29[B].—GEORGE E. REYNOLDS, *A New Method of Cube Root Extraction on Desk Calculators*, Electronics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, USAF, April 1958, (ASTIA Doc. No. AD 133760), vi + 27 p., multilithed.

The method consists of the use of one iteration of a third-order iterative process for computing $\sqrt[3]{N}$, with the aid of a table from which certain functions of an approximate root A can be read. The iterative formula is used in the form

$$\frac{(2A)\left(\frac{N}{A^3} + \frac{1}{2}\right)}{\frac{N}{A^3} + 2},$$

both $2A$ and A^3 being read from a table. Thus only one multiplication and two divisions, plus some minor auxiliary operations, are necessary.

Three tables are provided, these being so-called five-place, six-place, and seven-place cube root tables. The author states that a maximum error of one unit in the last significant digit of the rounded answer can be achieved. A few spot checks have revealed no cases in which the statement is false. It is worth noting that the only instance discovered by the reviewer in which the result obtained from the tables differed from the correct result by one unit in the last digit occurred in the beginning of the five-place table for the range in which the tabulated value of $2A$ has 5S instead of 6S. This agrees with the author's statements on p. 11–12 concerning the location of the maximum error in the tabulated values.

The instructions as to procedure are sufficiently explicit except in respect to the number of guard figures to be kept in the quotient N/A^3. One may infer from the corrected version (furnished with the review copy as an erratum) of the example on p. 15 that the author believes one guard figure will suffice, although there is no consideration of this issue in the paper.

Equation (21) is incorrect. Its right side should read

$$(2A)(N/A^3 + \tfrac{1}{2}) \div (N/A^3 + 2).$$

The reviewer is allergic to the use (p. 9) of the colloquial phrase "several times greater than" in lieu of "several times as large as". There is no such thing (p. 13, line 3 *fb*) as *the* Newton-Raphson Method for the cube root, since there are many equivalent equations with the cube root of N as a solution. The reviewer is unfamiliar with the term (p. 14, line 8) "increasing asymptotically". Is "monotonically" the adverb intended?

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30[G].—JOHN L. SELFRIDGE, *On Finite Semigroups*, Dissertation, University of California, Los Angeles, multilithed typescript, iv + 185 pp.

A *system* is a set of abstract elements, together with a binary operation [in this review called multiplication] defined from the cartesian product $S \times S$ to a set V. For $a, b \in S$, the value of the product is written ab. The system is *closed* whenever $V \subset S$. The *order* of the system is the number of elements in S.

Two systems S and T with sets of values U and V are called *isomorphic* whenever there is a one-to-one function φ from $S \cup U$ to $T \cup V$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S$. They are *anti-isomorphic* whenever $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in S$. "Using isomorphism as an equivalence relation $\cdots$ [systems] are divided into *classes*, all $\cdots$ [systems] in a class being isomorphic *copies*. If a pair of classes are anti-isomorphic (i.e., they have anti-isomorphic representatives) they are called a *type*. The remaining classes are types which are anti-isomorphic to themselves" (p. 3).

If a system is closed and multiplication is associative, it is called a *semigroup*. A semigroup in which, for some p, all products of p elements are equal is called *nilpotent*. The classification and enumeration of general systems of finite order is an exceedingly difficult problem, unsolved even for finite groups. In 1954 the reviewer (with Dr. Selfridge's able assistance) wrote a code [1] which caused SWAC to punch multiplication tables for all semigroups of order 4. One of the main tasks was to arrange the 3492 tables by types, and to select one table to represent each type. In order to represent the tables compactly and deal with them arithmetically, the elements of the semigroup were given the names 0, 1, 2, 3, and the sixteen entries in a multiplication table were written row after row in a single line of 16 base 4 digits. It was convenient to select as the representative of a type of semigroup that one whose table comes first in the lexicographic order by rows. Such a multiplication table is called a *row-normal table* by Selfridge. Similarly, *column-normal table* is defined.

The above considerations were purely matters of machine convenience, devised for a limited purpose. In his dissertation Selfridge has made a systematic study of multiplication tables, both for semigroups and for more general systems, and through an analysis of tables and row-normal tables has obtained new information about the structure of the algebraic systems. This is a good illustration of the feedback from digital computation to pure mathematics. Perhaps the greatest benefit of automatic computation for any problem is the resulting increased theoretical comprehension of the problem.

In the dissertation definitions and elementary properties of multiplication tables occupy p. 1–7. For example, it is proved that there are at least $(n^2 + n)^q/(n!)(n^2!)$ types of commutative systems of order n, where $q = n(n + 1)/2$. On p. 8–14 the author enumerates all possible initial rows of a row-normal semi-group table, and proves that there is an example of a semigroup corresponding to each possibility. Certain last rows are also enumerated.

Pages 15–19 contain a classification and enumeration of all tables corresponding to nilpotent semigroups in which every product of three elements if 0. As a result, it is proved that there are exactly Σ_n nilpotent semigroups of order n in which every triple product is 0, where

$$\Sigma_n = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-i)^{(n-k)^2}$$

The author then tabulates Σ_n and certain auxiliary quantities for $n = 1(1)7$. It is startling to find that $\Sigma_7 = 5{,}944{,}080{,}072$, a low lower bound for the number of semigroups of order 7.

