$\frac{1}{2\cdot 89}(n^4 + 1)$ for $n = 2747, 2771, 2885.$

$\frac{1}{2\cdot 97}(n^4 + 1)$ for $n = 2669, 2683, 2749.$

New factorizations are as follows:

$$938^4 + 1 = 809273\cdot 956569$$
$$1060^4 + 1 = 847577\cdot 1489513$$
$$1348^4 + 1 = 940169\cdot 3511993$$
$$1512^4 + 1 = 926617\cdot 5640361$$
$$1874^4 + 1 = 914561\cdot 13485457$$
$$2100^4 + 1 = 17\cdot 873553\cdot 1309601$$
$$2838^4 + 1 = 868841\cdot 74663657$$
$$2908^4 + 1 = 41\cdot 940369\cdot 1854793$$
$$\tfrac{1}{2}(1155^4 + 1) = 830233\cdot 1071761$$
$$\tfrac{1}{2}(1191^4 + 1) = 935353\cdot 1075577$$
$$\tfrac{1}{2}(1509^4 + 1) = 872369\cdot 2971849$$
$$\tfrac{1}{2}(2635^4 + 1) = 857569\cdot 28107577$$
$$\tfrac{1}{2}(2765^4 + 1) = 908353\cdot 32173321$$
$$\tfrac{1}{2}(2977^4 + 1) = 17\cdot 809041\cdot 2855393$$

The following factorization was omitted from my original table [1]:

$$\tfrac{1}{2}(2055^4 + 1) = 17\cdot 572233\cdot 916633.$$

The least integers still incompletely factored correspond to $n = 1038$ and 1229, for even and odd values of n, respectively.

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1. A. Gloden, "Table de factorisation des nombres $n^4 + 1$ dans l'intervalle $1000 < n < 3000$," Institut Grand-Ducal de Luxembourg, *Archives*, Tome XVI, Luxembourg, 1946, p. 71–88.

2. A. Gloden, *Table des Solutions de la Congruence $x^4 + 1 \equiv 0 \pmod p$ pour $800,000 < p < 1,000,000$*, published by the author, rue Jean Jaurès, 11, Luxembourg, 1959.

A Note on the Solution of Quartic Equations

By Herbert E. Salzer

For any quartic equation with real coefficients,

$$X^4 + AX^3 + BX^2 + CX + D = 0, \tag{1}$$

the following condensation of the customary algebraic solution is recommended as quickest and easiest for the computer to follow (no mental effort required). It works in every exceptional case.

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Denote the four roots of (1), by X_1, X_2, X_3, and X_4. With the aid of [1], solve the "resolvent cubic equation" $ax^3 + bx^2 + cx + d = 0$ *for the real root* x_1 *only*, where

$$(2) \qquad a = 1, \qquad b = -B, \qquad c = AC - 4D, \quad \text{and} \quad d = D(4B - A^2) - C^2.$$

Find

$$(3) \qquad m = +\sqrt{\tfrac{1}{4}A^2 - B + x_1}, \qquad n = \frac{Ax_1 - 2C}{4m}.$$

If $m = 0$, take $n = \sqrt{\tfrac{1}{4}x_1^2 - D}$ and proceed according to the following *Case I* or *Case II*, depending upon whether m is real or imaginary.

Case I: If m is real, let $(\tfrac{1}{2}A^2 - x_1 - B) = \alpha$, $4n - Am = \beta$, $\sqrt{\alpha + \beta} = \gamma$, $\sqrt{\alpha - \beta} = \delta$, and finally

$$(4\text{I}) \qquad X_1 = \frac{-\tfrac{1}{2}A + m + \gamma}{2}, \qquad X_2 = \frac{-\tfrac{1}{2}A - m + \delta}{2},$$
$$X_3 = \frac{-\tfrac{1}{2}A + m - \gamma}{2}, \quad \text{and} \quad X_4 = \frac{-\tfrac{1}{2}A - m - \delta}{2}.$$

