On Liouville's Function

By R. Sherman Lehman

1. Introduction. Liouville's function $\lambda(n)$ is defined by the equation $\lambda(n) = (-1)^r$ where r is the number of prime factors of n, multiple factors being counted according to their multiplicity. Pólya [6] conjectured that the function

$$L(x) = \sum_{n \leqq x} \lambda(n)$$

is negative or zero for all $x \geqq 2$, and in fact this is true within the range where this function has been previously calculated. Calculations connected with the present study show that $L(x) \leqq 0$ for $2 \leqq x \leqq 10^6$. Nevertheless Haselgrove [3] has shown that the Pólya conjecture is false and that there are infinitely many integers x for which $L(x) > 0$. However, his method does not furnish explicitly an x for which the conjecture fails; and in fact it does not give an upper bound for the first counterexample. In the present paper we shall describe calculations which lead to the result that $L(906{,}180{,}359) = +1$. We have not found a smaller value of x for which the conjecture fails, but also we have not proved that this is the smallest x greater than 2 for which $L(x)$ is positive.

2. Background and Heuristic Considerations. Liouville's function is connected with the Riemann zeta function by the equation

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

Let the zeros of $\zeta(s)$ on the line Re $s = \frac{1}{2}$ be $\rho_n = \frac{1}{2} + i\gamma_n$ $(n = \pm 1, \pm 2, \cdots)$ with $\gamma_{-n} = -\gamma_n$ and let $\gamma_0 = 0$. If it is assumed that these zeros are all simple, then the function $\zeta(2s)/(s\zeta(s))$ has simple poles at $\frac{1}{2} + i\gamma_n$ for $n = 0, \pm 1, \pm 2, \cdots$ with residues

$$\alpha_0 = \frac{1}{\zeta(\frac{1}{2})}, \qquad \alpha_n = \frac{\zeta(2\rho_n)}{\rho_n \zeta'(\rho_n)} \qquad (n = \pm 1, \pm 2, \pm 3, \cdots).$$

Fawaz [1] has obtained an explicit formula for $L(x)$ which is valid if the Riemann hypothesis holds and the zeros of $\zeta(s)$ are simple. Under these assumptions he showed that there is a sequence of numbers T_k, with $k < T_k < k + 1$, for which

$$L(x) = \lim_{k \to \infty} \sum_{|\gamma_n| \leqq T_k} \alpha_n x^{\frac{1}{2} + i\gamma_n} + O(1) \tag{1}$$

for $x > 0$.

Let

$$A_T(u) = \sum_{|\gamma_n| \leqq T} \alpha_n e^{i\gamma_n u}. \tag{2}$$

Fawaz's result suggests that one study numerically the behavior of $A_T(u)$ for different values of T. Since $A_T(u)$ should be an approximation to $e^{-\frac{1}{2}u}L(e^u)$, one might expect $L(e^u)$ to be positive for a u for which $A_T(u) > 0$.

Received April 22, 1960

In §5 we shall show that if the Riemann hypothesis holds, if the zeros of $\zeta(s)$ are simple, and if a conjectured estimate for $1/\zeta'(\rho_n)$ holds, then

$$(3)\qquad \lim_{\omega\to\infty}\left\{\int_{\frac{1}{2}\omega}^{\frac{3}{2}\omega} K_T(u-\omega)e^{-\frac{1}{2}u}L(e^u)\,du - \sum_{|\gamma_n|\leqq T}\alpha_n e^{i\gamma_n\omega}\right\} = 0,$$

where

$$K_T(t) = \frac{\sin Tt}{\pi t}.$$

This suggests that the sum $A_T(u)$ will represent a smoothing of the function $e^{-\frac{1}{2}u}L(e^u)$ which effectively operates over an interval of width about $2\pi/T$. However, since the kernel $K_T(t)$ is not always positive, the sum does not represent a true smoothing of $e^{-\frac{1}{2}u}L(e^u)$. Thus, if we find a maximum of the sum $A_T(u)$ we cannot always expect that there will be a value of $e^{-\frac{1}{2}u}L(e^u)$ in the vicinity which is as high. Instead it is possible for a rapid falling off of $e^{-\frac{1}{2}u}L(e^u)$ at a distance away of from π/T to $2\pi/T$ to be translated into an extra high peak of $A_T(u)$. On the other hand, high values of $A_T(u)$ for several different choices of T will make such behavior appear less likely.

