Explicit Solutions of the One-Dimensional Heat Equation for a Composite Wall

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1. Introduction. Explicit numerical solutions of the equation of heat conduction in a wall of one material have been widely discussed in the literature. Consideration of the forward difference equation studied in references [2], [3], [4], and [6] suggests a variety of ways to handle the solution for a composite wall. This paper is a study of the convergence, stability, comparative accuracy and comparative computing time of three explicit numerical solutions of the heat equation for a wall composed of two materials.

2. System of Equations. The equation for the one-dimensional flow of heat is:

$$c_s \rho_s \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_s \frac{\partial u}{\partial x}\right) \qquad \text{where } a_{s-1} \leqq x \leqq a_s \qquad s = 1, 2 \tag{1}$$

$$0 < t \leqq t_F$$

with the condition at the interface

$$k_1 \left(\frac{\partial u}{\partial x}\right)_1 = k_2 \left(\frac{\partial u}{\partial x}\right)_2 \tag{2}$$

where ρ_s, c_s, and k_s are constant with respect to time and temperature but may be different for each material.

We will assume the boundary conditions:

$$\begin{aligned} u(a_0, t) &= \text{constant}_1 \quad t \geqq 0 \\ u(a_2, t) &= \text{constant}_2 \quad t \geqq 0 \end{aligned} \tag{3}$$

and initial conditions:

$$u(x, 0) = \text{constant}_3 \quad a_0 < x < a_2 \tag{4}$$

Let each material's thickness, $a_s - a_{s-1}$, be divided into N_s equal parts of Δx_s, and t_F into equal parts of Δt_s. Let i denote the subscript associated with the space variable and j the subscript associated with the time variable. Let the solution of (1)–(4) be called $T(x, t)$.

Taylor series expansions of $T_{i\,j+1}$, $T_{i+1,j}$, and $T_{i-1,j}$, about T_{ij} are used to obtain

$$T_{i,j+1} = \frac{r_s \Delta t}{\Delta x_s^2} [T_{i+1,j} - 2T_{ij} + T_{i-1,j}] + T_{ij} + E_1 \tag{5}$$

where $r_s = k_s/\rho_s c_s$ and

$$E_1 = \frac{\Delta t^2}{2} \frac{\partial^2 T}{\partial t^2} - \frac{r_s \Delta t \Delta x_s^2}{12} \frac{\partial^4 T}{\partial x^4} + \text{terms of higher order.}$$

Omitting E_1, equation (5) gives a difference equation for finding the approximate

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solution of (1)–(4), $T_{i,j+1}$, when x_i, x_{i-1}, and x_{i+1} are in the same material. This is the same as the forward difference equation for a one-material wall.

The following equation for $T_{i,j+1}$ at the interface is derived in a manner similar to that used by M. Lotkin [5] in his discussion of an implicit method for a wall of two materials.

$$T_{ij} + \frac{\left[(T_{i+1,j} - T_{ij}) \dfrac{k_2}{\Delta x_2} + (T_{i-1,j} - T_{ij}) \dfrac{k_1}{\Delta x_1} \right] 2\Delta t}{\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1} = T_{i,j+1} + E_2 \tag{6}$$

where

$$E_2 = \frac{2\Delta t}{\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1} - \left[\frac{\Delta x_1 c_1 \rho_1 \Delta t}{4} \frac{\partial^2 T}{\partial t^2} - \frac{\Delta x_1^2 k_1}{6} \left(\frac{\partial^3 T}{\partial x^3} \right)_1 + \frac{k_1 \Delta x_1^3}{24} \left(\frac{\partial^4 T}{\partial x^4} \right)_1 - \frac{\Delta x_2 c_2 \rho_2 \Delta t}{4} \frac{\partial^2 T}{\partial t^2} + \frac{\Delta x_2^2 k_2}{6} \left(\frac{\partial^3 T}{\partial x^3} \right)_2 + \frac{k_2 \Delta x_2^3}{24} \left(\frac{\partial^4 T}{\partial x^4} \right)_2 \right] + \text{terms of higher order.}$$

Omitting E_2, equation (6) gives a difference equation for finding $T_{i,j+1}$ when x_1 is at the interface, x_{i-1} is in the first material, and x_{i+1} is in the second material.