Pages 20–22 list the results of the enumeration of certain algebraic systems of orders 1, 2, 3, 4, and 5. The four properties of closure, associativity, commutativity, and having a multiplicative unit are considered separately, and the list gives the number of types, classes, and tables. Some of the numbers come from the earlier theory, some come from a list of semigroups of order 5, and some apparently come from other enumerations. We find, for example, that there are 183,732 semigroups of order 5, of which (by looking at Σ_5) we see that only 11,725 are of the types considered earlier. We find also that there are 720 types of closed commutatives systems of order 4 which have a unit.

After a three-page history of the use of high-speed computers to seek and enumerate various systems, the author devotes p. 26–33 to indicating how in 1955 he and Professor T. S. Motzkin found and classified all semigroups of order 5 on SWAC, using only 256 cells of rapid-access storage. To reduce the time, only normal tables were computed. There follows a two-page study of the feasibility of carrying on to order 6. Exclusive of punch time, on SWAC the principal search code took 3 minutes for order 4, 40 minutes for order 5, and might be expected to take about 25 hours for order 6. In all cases the editing and preparation of the tables for publication remains the major part of the work.

The greatest bulk of the dissertation is the appendix on pp. 41–185, where the semigroups of order 5 are tabulated in the form of multiplication tables. It is difficult (though not impossible) for the reader to determine just what is listed; the explanation (p. 28 and 29) refers to lists L_1 and L_2, which are not the lists in the appendix! It is to be hoped that Dr. Selfridge will add an explanation to any existing copies of this appendix.

Here is the reviewer's explanation of the appendix, based partly on personal copies of L_1 and L_2. There are 1915 distinct classes of these semigroups. Consider each multiplication table to be in row-normal form, and the resulting 1915 tables to be given *class numbers* [reviewer's term] from 0 to 1914 in lexicographic order by rows. Now 405 of the 1915 semigroups (including the 325 commutative semigroups) have the property that each is anti-isomorphic to itself, while the other 1510 semigroups form 755 anti-isomorphic pairs. The author omits the table in each anti-isomorphic pair that has the larger class number, and thus obtains an ordered set of 1160 row-normal tables, representing each of the 1160 distinct types of semigroups of order 5. These are given *type numbers* [reviewer's term] from 0 to 1159 in lexicographic order by rows.

The appendix lists each of these 1160 semigroups in the order of the type numbers. For each semigroup the following five items are given across the page, as listed by an IBM tabulator on multilith masters:

1) The type number.
2) The multiplication table in row-normal form (a 5-by-5 array of digits 0 through 4).
3) The class number.

That is all the information for the commutative semigroups. For the non-commutative semigroups, there are two more pieces of information:

4) The class number of the semigroup to which the listed semigroup is anti-isomorphic. (This is omitted if the class number is the same as item 3. That a noncommutative semigroup can be anti-isomorphic to itself had not been observed by all workers with semigroups.)

5) The column-normal isomorph of the same multiplication table (omitted if the row-normal table is itself column-normal). The semigroup is anti-isomorphic to itself if and only if item 4 is omitted and item 5 is included.

There is a bibliography of 34 titles, including work of a Japanese group including Professor Takayuki Tamura. It is notable that the Japanese group, working by hand, obtained all 126 semigroup types of order 4 prior to SWAC (without error), and that they finished obtaining the semigroups of order 5 almost simultaneously with the American group of Motzkin, Selfridge, and SWAC (but with at least one error discovered by the American group). It appears that in computing semigroups it has been a reasonably fair match between Japanese without an abacus and Americans armed with an electronic digital computer! The text of the dissertation reviewed here does not refer to the Japanese computation for order 5.

Minor criticisms: On page 8, line 8, for "... an element of a semigroup..." read "... an element *a* of a semigroup ... ". In the footnote on page 8, for *m* read *a*. In reference [A2] the authors' names are permuted. In this and other references the authors' first names have been carefully omitted [why is this done so often?].

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1. George E. Forsythe, "SWAC computes 126 distinct semigroups of order 4," Amer. Math. Soc., *Proc.*, v. 6, 1955, p. 443–447.

31[K].—E. L. Lehmann, *Testing Statistical Hypotheses*, John Wiley & Sons, Inc., New York, 1959, xiii + 369 p., 24 cm. Price $11.00.

This long-awaited book is a welcome addition to Wiley's fine series of texts in the different areas of modern mathematical statistics, and the author is to be thanked and congratulated for a difficult but needed job exceedingly well done. The title does not convey an adequate idea of the scope of the book, which includes material from many branches of statistics, nor of the exceedingly helpful devices used that make it possible to catch up with the most recent advances. Going through it is a refreshing and stimulating experience.

The book provides "a systematic account of the theory of hypothesis testing and of the closely related theory of estimation by confidence sets. The principal applications of these theories are given, including the one- and two-sample problems concerning normal, binomial and Poisson distributions. There is also a treatment of permutation tests and of some aspects of the analysis of variance and of regression analysis. Introductions to multivariate and sequential analysis, and to non-parametric tests are offered. Methods based on large sample considerations (χ^2 and likelihood ratio tests) are sketched. The emphasis throughout is on the various optimum properties of the procedures. These are discussed in terms of the Neyman-Pearson formulation, but against a background of decision theory which frequently permits a broader justification of the results."

The level of the treatment is set by the fact that "the natural framework for a systematic treatment of hypothesis testing is the theory of measure in abstract spaces." By unrestricted use of the abstract approach, the author is enabled to bring the prepared reader abreast of the very latest developments. For this, one should have an appreciation, if not a knowledge, of concepts in measure theory,