Haselgrove's disproof of the Pólya conjecture was based on a numerical study of the similar sum

$$A_T^*(u) = \sum_{|\gamma_n|\leqq T}\alpha_n\left(1-\frac{|\gamma_n|}{T}\right)e^{i\gamma_n u}.$$

Ingham [5] had shown that for any u_0

$$A_T^*(u_0) \leqq \limsup_{u\to\infty} e^{-\frac{1}{2}u}L(e^u).$$

Hence to disprove the Pólya conjecture it was sufficient to find a T and u_0 for which $A_T^*(u_0) > 0$. Haselgrove found that $A_{1000}^*(831.847) = 0.00495$.

Using an IBM 701 at the University of California, Berkeley, we have independently computed approximations to the zeros of $\zeta(s)$ and the residues α_n and have confirmed Haselgrove's result. We have also obtained a smaller value of u_0 for which $A_{1000}^*(u_0)$ is positive. Our values to 4 decimal places are

$$A_{1000}^*(831.847) = 0.0050, \qquad A_{1000}(831.847) = 0.0526$$

$$A_{1000}^*(814.492) = 0.0782, \qquad A_{1000}(814.492) = 0.1102.$$

As a result of a search for smaller values of u for which $A_{1000}(u) > 0$ we found

$$A_{1000}^*(79.28) = -0.0418, \qquad A_{1000}(79.28) = 0.0075.$$

The number $e^{79.28}$ is still a very large number, and there does not seem to be any more hope of calculating $L(x)$ for $x = e^{79.28}$ than for $x = e^{831.847}$. On the other hand, it is possible to find a method for calculating $L(x)$ at isolated values which is quite feasible with present-day computers for x as large as 10^9. Therefore, we computed $A_T(u)$ for $u = 12.5(0.01)20.69$ with $T = 100$ and $T = 200$, covering approximately the range $2.7\cdot10^5 < x < 9.7\cdot10^8$. The vicinities of several high points were then selected for further study. The most promising of these was near