3. Definition of Methods. Stability is maintained in the explicit numerical solution for a wall of one material by choosing

$$\Delta t \leqq \frac{\rho c \Delta x^2}{2k}. \tag{7}$$

Using equations (5) and (6) as our basic computing equations, three different means of choosing Δt will be defined and thereby different computational schemes. We will arbitrarily assume that $r_1 > r_2$ and, for simplicity, will confine the discussion to cases in which r_1/r_2 is an integer R, and $\sqrt{r_1/r_2}$ is an integer.

If Δx is specified as the thickness of each lamina within the wall, equation (7) gives two different maximum usable time increments depending on the properties of each material;

$$\Delta t_1 = \frac{\Delta x^2}{2r_1} \quad \text{and} \quad \Delta t_2 = \frac{\Delta x^2}{2r_2}.$$

METHOD 1. In the first method an attempt is made to circumvent the difficulty of having two Δt's by letting $\Delta x_1 = \Delta x$ and redefining Δx_2 such that $\Delta x_2 = \Delta x_1/\sqrt{R}$. This increases the number of laminae in the second material but yields only one time increment, $\Delta t_1 = \Delta t_2 = \Delta t$. The computations would take place as follows:

Given $\Delta x = \Delta x_1$, r_1 and r_2

1. Compute $\Delta x_2 = \Delta x_1/\sqrt{R}$
2. Find $\Delta t = \Delta t_1 = \Delta t_2$
3. Set time equal to Δt
4. Use equation (5) to find the temperatures in material 1.
5. Use equation (6) to find the interface temperature.
6. Use equation (5) to find the temperatures in material 2.

7. Advance the time by Δt.
8. Repeat steps 4 to 7 until the temperatures at t_F have been computed.

METHOD 2. In the second method Δt is chosen by evaluating Δt_1 and Δt_2 and using whichever is less. The computations would then be:

Given $\Delta x = \Delta x_1 = \Delta x_2$, r_1 and r_2

1. Choose $\Delta t = \min(\Delta t_1, \Delta t_2)$
2. Proceed as in steps 3 to 8 of Method 1.

METHOD 3. In the third method both time increments are utilized by using the smaller increment only for those points at which it is necessary and the larger increment for the rest. The smaller time increment must be used for all points in the first material, at the interface, and for enough points in the second material to enable a smooth transition. For example, when $R = 4$, those points on the grid in Figure 1 denoted by dots are computed using Δt_1 and then those denoted by crosses are computed using Δt_2. The computations would proceed as follows:

Given $\Delta x = \Delta x_1 = \Delta x_2$, r_1 and r_2

1. Compute Δt_1
2. Compute Δt_2
3. Set time $= 0$
4. Set $Q = R - 1$
5. Advance time by Δt_1
6. Compute the temperatures in material 1 using equation (5) and Δt_1.
7. Compute the interface temperature using equation (6) and Δt_1.
8. If $Q = 0$ proceed to step 12.
9. Compute Q points in material 2 using equation (5) and Δt_1.
10. $Q - 1 \rightarrow Q$.
11. Repeat steps 5 to 10 until indicated by step 8.
12. Compute temperatures in material 2 using equation (5) and Δt_2.
13. Repeat steps 4 to 12 until the temperatures at t_F have been computed.

4. Convergence of Solutions.

THEOREM. *If there exists a solution of the system of equations* (1) *to* (4) *which has bounded derivatives* $\partial^2T/\partial t^2$, $\partial^3T/\partial x^3$, *and* $\partial^4T/\partial x^4$ *in* $0 \leqq t \leqq t_F$, $a_0 \leqq x < a_1$ *and*

