Further Evaluation of Khintchine's Constant

By John W. Wrench, Jr.

In his fundamental investigation of the metric theory of continued fractions Khintchine [1] proved that the limit, as n tends to infinity, of the geometric mean of the first n partial quotients in the simple continued fraction expansion of almost all real numbers is the absolute constant

$$K = \prod_{r=1}^{\infty} \left(1 + \frac{1}{r(r+2)}\right)^{\ln r/\ln 2}.$$

A different proof, by C. Ryll-Nardzewski, has been recently reproduced by M. Kac [2].

The numerical evaluation of Khintchine's constant was considered by D. H. Lehmer [3]. In addition to finding an approximation to K to 6 decimal places, whose accuracy was subsequently discussed by D. Shanks [4], Lehmer investigated the geometric mean of the first one hundred partial quotients of π.

Recently R. S. Lehman [5] computed the first 1986 partial quotients of π on ORDVAC in order to test the applicability of a similar theorem of Lévy [6], which asserts that, as n tends to infinity, the nth root of the denominator of the nth convergent tends to $\exp(\pi^2/12 \ln 2)$.

Shanks and the writer [7] have studied the representation of K by infinite series and by definite integrals. The computational effectiveness of these series was illustrated by the evaluation of K to 65 decimal places. This calculation has now been extended by me to 155 places, using the same series as previously, namely:

$$\ln 2 \ln K = \ln \frac{3}{2} + \ln 2 \ln \frac{3}{2} - \left\{\frac{1}{2.3} \sum_{k=2}^{\infty} \frac{S_{2k}''}{k} + \frac{1}{4.5} \sum_{k=3}^{\infty} \frac{S_{2k}''}{k} + \frac{1}{6.7} \sum_{k=4}^{\infty} \frac{S_{2k}''}{k} + \cdots\right\},$$

where S_{2k}'' represents

$$\sum_{n=3}^{\infty} n^{-2k} = \zeta(2k) - 1 - 2^{-2k}.$$

A preliminary step in this calculation consisted of the formation of a table of $\zeta(2k)$ to at least 155D for $k = 1(1)\,257$. The first 60 entries of this table were computed by the formula

$$\zeta(2k) = (-1)^{k-1} \frac{B_{2k}\,(2\pi)^{2k}}{2(2k)!},$$

where the notation for the Bernoulli numbers is that used by K. Knopp [8]. The numerical values of these numbers were taken from the tables of H. T. Davis [9]. The requisite decimal approximations to $\pi^{2k}/(2k)!$ were obtained from my manuscript table [10] of such data. The remaining entries were computed directly from the series defining $\zeta(2k)$, a maximum of eighteen terms being required initially.

From these values of $\zeta(2k)$ the approximations to S_{2k}'' and S_{2k}''/k were then computed to 155D. All these data were subjected to the following check relations:

Received June 4, 1960.

$$\sum_{k=1}^{\infty} [\zeta(2k) - 1] = \frac{3}{4},$$

$$\sum_{k=1}^{\infty} S_{2k}'' = \frac{5}{12},$$

$$\sum_{k=1}^{\infty} S_{2k}''/k = \ln \frac{3}{2},$$

$$\sum_{r=1}^{\infty} \sum_{k=r}^{\infty} S_{2k}''/k = \frac{5}{12}.$$

Substitution of the computed values in these formulas resulted in discrepancies all less than 3 units in the 155th decimal place.

The final results of this calculation when rounded to 155D are as follows:

$\ln 2 \ln K =$

0.68472 47885 63157 12329 91461 48755 77762 04606 75416 33744
88366 06289 86781 59568 82176 26936 10437 07681 43495 85810
09970 15696 93974 12470 41578 92227 14396 39612 78766 18097
72947 $\cdots$,

$\ln K =$

0.98784 90568 33810 78966 92547 27147 07295 43261 99254 96088
67354 27755 30068 72109 27094 18512 90938 20768 83372 75259
67479 51231 68801 78544 35925 75519 06227 59695 60965 06769
43483 $\cdots$,

$K =$

2.68545 20010 65306 44530 97148 35481 79569 38203 82293 99446
29530 51152 34555 72188 59537 15200 28011 41174 93184 76979
95153 46590 52880 90082 89767 77164 10963 05179 25334 83259
66838 $\cdots$.

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