Improved Formulas for Complete and Partial Summation of Certain Series

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Abstract. In two previous articles one of the authors gave formulas, with numerous examples, for summing a series either to infinity (complete) or up to a certain number n of terms (partial) by considering the sum of the first j terms S_j, or some suitable modification S_j', closely related to S_j, as a polynomial in $1/j$. Either S_∞ or S_n was found by m-point Lagrangian extrapolation from S_{j_0}, S_{j_0-1}, $\cdots$, S_{j_0-m+1} to $1/j = 0$ or $1/j = 1/n$ respectively. This present paper furnishes more accurate m-point formulas for sums (or sequences) S_j which behave as even functions of $1/j$. Tables of Lagrangian extrapolation coefficients in the variable $1/j^2$ are given for: complete summation, $m = 2(1)7$, $j_0 = 10$, exactly and 20D, $m = 11$, $j_0 = 20$, 30D; partial summation, $m = 7$, $j_0 = 10$, $n = 11(1)25(5)100$, 200, 500, 1000, exactly. Applications are made to calculating π or the semi-perimeters of many-sided regular polygons, Euler's constant,

$$1 + \sum_{r=1}^{j} \left\{ \frac{-1}{(4r-1)^2} + \frac{1}{(4r+1)^2} \right\} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots \quad \text{for} \quad j = \infty$$

(Catalan's constant), calculation of a definite integral as the limit of a suitably chosen sequence, determining later zeros of $J_\nu(x)$ from earlier zeros for suitable ν, etc. A useful device in many cases involving sums of odd functions, is to replace S_j by a trapezoidal-type S_j' which is seen, from the Euler-Maclaurin formula, to be formally a series in $1/j^2$. In almost every example, comparison with the earlier $(1/j)$-extrapolation showed these present formulas, for 7 points, to improve results by anywhere from around 4 to 9 places.

1. Introduction. In two earlier papers, [1, 2], one of the authors gave tables for both complete summation (all terms, to infinity) and partial summation (up to a certain number of terms) of certain kinds of slowly convergent series. In the case of partial summation, divergent series were also included, provided that a suitable auxiliary series could be found of the desired slowly convergent type and simply related to the original divergent series. The essential idea in both cases is to regard the sequence S_j, the sum of the first j terms of the series, as the values for $x = 1/j$ of an interpolable function $S(x)$ to which the slight extrapolation from specified S_j, to $j = \infty$ $(x = 0)$ or to $j = k$ $(x = 1/k)$, $k > j_0$ where S_{j_0} is the last specified S_j, yields good accuracy. The approximating formula for $S(x)$ was an m-point Lagrange polynomial of the $(m-1)$th degree in x which at $x = 1/j$ assumes the prescribed value S_j, for the last m values of j ending at $j_0 = 5$, 10, 15 or 20, from which we extrapolated to either $j = \infty$ $(x = 0)$ or $j = k > j_0(x = 1/k)$. Numerous examples which yielded surprisingly high accuracy for a variety of sequences S_j in both complete and incomplete cases, attested to the wide applicability of considering S_j a smooth function of $1/j$, even when we were in complete ignorance as to the

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actual analytic expression for $S_j = S(1/j)$ or of a theoretical justification for considering $S(1/j)$ as an approximate polynomial in $1/j$.

However, a still further improvement in m-point formulas for both complete and partial summation is applicable to a wide class of sequences where $S_j \equiv S(1/j)$ behaves like an even function of $1/j$. Thus by taking x^2 as argument instead of x, in an m-point Lagrangian extrapolation formula for $x = 0$ (complete summation) or a value close to 0 (partial summation) based upon those same final m values of S_j, we should get accuracy equivalent to $(2m - 1)$th degree instead of $(m - 1)$th degree. As will be seen from the illustrations below, the resulting improvement is often quite impressive.

There is no hard and fast classification of all the varied problems to which these newer formulas are applicable. The reason is that even if a problem does not seem offhand to involve a sequence of that even-function type, often with a very slight transformation, regrouping, or alteration, one sees that it really is amenable to this more accurate treatment.