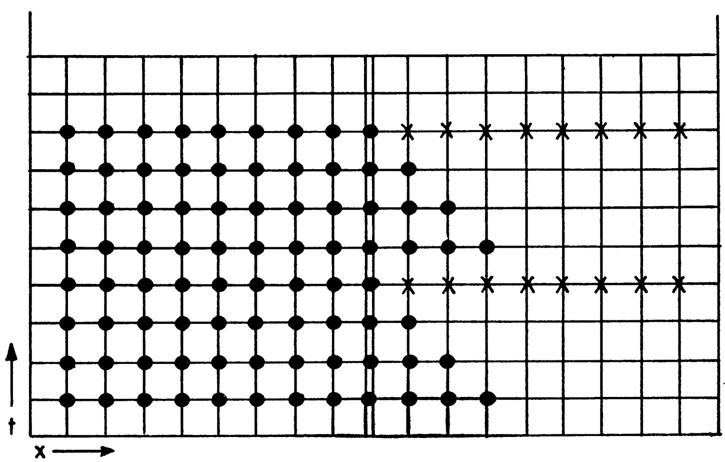

FIG. 1.—Grid points for $R = 4$.

$a_1 < x \leqq a_2$, *then the solutions obtained with methods* 1, 2, *and* 3 *converge to the true solution. The rate of convergence is* $O\ (\Delta x^2)$.

Proof. Let $B_1 = |$ upper bound on $\partial^2T/\partial t^2 |$, $B_2 = |$ upper bound on $\partial^4T/\partial x^4 |$, and $B_3 = |$ upper bound on $\partial^3T/\partial x^3 |$. A barred derivative denotes that it is evaluated somewhere within the interval $0 \leqq t \leqq t_F$, $a_0 \leqq x < a_1$, $a_1 < x \leqq a_2$. Define the error at the point x_i, t_j to be $e_{ij} = T_{ij} - u_{ij}$. Here u_{ij} is the true solution of (1)–(4). The error arising from the use of equation (5) satisfies the following equation:

$$(8) \quad e_{i,j+1} = \frac{r_s\Delta t}{\Delta x_s^2}[e_{i+1,j} - 2e_{ij} + e_{i-1,j}] + e_{ij} + \Delta t^2\,\overline{\frac{\partial^2 T}{\partial t^2}} - \frac{r_s\Delta t\Delta x_s^2}{12}\,\overline{\frac{\partial^4 T}{\partial x^4}}$$

$$(8') \quad e_{i,j+1} \leqq \left|\frac{r_s\Delta t}{\Delta x_s^2}\right| e_{i+1,j} + \left|1 - \frac{2r_s\Delta t}{\Delta x_s^2}\right| e_{ij} + \left|\frac{r_s\Delta t}{\Delta x_s^2}\right| e_{i-1,j} + \frac{\Delta t^2}{2}B_1 + \frac{r_s\Delta t\Delta x_s^2}{12}B_2.$$

Let $\alpha_j = \max_i | e_{ij} |$, then

$$(8'') \quad e_{i,j+1} \leqq \left[\left|\frac{r_s\Delta t}{\Delta x_s^2}\right| + \left|1 - \frac{2r_s\Delta t}{\Delta x_s^2}\right| + \left|\frac{r_s\Delta t}{\Delta x_s^2}\right|\right]\alpha_j + \frac{\Delta t^2}{2}B_1 + \frac{r_s\Delta t\Delta x_s^2}{12}B_2.$$

In methods 1 and 2, $\Delta t \leqq \Delta x_s^2/2r_s\ (s = 1, 2)$ and therefore $r_s\Delta t/\Delta x_s^2 \leqq \frac{1}{2}$. This causes each of the terms within the absolute value signs in equation (8″) to be positive and so they may be eliminated giving

$$(9) \qquad e_{i,j+1} \leqq \alpha_j + \frac{\Delta t^2}{2}B_1 + \frac{\Delta x_s^4}{24}B_2.$$

In method 3, the larger time increment, Δt_2, is only used at points for which $\Delta t_2 \leqq \Delta x_2^2/2r_2$ is satisfied and whenever Δt_1 is used, in the first material or for the transition values, $\Delta t_1 \leqq \Delta x_s^2/2r_s\ (s = 1, 2)$. Therefore, whenever equation (5) is used $\Delta t \leqq \Delta x^2/2r$ and so equation (9) also applies to method 3.