Case II: If m is imaginary, say $m = im'$, then n is also imaginary, say $n = in'$. Let

$$(\tfrac{1}{2}A^2 - x_1 - B) = \alpha, \qquad 4n' - Am' = \beta, \qquad +\sqrt{\alpha^2 + \beta^2} = \rho,$$
$$\sqrt{\frac{\alpha + \rho}{2}} = \gamma, \qquad \frac{\beta}{2\gamma} = \delta,$$

and finally

$$(4\text{II}) \qquad \begin{cases} X_1 = \dfrac{-\tfrac{1}{2}A + \gamma + i(m' + \delta)}{2}, \\ \qquad\qquad X_2 = \bar{X}_1, \text{ the complex conjugate of } X_1, \\ X_3 = \dfrac{-\tfrac{1}{2}A - \gamma + i(m' - \delta)}{2} \\ \qquad\qquad \text{and} \quad X_4 = \bar{X}_3, \text{ the complex conjugate of } X_3. \end{cases}$$

If $\gamma = 0$, we must have $\alpha = -\alpha'$, $\alpha' \geqq 0$, and formula (4II) still holds provided that in it we replace δ by $+\sqrt{\alpha'}$.

As an example consider the quartic equation $X^4 + X^3 + X^2 + X + 1 = 0$, where $A = B = C = D = 1$, so that from (2) the resolvent cubic equation is $x^3 - x^2 - 3x + 2 = 0$. From [1] we find $x_1 = 0.61803\ 400$. From (3), $m = +\sqrt{-0.13196\ 600} = +0.36327\ 125i$, so that $m' = +0.36327\ 125$. Then $n = \dfrac{-1.38196\ 600}{1.45308\ 500i} = +0.95105\ 655i$, so that $n' = +0.95105\ 655$. Proceeding according to *Case II*, $\alpha = -1.11803\ 400$, $\beta = 3.44095\ 495$, $\rho = 3.61803\ 41$, $\gamma = 1.11803\ 40$ and $\delta = 1.53884\ 18$. Then from (4II) we obtain $X_1 = 0.30901\ 70 + 0.95105\ 65i$, $X_2 = \bar{X}_1 = 0.30901\ 70 - 0.95105\ 65i$, $X_3 = -0.80901\ 70 - 0.58778\ 53i$ and

$X_4 = \bar{X}_3 = -0.80901\ 70 + 0.58778\ 53i$. These roots may be verified as correct, since they are known to be the complex fifth roots of unity, namely $X_1 = \cos 72° + i \sin 72°$, $X_2 = \cos 288° + i \sin 288°$, $X_3 = \cos 216° + i \sin 216°$, and $X_4 = \cos 144° + i \sin 144°$.

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1. H. E. SALZER, C. H. RICHARDS & I. ARSHAM, *Table for the Solution of Cubic Equations*, McGraw-Hill, New York, 1958.

A Conjugate Factor Method for the Solution of a Cubic

By D. A. Magula

1. Introduction. This paper gives a simple method for computing the real roots of the reduced cubic equation with real coefficients,

$$x^3 + Ax + B = 0, \tag{1}$$

having roots a, b, c. We assume a to be real, since every cubic equation has at least one real root.

The method consists in factoring B, and setting one factor equal to $\pm\sqrt{m}$, the other n. For all pairs m, n such that $m + n = -A$, $\pm\sqrt{m}$ is a root. If no such pair exists, a method of interpolation is shown.

2. Proof of Method. The reduced cubic equation (1) can be transformed, by using the relations between the roots and coefficients, into a complete cubic,

$$p^3 + 6Ap^2 + 9A^2p + 4A^3 + 27B^2 = 0, \tag{2}$$

where

$$p = (-3a^2 - 4A). \tag{3}$$

Equation (2) can be written in the form:

$$(p + A)^2(-p - 4A) = 27B^2 \tag{4}$$

or

$$\frac{(p + A)}{3}\sqrt{\frac{(-p - 4A)}{3}} = \pm B. \tag{5}$$

Let

$$m = \frac{-p - 4A}{3} \quad \text{and} \quad n = \frac{p + A}{3} \tag{6}$$

$$n\sqrt{m} = \pm B \tag{7}$$

and

$$m + n = -A. \tag{8}$$

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