Of course, every sequence to which these improved extrapolation formulas for arguments $1/j^2$ are particularly applicable can also be handled by the earlier formulas employing arguments $1/j$, because any polynomial in x^2 is also a polynomial in x, but with considerably less accuracy for the same number m of points and the same last $j = j_0$. But the converse is not true—we cannot in general expect these newer summation formulas to work well when applied indiscriminately to sequences where the earlier method may give very high accuracy. One way of realizing this is to think of the non-constant part of a well-behaved function of x near $x = 0$ being approximated by Cx. Extrapolation employing $x^2 = y$ as the variable, near $x = 0$, is like extrapolation for $\sqrt{y}$ based upon a polynomial approximation in the variable y. But, as anybody who has attempted to interpolate in a table of square roots near zero has found out, $\sqrt{y}$, although continuous at $y = 0$, possesses a singularity due to an infinite derivative.

2. Other Related Articles. The idea of the extrapolation to $x = 0$ for argument $y = x^2$ has been employed for just the linear case in the well-known "h^2-extrapolation process", or "deferred approach to the limit", which has been extensively treated in the literature on the numerical solution of differential equations (first introduced by L. F. Richardson [3, 4]). The argument x or h corresponds to two conveniently small values of a mesh-length, say h_1 and h_2. Richardson's process has been generalized to higher powers beyond h^2 by several writers, notably G. Blanch, [5] and H. C. Bolton and H. I. Scoins [6]. However, the only reference that was encountered by the writer which was concerned with problems where the approximation might be considered as a purely even function of h having more than a single term, has been M. G. Salvadori [7]. Besides some sets of 2-point coefficients for h^2- and h^4-extrapolation, Salvadori tabulates 3-point coefficients for (h^2, h^4)- and (h^4, h^6)- extrapolation, and 4-point coefficients for (h^2, h^4, h^6)- and (h^4, h^6, h^8)-extrapolation. The values of h are in the form $1/n_i$, where n_i are sets of small integers ranging from 2 to 8. Salvadori gives applications to numerical differentiation and integration, as well as to some boundary value problems and characteristic value problems.

3. Formulas for Complete Summation. In choosing a j_0 suitable for most complete summation purposes, we wish to obtain a substantial increase in accuracy over the use of the earlier formulas in [1], which has already been proved to be very accurate, without having coefficients that might be too cumbersome. It is also desirable to give exact values rather than decimal values, because in highly accurate formulas the theoretical or truncation error might be considerably smaller than the computing error arising from the use of rounded decimal entries. But we must also take account of the fact that the fixed points $1/j^2$ in place of the older $1/j$ makes the exact fractional form of the extrapolation coefficients have around twice as many digits in both numerator and denominator, which adds considerably to the amount of time to do an example.

In the present paper it seems that a very convenient choice is $j_0 = 10$, for all cases ranging from the 2-point through the 7-point. In other words we give formulas for linear through sextic Lagrangian extrapolation formulas for functions of the variable $y = x^2$ taken at $x = 1/j$, or arguments $y = 1/j^2$ at $j = 10, 9, \cdots, 10 - m + 1$ for $m = 2(1)7$. This is equivalent to quadratic through twelfth degree accuracy for even functions in $x = 1/j$. The extrapolation formula to obtain the complete sum S from the partial sums $S_{10}, S_9, \cdots, S_{10-m+1}$ is the very simple

$$(1) \qquad S \sim \sum_{i=0}^{m-1} A_{10,10-i}^{(m)} S_{10-i}\,.$$

The coefficients $A_{10,10-i}^{(m)}$ are given in Table 1 in exact fractional form $B_{10,10-i}^{(m)}/D_{10}^{(m)}$, so that (1) may be most conveniently employed as

$$(2) \qquad S \sim (1/D_{10}^{(m)}) \sum_{i=0}^{m-1} B_{10,10-i}^{(m)} S_{10-i}\,.$$

In no case through $m = 7$, does $D_{10}^{(m)}$ have more than ten digits exclusive of final zeros, which is convenient in the division. The values of $A_{10,10-i}^{(m)}$ are given also to 20 decimals in Table 2.