The error in the evaluation of the interface temperature from equation (6) satisfies

$$(10) \quad e_{i,j+1} = e_{i,j} + \frac{\left[(e_{i+1,j} - e_{ij})\dfrac{k_2}{\Delta x_2} + (e_{i-1,j} - e_{ij})\dfrac{k_1}{\Delta x_1}\right]2\Delta t}{\Delta x_2 c_2\rho_2 + \Delta x_1 c_1\rho_1} - E_2$$

$$(11) \quad \begin{aligned} e_{i,j+1} \leqq\ & \alpha_j\left[\frac{k_2}{\Delta x_2} + \frac{k_1}{\Delta x_1}\right]\left[\frac{2\Delta t}{\Delta x_2 c_2\rho_2 + \Delta x_1 c_1\rho_1}\right] \\ & + \left|1 - \frac{2\Delta t\left[\dfrac{k_2}{\Delta x_2} + \dfrac{k_1}{\Delta x_1}\right]}{\Delta x_2 c_2\rho_2 + \Delta x_1 c_1\rho_1}\right|\alpha_j + \frac{\Delta t^2}{2}B_1 + \frac{\Delta t B_3(\Delta x_1^2 k_1 + \Delta x_2^2 k_2)}{3(\Delta x_2 c_2\rho_2 + \Delta x_1 c_1\rho_1)} \\ & + \frac{\Delta t B_2(k_1\Delta x_1^3 + k_2\Delta_2^3)}{12(\Delta x_2 c_2\rho_2 + \Delta x_1 c_1\rho_1)}. \end{aligned}$$

Since in all three methods the Δt used in the interface equation is min $(\Delta t_1, \Delta t_2)$,

$$\frac{\Delta x_2 c_2\rho_2}{\Delta t} + \frac{\Delta x_1 c_1\rho_1}{\Delta t} \geqq 2\left[\frac{k_2}{\Delta x_2} + \frac{k_1}{\Delta x_1}\right]$$

and, therefore, (11) can be rewritten as

$$e_{i,j+1} \leqq \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{B_3}{6}\left[\frac{k_1 \Delta x_2(\Delta x_1^3) + k_2 \Delta x_1(\Delta x_2^3)}{\Delta x_1 k_2 + \Delta x_2 k_1}\right] + \frac{B_2}{24}\left[\frac{k_1 \Delta x_2(\Delta x_1^4) + k_2 \Delta x_1(\Delta x_2^4)}{\Delta x_1 k_2 + \Delta x_2 k_1}\right]. \tag{12}$$

Let $\Delta x = \max(\Delta x_1, \Delta x_2)$, then (12) becomes

$$e_{i,j+1} \leqq \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{\Delta x^3}{6} B_3 + \frac{\Delta x^4}{24} B_2. \tag{13}$$

Comparing equations (9) and (13), it can be seen that

$$\alpha_{j+1} \leqq \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{\Delta x^3}{6} B_3 + \frac{\Delta x^4}{24} B_2 \quad \text{where} \quad \Delta t = \max(\Delta t_1, \Delta t_2) \tag{14}$$

and $\Delta x = \max(\Delta x_1, \Delta x_2)$. At any point $t = j\Delta t$

$$\alpha_j \leqq \alpha_0 + j\left[\frac{\Delta t^2}{2} B_1 + \frac{\Delta x^3}{6} B_3 + \frac{\Delta x^4}{24} B_2\right]. \tag{15}$$

The rate of convergence is, therefore, of the order $0(\Delta t \;\&\; \Delta x^2)$. Since Δt is of the order $0(\Delta x^2)$, the rate of convergence is $0(\Delta x^2)$.

5. Analytical Example. In order to examine the performance of the three methods, a test case will be used for which some analytical solutions are known. The equations for the composite wall will first be reduced, by transformations of the variables, to the equations for a wall of one material. To do this we will impose the conditions:

$$\text{(16)} \quad \begin{array}{ll} \text{(i)} & k_1 c_1 \rho_1 = k_2 c_2 \rho_2 \\ \text{(ii)} & a_0 = 0. \end{array}$$