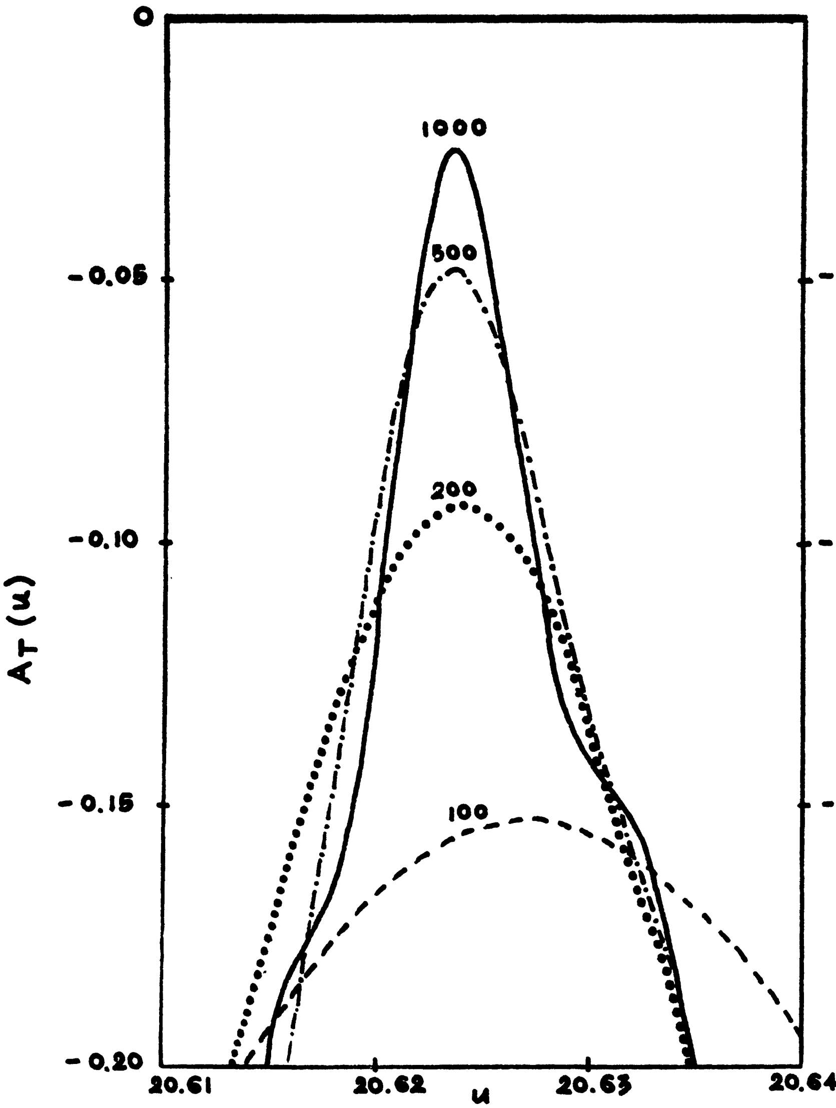

FIG. 1.—The functions $A_T(u)$ for $T = 100, 200, 500, 1000$. The maximum of $A_{1000}(u)$ is near $u = 20.623 \approx \log(9.05 \cdot 10^8)$.

$u = 20.62$. Figure 1 shows a graph of $A_T(u)$ for u in this vicinity with four different values of T. Although none of the functions $A_T(u)$ graphed is positive there, the strong upward trend suggested the conjecture that $L(x)$ is positive for some x near $9.05 \cdot 10^8$.

3. A Formula for Calculating $L(x)$. A direct calculation of $L(x)$ for all $x \leqq 10^9$ by factoring each number would require far too much machine time to be feasible on the IBM 704—at least 1000 hours. Fortunately it is possible to devise an appropriate method for calculating $L(x)$ at isolated values.

We begin with the known formula

$$\sum_{n \leqq x} L\left(\frac{x}{n}\right) = [\sqrt{x}] \tag{4}$$

where by $[y]$ we mean the greatest integer less than or equal to y. This formula is not suitable for calculating $L(x)$ for two reasons. First, there are $[x]$ terms in it; and second, to calculate $L(x)$ requires knowledge of values of the function L at large arguments such as $x/2$, $x/3$, etc. The formula can, however, be modified to get around both of these difficulties.

THEOREM 1. *Let k, l, and m be variables ranging over the positive integers. Let $\mu(n)$ be the Möbius function. Let v, w, and x be positive real numbers with $v < w < x$. Then*

$$L(x) = \sum_{m \leqq \frac{x}{w}} \mu(m) \left\{ \left[\sqrt{\frac{x}{m}}\right] - \sum_{k < v} \lambda(k) \left(\left[\frac{x}{km}\right] - \left[\frac{x}{mv}\right] \right) \right\} - \sum_{\frac{x}{w} < l \leqq \frac{x}{v}} L\left(\frac{x}{l}\right) \sum_{\substack{m \mid l \\ m \leqq \frac{x}{w}}} \mu(m).$$

Proof: Replacing x by x/m in (4) and breaking the sum into three parts, we obtain

$$\left[\sqrt{\frac{x}{m}}\right] = \sum_{n \leqq \frac{x}{mw}} L\left(\frac{x}{mn}\right) + \sum_{\frac{x}{mw} < n \leqq \frac{x}{mv}} L\left(\frac{x}{mn}\right) + \sum_{\frac{x}{mv} < n \leqq \frac{x}{m}} L\left(\frac{x}{mn}\right).$$