Although the 7-point formulas for $j_0 = 10$ are very accurate, as will be apparent from the examples below, we give also in Table 3 for possible use in some kind of isolated key calculation where extreme accuracy is sought, even at the expense of considerable computing labor, the coefficients in the 11-point formula, ending at S_{20}, given exactly, to be employed in

$$(3) \qquad S \sim (1/D_{20}^{(11)}) \sum_{i=0}^{10} B_{20,20-i}^{(20)} S_{20-i}\,.$$

Formula (3) is exact for any even polynomial in $x = 1/j$ up to the 20th degree. To avoid too much non-essential numerical work, no illustrations were given of the use of Table 3, since the resulting accuracy is so high by comparison with the results of using Table 1 or 2, that an excessively large number of significant digits is needed to reveal its full extent. But Table 3 should be kept in reserve for a summation problem requiring unusual precision.

The formula for $A_{j_0,j_0-i}^{(m)}$ is obtained rather simply from the well-known definition of the m-point Lagrangian interpolation coefficients where we have fixed points $1/j_0^2, 1/(j_0 - 1)^2, \cdots, 1/(j_0 - m + 1)^2$ and set the variable $y = x^2 = 1/j^2$ equal to 0 to correspond to $j = \infty$.

TABLE 1

$$A^{(m)}_{10,10-i} = B^{(m)}_{10,10-i}/D^{(m)}_{10}$$

$m = 2$	**$m = 5$**
$B^{(2)}_{10,9} = -81$	$B^{(5)}_{10,6} = 2034\ 43488$
$B^{(2)}_{10,10} = 100$	$B^{(5)}_{10,7} = -23001\ 55599$
$D^{(2)}_{10} = 19$	$B^{(5)}_{10,8} = 82879\ 44704$
$m = 3$	$B^{(5)}_{10,9} = -1\ 17517\ 54833$
$B^{(3)}_{10,8} = 19456$	$B^{(5)}_{10,10} = 56875\ 00000$
$B^{(3)}_{10,9} = -59049$	$D^{(5)}_{10} = 1269\ 77760$
$B^{(3)}_{10,10} = 42500$	**$m = 6$**
$D^{(3)}_{10} = 2907$	$B^{(6)}_{10,5} = -75703\ 12500$
$m = 4$	$B^{(6)}_{10,6} = 17\ 57751\ 73632$
$B^{(4)}_{10,7} = -67\ 05993$	$B^{(6)}_{10,7} = -123\ 97838\ 67861$
$B^{(4)}_{10,8} = 398\ 45888$	$B^{(6)}_{10,8} = 359\ 05926\ 59456$
$B^{(4)}_{10,9} = -717\ 44535$	$B^{(6)}_{10,9} = -448\ 74915\ 24087$
$B^{(4)}_{10,10} = 400\ 00000$	$B^{(6)}_{10,10} = 200\ 20000\ 00000$
$D^{(4)}_{10} = 13\ 95360$	$D^{(6)}_{10} = 3\ 35221\ 28640$

$m = 7$

$B^{(7)}_{10,4}$	$= 54190\ 40768$
$B^{(7)}_{10,5}$	$= -31\ 54296\ 87500$
$B^{(7)}_{10,6}$	$= 474\ 59296\ 88064$
$B^{(7)}_{10,7}$	$= -2761\ 33679\ 65995$
$B^{(7)}_{10,8}$	$= 7181\ 18531\ 89120$
$B^{(7)}_{10,9}$	$= -8388\ 15723\ 34857$
$B^{(7)}_{10,10}$	$= 3575\ 00000\ 00000$
$D^{(7)}_{10}$	$= 50\ 28319\ 29600$

$$A^{(m)}_{j_0,j_0-i} = \frac{(-1)^{m-1}(j_0 - i)^{2m-2}}{\prod_{k=0}^{m-1}{}' \, [(j_0 - k)^2 - (j_0 - i)^2]}, \tag{4}$$

where in $\prod'$, $k = i$ is omitted.