Define the transformations

$$\text{(17)} \quad \begin{array}{lll} \text{(i)} & y = \begin{cases} \left[b + (1 - b)\dfrac{a_2}{a_1}\right] x & 0 \leqq x \leqq a_1 \\ bx + (1 - b)a_2 & a_1 \leqq x \leqq a_2 \end{cases} & \\ \text{(ii)} & \tau = r_2 b^2 t & 0 \leqq t \leqq t_F \end{array}$$

$$\text{where} \quad b = \frac{k_1 a_2}{k_1(a_2 - a_1) + k_2 a_1}.$$

This reduces equations (1) to (4) to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2} \qquad 0 \leqq y \leqq a_2, 0 \leqq \tau \leqq t_F r_2 b^2 \tag{18}$$

$$\text{(19)} \quad \begin{array}{l} u(0, \tau) = \text{constant}_1 \quad \tau \geqq 0 \\ u(a_2, \tau) = \text{constant}_2 \quad \tau \geqq 0 \end{array}$$

$$u(y, 0) = \text{constant}_3 \quad 0 < y < a_2. \tag{20}$$

The solution to this set of equations is found in reference [6] for $\tau = 0(.005).1$, $y = .4$, $\text{constant}_1 = \text{constant}_2 = 0$, $\text{constant}_3 = 1$, $a_2 = 1$.

6. Test Case Results. Each method was programmed for the IBM 704 EDPM. The test data used was $k_1 = 10$, $c_1 = 5$, $\rho_1 = 2$, $k_2 = 5$, $c_2 = 4$, $\rho_2 = 5$, $a_0 = 0$, $a_1 = .5$, and $a_2 = 1$. Each method was run for three cases: Case A, $\Delta x = .1$; case B, $\Delta x = .05$, and case C, $\Delta x = .025$.

To examine the rate of convergence, the maximum differences for a given time point were found between the results for case A and case C, and the results for case B and case C. The ratios of these maximum differences ranged between 3.8 and 5.0 for each of the three methods. Since the ratios of the Δx's were 2.0, this would seem to corroborate that the rate of convergence is $0(\Delta x^2)$.

To compare the accuracy of the three methods, the data presented in reference [6] was used. Their values correspond, according to the transformation presented in equation (17), to $x = .55$ and $x = .7$ for $t = 0\ (.01125).225$. These values, as well as those obtained for case C for the three methods, are presented in Table 1. It can be seen from the table that, although they all showed close agreement, method 2 gave the most accuracy with a maximum of .08% error, method 1 the next with a maximum of .14% error, and method 3 the least with a maximum of .28% error. The symmetry of the transformed equation indicates that for this case the temperatures should be the same for $x = .15$ and $x = .925$, $x = .3$ and $x = .85$, $x = .45$ and $x = .775$, and for $x = .55$ and $x = .7$. When comparing the results at these points at $t = .1$, method 1 has them all the same, method 2 has a greatest difference of .04%, and method 3 has a greatest difference of .53%. A comparison at $t = .225$ shows method 1 has them all the same, method 2 still with a greatest difference of .04%, and method 3 with a greatest difference of .20%. These differences are reasonable in terms of the methods of choosing Δt for computation and illustrate that in method 1 the results for both materials is equally accurate, for method 2 the results for the second material is a bit more accurate than for the first material, while for method 3 the result for the second material is less accurate than for the first but their differences decrease as more time steps are taken.

TABLE 1

Time	*Exact Solution*	*Method 1*, $x = .55$	*Method 1*, $x = .7$	*Method 2*, $x = .55$	*Method 2*, $x = .7$	*Method 3*, $x = .55$	*Method 3*, $x = .7$
.0225	.9953	.99536	.99536	.99551	.99499	.99579	.99557
.0450	.9518	.95145	.95145	.95188	.95127	.95254	.95055
.0675	.8832	.88261	.88261	.88306	.88275	.88354	.88136
.0900	.8088	.80807	.80807	.80851	.80840	.80868	.80686
.1125	.7363	.73556	.73556	.73598	.73597	.73590	.73448
.1350	.6686	.66782	.66782	.66821	.66826	.66797	.66689
.1575	.6063	.60561	.60561	.60597	.60605	.60562	.60481
.1800	.5496	.54891	.54891	.54924	.54933	.54882	.54821
.2025	.4981	.49739	.49739	.49770	.49780	.49724	.49678
.2250	.4513	.45067	.45067	.45095	.45105	.45047	.45011