We multiply each of these sums by $\mu(m)$ and sum for $m \leqq x/w$, obtaining

$$\sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{n \leqq \frac{x}{mw}} L\left(\frac{x}{mn}\right) = \sum_{l \leqq \frac{x}{w}} L\left(\frac{x}{l}\right) \sum_{m \mid l} \mu(m) = L(x),$$

$$\sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{\frac{x}{mw} < n \leqq \frac{x}{mv}} L\left(\frac{x}{mn}\right) = \sum_{\frac{x}{w} < l \leqq \frac{x}{v}} L\left(\frac{x}{l}\right) \sum_{\substack{m \mid l \\ m \leqq \frac{x}{w}}} \mu(m),$$

$$\begin{aligned} \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{\frac{x}{mv} < n \leqq \frac{x}{m}} L\left(\frac{x}{mn}\right) &= \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{\frac{x}{mv} < n \leqq \frac{x}{m}} \sum_{k \leqq \frac{x}{mn}} \lambda(k) \\ &= \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{k < v} \lambda(k) \sum_{\frac{x}{mv} < n \leqq \frac{x}{km}} 1 \\ &= \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{k < v} \lambda(k) \left(\left[\frac{x}{km}\right] - \left[\frac{x}{mv}\right] \right). \end{aligned}$$

Rearrangement of terms then yields the theorem.

We observe that if $v \approx x^{1/3}$ and $w \approx x^{2/3}$ then the number of operations is proportional to $x^{2/3}$ if one has available a table of $\lambda(n)$ for $n \leqq w$, a table of

$$\xi(l) = \sum_{\substack{m \mid l \\ m \leqq \frac{x}{w}}} \mu(m) \tag{5}$$

for $l \leqq x/v$, and a table of $\mu(m)$ for $m \leqq x/w$.

In practice it turns out to be more efficient to use a somewhat more complicated formula. Let k range over positive integers, and let l', m', and n' range over positive *odd* integers. From (4) we obtain

$$\sum_{n' \leqq x} L\left(\frac{x}{n'}\right) = [\sqrt{x}] - \left[\sqrt{\frac{x}{2}}\right].$$

If x is replaced by x/m' and the sum is treated as in the proof of Theorem 1, we obtain

$$\begin{aligned} L(x) = &\sum_{m' \leqq \frac{x}{w}} \mu(m') \\ &\cdot \left\{\left[\sqrt{\frac{x}{m'}}\right] - \left[\sqrt{\frac{x}{2m'}}\right] + \left[\frac{x}{2m'v} - \frac{1}{2}\right] \sum_{k<v} \lambda(k) - \sum_{k<v} \lambda(k) \left[\frac{x}{2km'} - \frac{1}{2}\right]\right\} \\ &- \sum_{\frac{x}{w} < l' \leqq \frac{x}{v}} L\left(\frac{x}{l'}\right) \sum_{\substack{m \mid l' \\ m' \leqq \frac{x}{w}}} \mu(m'). \end{aligned} \tag{6}$$

We remark that similar formulas can be obtained for

$$M(x) = \sum_{n \leqq x} \mu(n)$$

by modifying the formula

$$\sum_{n \leqq x} M\left(\frac{x}{n}\right) = 1 \qquad \text{for } x \geqq 1.$$

4. Numerical Computations. The computations described in this section were performed on an IBM 704 at the University of California, Berkeley.

Formula (6) contains two parameters v and w which can be chosen to minimize computation time. We chose to fix $x/w = 1000$ and take $v = (10^{-3}x)^{1/2}$ in order to make the program suitable for values of x near 10^9. Two preliminary programs were then used to compute tables of $\lambda(n)$ for $n \leqq 10^6$ and $\xi(l)$, defined by (5), for odd $l < 10^6$.