4. Illustrations of Complete Summation.

A. *Example* 1. Considering the circle as the limiting case of inscribed regular polygons of j sides, as $j \to \infty$, the quantity π is the limit of the semi-perimeter, $j \sin \alpha$, where $\alpha = 180°/j = \pi/j$, as $j \to \infty$.* Now the approximation $S_j = j \sin \alpha =$

* Although this example affords a splendid illustration of the improvement of $(1/j^2)$-extrapolation over $(1/j)$-extrapolation, it suffers from the aesthetic defect of having the value of π occurring implicitly in every S_j in the various powers of α needed to compute $\sin \alpha$. In other words, there is definitely something "circular" in this example.

TABLE 2

$A^{(m)}_{10,10-i}$ *in Decimal Form*

m = 2		
$A^{(2)}_{10,9}$	=	−4.26315 78947 36842 10526
$A^{(2)}_{10,10}$	=	5.26315 78947 36842 10526
m = 3		
$A^{(3)}_{10,8}$	=	6.69281 04575 16339 86928
$A^{(3)}_{10,9}$	=	−20.31269 34984 52012 38390
$A^{(3)}_{10,10}$	=	14.61988 30409 35672 51462
m = 4		
$A^{(4)}_{10,7}$	=	−4.80592 32026 14379 08497
$A^{(4)}_{10,8}$	=	28.55599 12854 03050 10893
$A^{(4)}_{10,9}$	=	−51.41650 54179 56656 34675
$A^{(4)}_{10,10}$	=	28.66643 73351 67985 32278
m = 5		
$A^{(5)}_{10,6}$	=	1.60219 78021 97802 19780
$A^{(5)}_{10,7}$	=	−18.11463 36098 54198 08949
$A^{(5)}_{10,8}$	=	65.27083 72237 78400 24899
$A^{(5)}_{10,9}$	=	−92.54970 97523 21981 42415
$A^{(5)}_{10,10}$	=	44.79130 83361 99977 06685
m = 6		
$A^{(6)}_{10,5}$	=	−0.22583 03039 55303 95530
$A^{(6)}_{10,6}$	=	5.24355 64435 56443 55644
$A^{(6)}_{10,7}$	=	−36.98404 36201 18987 76605
$A^{(6)}_{10,8}$	=	107.11111 74954 31220 92142
$A^{(6)}_{10,9}$	=	−133.86654 44631 80008 84564
$A^{(6)}_{10,10}$	=	59.72174 44482 66636 08913
m = 7		
$A^{(7)}_{10,4}$	=	0.01077 70418 88152 99926
$A^{(7)}_{10,5}$	=	−0.62730 63998 75844 32029
$A^{(7)}_{10,6}$	=	9.43840 15984 01598 40160
$A^{(7)}_{10,7}$	=	−54.91570 11329 03951 53140
$A^{(7)}_{10,8}$	=	142.81482 33272 41627 89522
$A^{(7)}_{10,9}$	=	−166.81830 92541 16626 40765
$A^{(7)}_{10,10}$	=	71.09731 48193 65042 96325

TABLE 3

$A^{(11)}_{20,20-i} = B^{(11)}_{20,20-i}/D^{(11)}_{20}$

$B^{(11)}_{20,10}$	=	74096 20000 00000 00000 00000
$B^{(11)}_{20,11}$	=	−35 37615 48335 31708 54782 90644
$B^{(11)}_{20,12}$	=	649 45974 08685 61313 24915 22048
$B^{(11)}_{20,13}$	=	−6200 60319 26092 91850 74192 35023
$B^{(11)}_{20,14}$	=	34801 60376 52150 35629 23772 47744
$B^{(11)}_{20,15}$	=	−1 21941 46052 37160 60638 42773 43750
$B^{(11)}_{20,16}$	=	2 73659 70208 28851 47761 53823 64160
$B^{(11)}_{20,17}$	=	−3 92511 27655 98026 11495 97941 97770
$B^{(11)}_{20,18}$	=	3 47343 22454 05086 94470 03616 05120
$B^{(11)}_{20,19}$	=	−1 72481 59320 99496 29170 21217 51885
$B^{(11)}_{20,20}$	=	36718 51008 00000 00000 00000 00000
$D^{(11)}_{20}$	=	(32124 40751)(38698 35264)(23 58125)
	=	2 93153 05663 14310 15219 20000

