{-2}(r^2+4)^{1/2}]\cos(P-Q) - [1 - cr^{-2}(r^2-4)^{1/2}]\sin(P-Q)$,

$B = [1 + cr^{-2}(r^2+4)^{1/2}]\cos(P+Q) + [1 + cr^{-2}(r^2-4)^{1/2}]\sin(P+Q)$,

$P = y(r^2+4)^{1/2}/2\sqrt{2}$, $\qquad Q = yc/2\sqrt{2}$.

In the degenerate case when $y = 0$ the general result (4), (5), becomes

$$(8) \qquad \int_{-\infty}^{\infty} [c^{2k} + (x - x^{-1})^{2k}]^{-1} \, dx = \pi k^{-1} c^{1-2k} \csc (\pi/2k).$$

Special results for $k = 1, c = \frac{1}{2}$ and $k = 2, c = 1/\sqrt{2}$ are

$$(9) \qquad \int_0^{\infty} \frac{dx}{\frac{1}{4} + (x - x^{-1})^2} = \int_0^{\infty} \frac{dx}{\frac{1}{4} + (x - x^{-1})^4} = \pi,$$

which may be verified independently.

Dalhousie University, Halifax, N. S., Canada
Naval Research Establishment, Dartmouth, N. S., Canada

1. A. Erdelyi, et al., *Tables of Integral Transforms*, McGraw-Hill, New York, 1954.
2. F. Oberhettinger, *Tabellen zur Fourier Transformationen*, Springer-Verlag, Berlin, 1957.
3. A. Van der Ziel, *Noise*, Prentice-Hall, Inc., New York, 1954, p. 316.

On the Evaluation of Certain Complex Elliptic Integrals

By H. A. Lang and D. F. Stevens

1. Introduction. Elliptic integrals of the third kind are occasionally encountered in the form

$$a\Pi(\phi, \alpha^2, k) + \bar{a}\Pi(\phi, \bar{\alpha}^2, k)$$

or

$$a\Pi(\phi, \alpha^2, k) - \bar{a}\Pi(\phi, \bar{\alpha}^2, k)$$

where a, $\bar{a}$ and α^2, $\bar{\alpha}^2$ are complex conjugates and the modulus k is real such that $0 < k^2 < 1.0$. It is usually desirable to rewrite these integrals using only real coefficients and parameters. This paper gives an elementary procedure for the evaluation of these expressions, together with the correction of some existing formulas.

2. Modified Development. We follow the development suggested by Hoüel [1], but use the notation of Byrd and Friedman [2] wherever applicable. Since the modulus k is the same in all of the elliptic integrals considered here, it will be omitted throughout.

For convenience, set

$$(1) \qquad \begin{aligned} 2\Pi_1 &= \Pi(\phi, \alpha^2) + \Pi(\phi, \bar{\alpha}^2) = \int_0^{\phi} \frac{d\phi}{(1 - \alpha^2 \sin^2 \phi)\Delta} + \int_0^{\phi} \frac{d\phi}{(1 - \bar{\alpha}^2 \sin^2 \phi)\Delta} \\ 2i\Pi_2 &= \Pi(\phi, \alpha^2) - \Pi(\phi, \bar{\alpha}^2) = \int_0^{\phi} \frac{d\phi}{(1 - \alpha^2 \sin^2 \phi)\Delta} - \int_0^{\phi} \frac{d\phi}{(1 - \bar{\alpha}^2 \sin^2 \phi)\Delta} \end{aligned}$$

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