In 1955 using the ORDVAC at Aberdeen Proving Ground, W. G. Spohn and the author computed Liouville's function for $n \leqq 802{,}000$ and verified that the Pólya conjecture holds up to this limit. In the present computation, the same method was used to obtain a table of $\lambda(n)$ for $n \leqq 10^6$. If one is given a table of $\lambda(n)$ for $n \leqq N/2$, then the following sieving process will allow the extension of the table to N. One begins by entering -1 as the value for each integer from $N/2$ to N. One then considers in turn each of the primes $p \leqq \sqrt{N}$, and one runs through the multiples of p among the integers from $N/2$ to N. If n is such a multiple, one sets $\lambda(n) = -\lambda(n/p)$ after erasing the value already recorded for n. When this is done for all multiples of primes $\leqq \sqrt{N}$, the table is complete. For a machine with enough storage space to hold the table of $\lambda(n)$ for $n \leqq N/2$, this method is much more efficient than factoring each integer in succession.

The table of $\lambda(n)$ was placed on magnetic tape with each value taking up one bit. This table was then summed to compute $L(x)$ for $x = 100(100)1{,}000{,}000$. The values for $x = 1000(1000)802{,}000$ were compared with the ORDVAC computation and all were found to agree. Also a comparison for $x = 100{,}000(100{,}000)600{,}000$ was made with a computation of D. H. Lehmer. Finally as a further check, the formula (6) was later used to compute $L(x)$ for $x = 200{,}000(200{,}000)1{,}000{,}000$. The values found agreed with those obtained by summing the table of $\lambda(n)$. (The circularity here is only apparent; in these cases the formula (6) makes no use of the table for $\lambda(n)$ beyond $n = 1000$). To compute the table for $\lambda(n)$ and sum it to obtain $L(x)$ required approximately 30 minutes of machine time.

A table of the function $\xi(l)$ was also needed for odd $l < 10^6$. By a combinatorial argument, which is easy but requires consideration of a number of cases, one can show that if $l < 10^6$, then $-7 \leqq \xi(l) \leqq 7$. Hence each value requires just 4 bits of storage. Again the table was put on magnetic tape. The method for computation was quite straightforward. Each odd square-free number $m < 1000$ was considered, and for each of its multiples $\mu(m)$ was added to a corresponding accumulator. A check of the accuracy of the computation was made by using the formula

$$\sum_{l' \leqq x}^{!} \xi(l') = \sum_{m' < 10^3} \mu(m') \left[\frac{x}{2m'} + \frac{1}{2} \right]$$

with $x = 10^6$ and l' and m' running over positive odd integers. The program for $\xi(l)$ required approximately 20 minutes.

Next a program for computing $L(x)$ by the formula (6) was constructed. The odd square-free numbers $m' < 1000$ together with the values $\mu(m')$ were stored as constants. Newton's method was used to compute $[\sqrt{y}]$ with especial care taken to avoid error due to round-off when $\sqrt{y}$ is an integer. The tables of $\lambda(n)$ and $\xi(l)$ on magnetic tapes were used as inputs, and $L(x/l')$ was obtained by summing the table of $\lambda(n)$. To obtain $L(x)$ for a value of x near $9 \cdot 10^8$ required approximately 16 minutes.

Table 1 contains values of $L(x)$ computed in connection with the search for a positive value. The order of computation is indicated in the last column. The values of x were chosen partly by guess and partly by heuristic considerations based on (3). There seem to be two separate peaks which for the functions $A_T(u)$ were smoothed into a single maximum. After we found a positive value on the seventh trial, it did not seem worthwhile to pursue an investigation of the other peak. Hence we do not know whether the maximum for the other peak is also positive.

TABLE 1

x	$L(x)$	Order of computation
903 000 000	−952	3
904 000 000	−1144	2
905 000 000	−1902	1
906 000 000	−584	4
906 170 000	−230	10
906 200 000	90	8
906 300 000	648	9
906 400 000	708	7
906 470 000	226	11
906 500 000	−120	5
907 000 000	−1920	6

Some other values of $L(x)$ computed by means of the main program are listed in Table 2. The values for $x \leqq 10^6$ are all confirmed by agreement with those obtained by directly summing the table of $\lambda(n)$.