$S(1/j)$ is seen to be an even function of $1/j$ which equals π for $1/j = 0$. Therefore we expect an m-point Lagrange polynomial approximation for variable $1/j^2$ to be considerably more accurate than a polynomial in $1/j$. Following are the values of the semi-perimeters $j \sin \alpha$ to 25D, which were obtained from a table of $\sin \alpha$ to 30D originally published by Herrmann [8]. For $j = 4(1)6, 9, 10$, $\sin \alpha$ was copied from Herrmann's table, and for $j = 7, 8$, $\sin \alpha$ was computed by Taylor's theorem employing Herrmann's entries as key values:

j	S_j: Semi-perimeter $= j \sin \alpha$
4	2.82842 71247 46190 09760 33774
5	2.93892 62614 62365 64584 35298
6	3.00000 00000 00000 00000 00000
7	3.03718 61738 22906 84333 03783
8	3.06146 74589 20718 17382 76799
9	3.07818 12899 31018 59739 68965
10	3.09016 99437 49474 24102 29342

In the above values of S_j, as well as S_j given in the other examples, the accuracy of the last few places, although highly probable, is still not absolutely guaranteed. However, in every example the values of S_j are certainly correct up to the number of places needed to guarantee that the "computational error" in the final answer (which is due to the error in the S_j multiplied by the extrapolation coefficients $A_{10,j}^{(m)}$) is appreciably less than the deviation of the answer from the true value. This latter "truncating error" is thus made to stand out clearly, and it indicates the theoretical accuracy of the extrapolation formula, regardless of the number of places carried in the work. In practice we do not often know at the outset of an example just how many places are needed in the S_j to assure us that the computing error will be dominated by the truncating error. Sometimes when the theoretical accuracy turns out to be unexpectedly fine, the example must be done again, carrying more places, to prevent the computing error from obscuring the truncating error.

The results of the extrapolations employing (1) or (2), for $m = 7$, gave for π, (whose true value to 20D is 3.14159 26535 89793 23846), the answer 3.14159 26535 89793 179 ... which is correct to within a unit in the 16th decimal. The extent of the improvement over the earlier $(1/j)$-extrapolation formulas is apparent from the result of 3.14159 280 ... obtained by the corresponding 7-point $(1/j)$-extrapolation coefficients, which deviates from π by $1\frac{1}{2}$ units in the 7th decimal. In other words, the error in the use of this newer formula is only around $0.4 \cdot 10^{-9}$ of that in the older one. The greater power of this newer method in this present example may be further illustrated even for $m = 4$, where $(1/j^2)$-extrapolation yields 3.14159 2650 ..., or accuracy to around $\frac{1}{3}$ of a unit in the 8th decimal, whereas the corresponding 4-point $(1/j)$-extrapolation formula gives no better than 3.1411 ..., which is off by $\frac{1}{2}$ of a unit in the 3rd decimal. In fact, the answer even by 2-point $(1/j^2)$-extrapolation, namely 3.1413 ..., is still better than the above 3.1411. ...

It is interesting to note that the use of $(1/j^2)$-extrapolation on the semi-perimeters gives this great improvement only for the *inscribed* polygons, and it will not work well for the *circumscribed* polygons, upon which it was also tried. A reason that would lead us to expect poor extrapolation results, even though the corresponding semi-perimeter $j \tan \alpha$ is still an even function of $1/j$, is that the series for $\tan \alpha$ converges poorly by comparison with $\sin \alpha$. Thus for $\alpha = \pi/4$, occurring in $S_j = S_4$, the remainder after the term involving the sixth power of $1/j^2$, is considerably greater for $j \tan \alpha$, so that the use of (1) or (2) for $m = 7$ is not nearly so good as for $j \sin \alpha$.