Although the time increments were chosen on the basis of equation (7), it is pointed out in reference [6] that a slightly larger increment is possible, namely

$$\Delta t_{\max} \leqq \frac{c\rho\, \Delta x^2}{2k \sin^2\left[\dfrac{(N-1)\pi}{2N}\right]}. \tag{21}$$

Using case A, the programs were run until instability appeared in an attempt to see what the maximum increment actually was for two materials. For method 3, if the maximum Δt's are computed separately for each material of 5 laminae with equation (21), the result is $\Delta t_1 \leqq .005528$ and $\Delta t_2 \leqq .022112$. The experimental results corroborated this since it was stable up to $\Delta t_1 = .0055$ and $\Delta t_2 = .0220$ but unstable for $\Delta t_1 = .0056$ and $\Delta t_2 = .0230$. For method 2, computing the maximum time increment for each material of 5 laminae and then choosing the smaller, one gets $\Delta t \leqq .005528$. The test showed the same result as it was stable up to $\Delta t = .0055$ but unstable for $\Delta t = .0056$. In method 1, when using equation (7), the adjustment of the thicknesses of the laminae in the second material led to equal Δt's. However, when using equation (21), the Δt's obtained are $\Delta t_1 \leqq .005528$ and $\Delta t_2 \leqq .005125$ since the number of laminae in each material is different. The test runs showed that it remained stable until $\Delta t = .0053$ but was unstable with $\Delta t = .0054$. This might indicate that a maximum time increment was used which is the average of Δt_1 and Δt_2 but no conclusion is possible since the stability condition states that it should be stable below the computed $\Delta t_{\max}$ but it is not necessarily unstable for a Δt above it. However, all these stability test runs seem to indicate that when there are two materials in a wall, the maximum usable time increment is quite closely related to the maximum increments computed for each material separately.

7. Comparison of Computing Time Required. The amount of computing time required for each method can be compared by comparing the number of temperatures that must be evaluated.

Let us assume that the thicknesses of the first and second materials are equal and that $t_F = PR\Delta t_1$ (where P is any integer). It should be noted that given Δx, r_1, and r_2, each method will compute the same value for Δt_1 and Δx_1.

In method 1 there are $N_1(1 + \sqrt{R})$ laminae and PR time points. Therefore, the number of temperatures computed equals $P[RN_1(1 + \sqrt{R}) - R]$. In method 2 there are $2N_1$ laminae and PR time points and so $P[2N_1R - R]$ temperatures are computed. In method 3 each of the $N_1 - 1$ temperatures in the first material

TABLE 2

	Case A	*Case* B	*Case* C
Method 1	P (56)	P (116)	P (236)
Method 2	P (36)	P (76)	P (156)
Method 3	P (30)	P (55)	P (105)
Ratio M1:M2:M3	1.87:1.20:1.00	2.11:1.38:1.00	2.25:1.49:1.00

is computed PR times, the interface is computed PR times, and for each step of Δt_2, which occurs P times, $N_1 - 1$ values are computed plus the additional number of transition values $= \sum_{i=1}^{R-1} i$. Method 3, therefore, computes

$$P\left[N_1 R + N_1 - 1 + \frac{R^2}{2} - \frac{R}{2}\right]$$

values. From this it can be seen that for all $R > 1$ method 2 is faster than method 1. For $2N_1 - 2 > R > 1$ method 3 is faster than method 2. This comparison is illustrated in Table 2 for cases A, B and C of the test data used.

8. Conclusions. The analysis and test cases used considered constant boundary and initial conditions. Since the stability and convergence depend also on the boundary and initial conditions, as has been pointed out in references [1], [3], and [4], it is quite possible that the introduction of varying conditions would lead to different results as to the usefulness of each method.

From this study it seems that method 1 is the least acceptable since it takes the most computing time, gives less accuracy than method 2, and presents the most difficulty when R is not an integer. Depending on the amount of accuracy desired, methods 2 and 3 seem of equal usefulness since method 2 gives the most accuracy but method 3 takes less computing time.

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