The values of $L(x)$ for $x = 10^6$ and $4 \cdot 10^6$ are confirmed by results of a hand computation of D. H. Lehmer using the formula

$$L(x) = \sum_{k \leqq g} M\left(\frac{x}{k^2}\right) + \sum_{l \leqq \frac{x}{g^2}} \mu(l) \left[\sqrt{\frac{x}{l}}\right] - M\left(\frac{x}{g^2}\right)\left[\sqrt{\frac{x}{g^2}}\right]$$

with g chosen to minimize computation. In this computation the values for $M(x/k^2)$ were taken from a corrected version of von Sterneck's tables of $M(x)$ (see [7]) with the following values differing from those given by von Sterneck: $M(444{,}444) = -37$, $M(10^6) = 212$, $M(4 \cdot 10^6) = 192$. The first two of these values were obtained by factoring all numbers $\leqq 10^6$. The value for $4 \cdot 10^6$ was obtained by making small adjustments of von Sterneck's computation which are required because of errors in the earlier tables.

After finding positive values of $L(x)$ we next took up the question of determining zeros of $L(x)$. The results given in Table 1 indicated that such zeros must occur between 906,170,000 and 906,200,000 and between 906,470,000 and 906,500,000. Consequently a program was constructed to factor all numbers in these ranges. This program required approximately one minute for each 1000 numbers factored. As a byproduct of this computation we obtained a further check of the program for computing $L(x)$ at isolated values.

One of the results of this computation was a listing of all zeros of $L(x)$ in the intervals from 906,170,000 to 906,200,000 and from 906,470,000 to 906,500,000. In all, 167 zeros of the function $L(x)$ were found in these intervals. We list the first 16 occurring in the first interval:

906180358, 906180362, 906180364, 906180366, 906180370,
906180374, 906180376, 906180386, 906180388, 906180390,
906180518, 906180520, 906180524, 906180534, 906180536,
906180554.

There are 34 zeros from 906,192,698 to 906,193,478 inclusive; 22 zeros from 906,-194,914 to 906,195,298; 19 zeros from 906,195,986 to 906,196,098; 15 zeros from 906,477,702 to 906,477,936; 43 zeros from 906,486,640 to 906,487,288; and 18 zeros from 906,487,932 to 906,488,080.

TABLE 2

x	$L(x)$	x	$L(x)$
200 000	−294	10 400 000	−394
400 000	−460	10 410 000	−330
600 000	−802	10 420 000	−384
800 000	−600	10 430 000	−300
1 000 000	−530	10 440 000	−292
4 000 000	−1098	10 450 000	−522
		10 460 000	−588
		453 200 000	−27088

The first value of x greater than 906,170,000 for which $L(x)$ is positive was found to be 906,180,359. We, of course, are not able to say whether this is the smallest x greater than 2 for which $L(x)$ is positive. To decide this question without the use of essentially new ideas might very well require an enormous amount of computation.

5. Derivation of an Explicit Formula. In this section we give a derivation of equation (3), which was used heuristically in finding where $L(x)$ is positive; this derivation proceeds from several unproved assumptions. We shall assume that the Riemann hypothesis holds and that the zeros $\rho_n = \frac{1}{2} + i\gamma_n$ ($n = \pm 1, \pm 2, \cdots$) of $\zeta(s)$ are all simple. In addition we shall assume that there is a real number $\nu < 1$ such that

$$\frac{1}{\zeta'(\rho_n)} = O(\rho_n^{\nu}) \qquad (n = \pm 1, \pm 2, \cdots). \tag{7}$$

Numerical evidence makes this conjectured estimate appear quite plausible. The twelve largest values of $| 1/\zeta'(\rho_n) |$ for $0 < \gamma_n < 1000$ are listed in Table 3.

As in §2 let $K_T(t) = (\sin Tt)/(\pi t)$.