B. *Example* 2. The sequence for Euler's constant

$$\gamma = \lim_{j\to\infty}\left\{\sum_{r=1}^{j}(1/r) - \log j\right\} = 0.57721\ 56649\ 01532\ 86061 \text{ to 20D}$$

has been treated earlier by $(1/j)$-extrapolation ([1], p. 358). Applying (1) or (2), for seven points, directly to $S_j = \sum_{r=1}^{j}(1/r) - \log j$ yields the very inaccurate 0.593, the reason being that S_j does not behave like an even function of $1/j$. The older $(1/j)$-extrapolation formulas, employing $j_0 = 10$, gave 0.57721 41 ... and 0.57721 56695 ... by the 4- and 7-point formulas with respective errors of around $1\frac{1}{2}\cdot 10^{-6}$ and $\frac{1}{2}\cdot 10^{-8}$. To improve upon these results we must modify our S_j sequence into an even function of $1/j$ having the same limit γ. This is easily accomplished by replacing the last $1/r$ in the summation, namely $1/r = 1/j$, by half its value, or $1/2j$. At first sight there is an apparent motivation in that the new finite summation is suggestive (at one end anyhow) of the more accurate trapezoidal rather than rectangular approximation to the integral $\int_1^j (1/r)\,dr$. This trapezoidal motivation happens to lead to the correct choice in this present example, but in general it does not yield a sequence that is even in $(1/j)$. The true motivation lies in the Euler-Maclaurin summation formula applied to $\log j$. The general formula is expressible as

$$(5)\quad \frac{1}{w}\int_a^{a+jw} f(x)dx = \left(\frac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{j-1} + \frac{1}{2}f_j\right) - \frac{w}{12}(f_j' - f_0') + \frac{w^3}{720}(f_j''' - f_0''') - \frac{w^5}{30240}(f_j^{v} - f_0^{v}) + \cdots [9].$$

Now (5) does not denote a complete equality, since the Euler-Maclaurin formula is an asymptotic expression that is given with a remainder term. Employing (5) heuristically for $w = 1$, $a = 1$ and $f(x) = 1/x$, the right member of (5), exclusive of the $(\frac{1}{2}f_0 + f_1 + \cdots + f_{j-1} + \frac{1}{2}f_j)$ and an undisclosed remainder term, is an even function of $1/(j + 1)$, from which, replacing j by $j - 1$,

$$\int_1^j (1/x)dx - \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j-1} + \frac{1}{2j}\right)$$

is an even function of $1/j$, so that the same is true of the sequence

$$S_j' \equiv \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j-1} + \frac{1}{2j}\right) - \log j$$

whose limit, as $j \to \infty$, is also equal to γ.*

Since the older m-point $(1/j)$-extrapolation formula is linear in S_j (or S_j') and

* The reader is cautioned that the above heuristic demonstration is not to be understood as a proof that we have a convergent infinite series in $(1/j^2)$ from which we can "prove" that the "constant" term in S_j' is γ by taking the limit as $j \to \infty$. The fallacy there would be in that there is no "constant" term because the $f_0, f_0', f_0''', \cdots$ terms in (5) yield for $f(x) = 1/x$ a divergent sequence. Actually S_j' is defined only up to any fixed order derivative, say $f_{j-1}^{(p)}$, and it then consists of terms in $1/j^2$, constant terms and an integral formula for the remainder.

yields exactly zero for any polynomial in $1/j$ having no constant term, up to the $(m-1)$th degree, the above-mentioned 4- and 7-point results will not be changed by use of S_j' instead of S_j. But the improvement is very noticeable when S_j' is employed with $(1/j^2)$-extrapolation. Following are the terms in the modified sequence S_j' to 20D:

j	$S_j' = \sum_{r=1}^{j-1} \frac{1}{r} + \frac{1}{2j} - \log j$
4	0.57203 89722 13442 71450
5	0.57389 54208 99232 95873
6	0.57490 71974 38611 66585
7	0.57551 84223 73258 12347
8	0.57591 56011 77306 92889
9	0.57618 81210 76479 02991
10	0.57638 31609 74208 28424

The use of the 7-point formula in (1) or (2), where $j_0 = 10$, upon S_j', gave an answer of 0.57721 56649 0143 ... which is correct to a unit in the 13th decimal (i.e., 5 places more than $(1/j)$-extrapolation). Use of just the 4-point formula in (1) or (2) gave an answer as good as 0.57721 56647 5 ... which is correct to within $1\frac{1}{2}$ units in the 10th decimal (i.e., 4 places more than $(1/j)$-extrapolation).

