Integrals of Products of Laguerre Polynomials

By R. D. Lord

In a recent note in this journal Gillis and Weiss [1] have evaluated as a finite sum, and found a recurrence relation for, the integral

$$(1) \qquad C_{rst} = \int_0^\infty e^{-x} L_r(x) L_s(x) L_t(x) \, dx,$$

which gives the coefficients in the expansion

$$L_r(x) L_s(x) = \sum C_{rst} L_t(x),$$

where $L_n(x)$ denotes the Laguerre polynomial,

$$(2) \qquad L_n(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \nu^r / r!$$

This attracted my interest since about ten years ago I considered the more general integral of the product of generalized Laguerre polynomials

$$(3) \qquad \int_0^\infty e^{-x} x^\beta L_{n_1}^{(\alpha_1)}(a_1 x) L_{n_2}^{(\alpha_2)}(a_2 x) \cdots L_{n_m}^{(\alpha_m)}(a_m x) \, dx,$$

with the object of finding cases which simplified. I tried methods which I had used with the corresponding problem for Hermite polynomials [2], but met with little success. But the special case (1) is easily dealt with as follows.

The generating function for Laguerre polynomials is

$$(4) \qquad \sum_{r=0}^{\infty} (-1)^r L_r(x) \, u^r = \frac{1}{1+u} \exp\left(\frac{xu}{1+u}\right),$$

and hence that for the product of three polynomials is

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{r+s+t} L_r(x) L_s(x) L_t(x) \, u_1^r u_2^s u_3^t$$

$$= [(1+u_1)(1+u_2)(1+u_3)]^{-1} \exp x \left(\frac{u_1}{1+u_1} + \frac{u_2}{1+u_2} + \frac{u_3}{1+u_3}\right).$$

On multiplying by e^{-x} and integrating from 0 to ∞, we get

$$(5) \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{r+s+t} C_{rst} u_1^r u_2^s u_3^t = (1 - u_2 u_3 - u_3 u_1 - u_1 u_2 - 2 u_1 u_2 u_3)^{-1}$$

as the generating function for C_{rst}. If we write the right hand side of (5) as

$$\sum (u_2 u_3 + u_3 u_1 + u_1 u_2 + 2 u_1 u_2 u_3)^\nu,$$

we immediately see that C_{rst} is an integer with the sign $(-)^{r+s+t}$. Further expansion of each term by the multinomial theorem gives Watson's formula [3]

$$(6) \qquad C_{rst} = (-2)^{r+s+t} \sum_n 2^{-2n} \frac{n!}{(n-r)!\,(n-s)!\,(n-t)!\,(r+s+t-2n)!} ,$$

where in the summation all factorials have non-negative arguments.

If we transpose the right hand side of (5) and equate coefficients of $u_1{}^r u_2{}^s u_3{}^t$, we have

$$(7) \qquad C_{rst} = C_{r,s-1,t-1} + C_{r-1,s,t-1} + C_{r-1,s-1,t} + 2C_{r-1,s-1,t-1} ,$$

which is the recurrence formula with simplest coefficients, especially as the right hand side can be treated as the sum of five terms. Even though all suffixes vary, it probably provides the quickest way of computing all values of C_{rst} up to a given set of r, s, t. For some machines at least, it may well give the quickest way for calculating a given C_{rst} , and it provides an easy method for desk-machine computation when r, s and t are small. For computations by other methods it provides a simple check. Other checks may be obtained by giving u_1 special values in (5) and equating coefficients of $u_2{}^s u_3{}^t$. Putting $u_1 = -1$, we get

$$(8) \qquad \sum_r C_{rst} = 1,$$

given by Gillis and Weiss; putting $u_1 = -\frac{1}{2}$, we get

$$(9) \qquad \sum_r 2^{s+t-r} C_{rst} = \binom{s+t}{s} .$$

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1. J. Gillis & G. Weiss, "Products of Laguerre polynomials," *Math. Comp. (MTAC)*, v. 14, 1960, p. 60–63.
2. R. D. Lord, "Some integrals involving Hermite polynomials," London Math. Soc., *Jn.*, v. 24, 1949, p. 101–112.
3. G. N. Watson, "A note on the polynomials of Hermite and Laguerre," London Math. Soc., *Jn.*, v. 13, 1938, p. 29.

The Evaluation of Certain Probability Integrals

By Irwin Greenberg

A problem which often arises in the field of probability and statistics is the following:

Assume that there are n independent stochastic processes and that the kth process has an output distribution $f_k(x_k)$. The probability that the jth process yields a higher output than any of the others is

$$(1) \qquad P_j = \int_{-\infty}^{\infty} f_j(x_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} \int_{-\infty}^{x_j} f_k(x_k) \, dx_k \, dx_j.$$

In certain special cases, equation (1) is easily integrated; for example, if the various $f_k(x_k)$ are all uniform or exponential distributions.