Lemma 1. *If $R > 0$ and $T > 0$ and γ is a real number, then*

$$\int_{-R}^{R} K_T(t) e^{i\gamma t}\, dt = \begin{cases} 1 + O\left(\dfrac{1}{R(T - |\gamma|)}\right) & \text{if } |\gamma| < T, \\[2ex] O\left(\dfrac{1}{R(|\gamma| - T)}\right) & \text{if } |\gamma| > T. \end{cases}$$

Proof: We have

$$\begin{aligned} \int_{-R}^{R} K_T(t) e^{i\gamma t}\, dt &= \frac{2}{\pi} \int_0^R \frac{\sin Tt}{t} \cos \gamma t\, dt \\ &= \frac{1}{\pi} \int_0^R \frac{\sin (T + \gamma)t}{t}\, dt + \frac{1}{\pi} \int_0^R \frac{\sin (T - \gamma)t}{t}\, dt \\ &= \frac{1}{\pi} \operatorname{Si}(R(T + \gamma)) + \frac{1}{\pi} \operatorname{Si}(R(T - \gamma)), \end{aligned}$$

where

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt = \begin{cases} \dfrac{\pi}{2} + O\left(\dfrac{1}{x}\right) & \text{for } x > 0, \\[2ex] -\dfrac{\pi}{2} + O\left(\dfrac{1}{x}\right) & \text{for } x < 0. \end{cases}$$

The conclusion of the lemma follows immediately.

We shall also use the following estimate for $R > 0$ and $T > 0$:

$$\begin{aligned} \int_{-R}^{R} | K_T(t) |\, dt &= \frac{2}{\pi} \int_0^{RT} \frac{|\sin t|}{t}\, dt < \int_0^1 dt + \int_1^{RT} \frac{dt}{t} \\ &= 1 + \log (RT). \end{aligned} \tag{8}$$

TABLE 3

n	γ_n	$\lvert 1/\zeta'(\frac{1}{2}+i\gamma_n)\rvert$
1	14.134725	1.2608
2	21.022040	0.8796
72	185.598784	0.8109
135	294.965370	0.8029
298	540.213166	0.8357
299	540.631390	0.8892
363	630.473887	0.8334
364	630.805781	0.9106
436	728.405482	0.8371
437	728.758750	0.8491
606	946.765842	0.9744
607	947.079183	0.9914

We are now ready to derive (3). We shall assume throughout that $R > 0$, $T > 0$, and $T \neq \gamma_n$ for $n = 1, 2, 3, \cdots$. The remainder of the notation is explained in §2.

Fawaz [2, p. 284] has shown (assuming the Riemann hypothesis and the simplicity of the zeros) that if u is restricted to a finite interval, then there is a positive constant C independent of k and u such that

$$\Big| \sum_{|\gamma_n| \leqq T_k} \alpha_n e^{i\gamma_n u} \Big| < C.$$

Also $| K_T(u - \omega) | \leqq T/\pi$. Consequently we can apply the Lebesgue bounded convergence theorem to obtain

$$\lim_{k\to\infty} \int_{\omega-R}^{\omega+R} \Big(K_T(u-\omega) \sum_{|\gamma_n|\leqq T_k} \alpha_n e^{i\gamma_n u}\Big)\, du$$
$$= \int_{\omega-R}^{\omega+R} K_T(u-\omega) \Big(\lim_{k\to\infty} \sum_{|\gamma_n|\leqq T_k} \alpha_n e^{i\gamma_n u}\Big)\, du.$$

Equation (1) can be written in the form

$$e^{-\frac{1}{2}u} L(e^u) = \lim_{k\to\infty} \sum_{|\gamma_n|\leqq T_k} \alpha_n e^{i\gamma_n u} + O(e^{-\frac{1}{2}u}).$$

Hence we obtain

$$\begin{aligned}\int_{\omega-R}^{\omega+R} K_T(u-\omega) e^{-\frac{1}{2}u} L(e^u)\, du &= \lim_{k\to\infty} \sum_{|\gamma_n|\leqq T_k} \Big(\alpha_n e^{i\gamma_n\omega} \int_{-R}^{R} K_T(t) e^{i\gamma_n t}\, dt\Big) \\ &\quad + \int_{\omega-R}^{\omega+R} K_T(u-\omega)\, O(e^{-\frac{1}{2}u})\, du.\end{aligned} \tag{9}$$

If $T_k \geqq 2T$, then by Lemma 1 we have

$$\begin{aligned}\sum_{|\gamma_n|\leqq T_k} \alpha_n e^{i\gamma_n\omega} \int_{-R}^{R} K_T(t) e^{i\gamma_n t}\, dt &= \sum_{|\gamma_n|\leqq T} \alpha_n e^{i\gamma_n\omega} \\ &\quad + O\Big(\sum_{0\leqq\gamma_n\leqq 2T} \frac{|\alpha_n|}{R\,|T-|\gamma_n||}\Big) + O\Big(\sum_{2T<\gamma_n\leqq T_k} \frac{|\alpha_n|}{R(|\gamma_n|/2)}\Big).\end{aligned} \tag{10}$$

Since $\zeta(1+it) = O(\log t)$ for $t > 2$, we obtain the following estimate from the assumption (7):

$$\frac{|\alpha_n|}{|\gamma_n|} = \frac{|\zeta(2\rho_n)|}{|\gamma_n\rho_n\zeta'(\rho_n)|} = O(\gamma_n^{\nu-2}\log\gamma_n)$$

for $n = 1, 2, 3, \cdots$. Since the series $\sum_{\gamma_n>0} \gamma_n^{-\beta}$ converges for $\beta > 1$, (see [4, p. 57]), it follows that the series

$$\sum_{\gamma_n>0} \frac{|\alpha_n|}{\gamma_n}$$

converges; hence the last term on the right-hand side of (10) is $O(1/R)$ uniformly in k. For *fixed* T the second term on the right-hand side of (10) is also $O(1/R)$ since it is a finite series.

Estimating the last term on the right-hand side of (9), we obtain

$$\int_{\omega-R}^{\omega+R} K_T(u-\omega)\, O(e^{-\frac{1}{2}u})\, du = O(e^{-\frac{1}{2}(\omega-R)}) \int_{-R}^{R} |K_T(t)|\, dt$$
$$= O(e^{-\frac{1}{2}(\omega-R)}\log(RT))$$

for $R > 2/T$.

Combining (9) and (10) and letting $k \to \infty$, we conclude that for fixed T,

$$\int_{\omega-R}^{\omega+R} K_T(u-\omega)e^{-\frac{1}{2}u}L(e^u)\, du = \sum_{|\gamma_n|\leqq T} \alpha_n e^{i\gamma_n\omega} + O\left(\frac{1}{R}\right) + O(e^{-\frac{1}{2}(\omega-R)}\log(RT)),$$

provided $R > 2/T$. Equation (3) can now be obtained by taking $R = \omega/2$ and letting $\omega \to \infty$.

University of California
Berkeley, California

1. A. Y. Fawaz, "The explicit formula for $L_0(x)$," *Proc.*, London Math. Soc., v. 1, s. 3, 1951, p. 86–103.
2. A. Y. Fawaz, "On an unsolved problem in the analytic theory of numbers," *Quart. Jn. of Math.*, v. 3, 1952, p. 282–295.
3. C. B. Haselgrove, "A disproof of a conjecture of Pólya," *Mathematika*, v. 5, 1958, p. 141–145.
4. A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge University Press, Cambridge, 1932.
5. A. E. Ingham, "On two conjectures in the theory of numbers," *Am. Jn. of Math.*, v. 64, 1942, p. 313–319.
6. G. Pólya, "Verschiedene Bemerkungen zur Zahlentheorie," *Jahresbericht*, Deutsch. Math. Verein., v. 28, 1919, p. 31–40.
7. R. Daublesky von Sterneck, "Die zahlentheoretische Funktion $\sigma(n)$ bis zur Grenze 5000000," *Wiener Sitzungsberichte*, v. 121:IIa, 1912, p. 1083–1096.