C. *Example* 3. A different type of sequence is encountered in the evaluation of the definite integral $\int_0^1 \frac{1}{1+x}\,dx = \log 2$, whose value to 20D is 0.69314 71805 59945 30942. One obvious sequence to consider is S_j which is formed by dividing the interval (0, 1) into j equally spaced intervals and letting S_j be the sum of the rectangles of height $1/[1 + (r-1)/j]$ and width $1/j$, for $r = 1(1)j$, but that fails to behave as an even function of $1/j$. However, the trapezoidal rule, or

$$S_j' = \frac{1}{j}\left(\frac{1}{2} + \frac{1}{1+1/j} + \frac{1}{1+2/j} + \cdots + \frac{1}{1+(j-1)/j} + \frac{1}{4}\right),$$

according to the Euler-Maclaurin formula (5), where now $w = 1/j$, $a = 0$, and both $f_j^{(p)}$ is fixed as well as $f_0^{(p)}$, being at the endpoints 1 and 0, is seen to have a truncating error that is formally a series in $1/j^2$. The values of S_j', in either exact form, or to 20D, are as follows:

j	$S_j' = \frac{1}{j}\left(\frac{1}{2} + \sum_{r=1}^{j-1} \frac{1}{1+r/j} + \frac{1}{4}\right)$
4	1171/1680 = 0.69702 38095 23809 52381
5	1753/2520 = 0.69563 49206 34920 63492
6	9631/13860 = 0.69487 73448 77344 87734
7	2 50241/3 60360 = 0.69441 94694 19469 41947
8	2 00107/2 88288 = 0.69412 18503 71850 37185
9	5 66803/8 16816 = 0.69391 76020 05837 29995
10	1615 04821/2327 92560 = 0.69377 14031 75427 94323

The 4- and 7-point $(1/j)$-extrapolation, $j_0 = 10$, gave values of 0.69314 86 ... and 0.69314 7176 ... , correct to $1\frac{1}{2}$ units in the 6th decimal and $\frac{1}{2}$ unit in the 8th decimal respectively. The $(1/j^2)$-extrapolation was performed for every m-point formula from $m = 2$ through $m = 7$, with the following results:

m	value of S	deviation	m	value of S	deviation
2	0.69314 81...	10^{-6}	5	0.69314 71805 67...	10^{-11}
3	0.69314 7188...	10^{-8}	6	0.69314 71805 6054...	$\frac{4}{5}\cdot 10^{-12}$
4	0.69314 71807 1...	$1\frac{1}{2}\cdot 10^{-10}$	7	0.69314 71805 60046...	10^{-13}

The improvement over $(1/j)$-extrapolation in the 4- and 7-point results is by four and five places respectively.

D. *Example* 4. A somewhat more sophisticated application of $(1/j^2)$-extrapolation is in the summation of the series for Catalan's constant, or

$$T_2 = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots.$$

H. T. Davis [10] gives a full discussion of Catalan's constant, including an account of the earlier work of J. W. L. Glaisher, and he also reprints Glaisher's 32-decimal value of $T_2 = 0.91596\ 55941\ 77219\ 01505\ 46035\ 14932\ 38$. Since the series for T_2 is absolutely convergent, it may be regrouped as

$$T_2 = 1 + \left(-\frac{1}{3^2} + \frac{1}{5^2}\right) + \left(-\frac{1}{7^2} + \frac{1}{9^2}\right) + \cdots + \left(-\frac{1}{(4r-1)^2} + \frac{1}{(4r+1)^2}\right) + \cdots.$$

The general term u_r, $r > 0$, of T_2 is equal to $\dfrac{-16r}{(16r^2-1)^2}$, which is an odd function of r or $1/r$. Thus, as in the preceding example, employing (5) with $w = a = 1$, the modified sum

$$S_j' = S_j - \tfrac{1}{2} u_j = 1 - \sum_{r=1}^{j-1} \frac{16r}{(16r^2-1)^2} - \frac{1}{2}\left(\frac{16j}{(16j^2-1)^2}\right)$$

is again seen to be formally an even function of $1/j$, having the same limit S which is approached by S_j.* The values of S_j' to 20D are as follows:

	$S_j' = 1 - \sum_{r=1}^{j-1} \frac{16r}{(16r^2-1)^2} - \frac{1}{2}\left(\frac{16j}{(16j^2-1)^2}\right)$
4	0.91798 69831 73330 85103
5	0.91724 36100 54163 02747
6	0.91684 71757 66868 06945
7	0.91661 06554 47552 03321
8	0.91645 81601 71966 79489
9	0.91635 40724 61230 31205
10	0.91627 98501 91732 37910

* Although in Example 4 we know the explicit formula for $\int_1^j f(x)\,dx$, we may expect this principle to be applicable also in cases where $\int_1^j f(x)\,dx$, $f(x)$ odd, or for that matter also $f_j^{(p)}$ for odd p, is not known in closed form, and where S_j' may still be regarded formally as a series in $1/j^2$.

Use of the 7-point $(1/j)$-extrapolation, $j_0 = 10$, upon either S_j or S_j', while not identical in accuracy, because now the difference of $\frac{1}{2}(16j/(16j^2 - 1)^2)$ is no longer an exact polynomial in $1/j$, gave results very close to each other, namely 0.91596 55973 ... and 0.91596 55980 ... with respective deviations of $\frac{1}{3} \cdot 10^{-8}$ and $\frac{2}{3} \cdot 10^{-8}$. The use of $(1/j^2)$-extrapolation, i.e., (1) or (2), for $m = 7$, while giving the poorer answer of 0.91596 74 ... with a deviation of $2 \cdot 10^{-6}$ in working with the S_j sequence (as was to be expected), gave upon working with the S_j' sequence the highly accurate 0.91596 55941 7714 ... , which is correct to $\frac{4}{5} \cdot 10^{-13}$, showing a gain in accuracy of around 5 places.

5. Formulas for Partial Summation. Given the first ten terms of a sequence S_j which behaves as an even function of $1/j$, we might wish to find by $(1/j^2)$-extrapolation S_n, $n > 10$, instead of going to the limit as $j \to \infty$. The purpose of this section is to improve what was accomplished in [2] where just $(1/j)$-extrapolation was employed. The m-point formula for S_n which occurs usually as a sum of the form $\sum_{r=0}^{n} u_r$, is obtained by setting $x = 1/n^2$ in the Lagrange interpolation coefficients whose fixed points are $1/j_0^2$, $1/(j_0 - 1)^2, \cdots, 1/(j_0 - m + 1)^2$. In the present instance, in order to avoid too much tabulation, since now besides j_0 and m, n is also a variable, being no longer just ∞, we consider a choice of j_0 and m which shall be suitable for most problems and which shall give a substantial increase in accuracy over the $(1/j)$-extrapolation formulas previously given which were based upon $j_0 = 10$ and $m = 7$ [2]. Thus it is natural to take $j_0 = 10$ and $m = 7$ for these present formulas also. The argument $n = 11(1)25(5)100, 200, 500, 1000$, and all coefficients are given exactly. This range of n is not quite so extensive as in the previous paper because the arguments $1/j^2$ in place of $1/j$, $j = 4, 5, \cdots, 10, n$, increase the labor in computing the exact forms, which also have considerably greater bulk in figures. To find $S_n \equiv S(n)$, we employ the extrapolation formula in the following form:

$$S(n) = \sum_{j=4}^{10} A_j(n) S_j . \tag{6}$$

Every set of coefficients $A_j(n)$ is given in the exact fractional form of $C_j(n)/D(n)$ where $D(n)$ is the least common denominator for each n. Thus it may help the computer to have

$$S(n) = (1/D(n)) \sum_{j=4}^{10} C_j(n) S_j . \tag{7}$$

In (6) and (7) the $j_0 = 10$ is understood as well as $m = 7$. When also n is understood, we may employ for (7) the somewhat more concise

$$S_n = (1/D) \sum_{j=4}^{10} C_j S_j . \tag{7'}$$

In (7), or (7′), the $D(n)$, or D, is given also in the form of factors having no more than 10 digits, exclusive of terminal 0's, to facilitate the divisions on a ten-bank desk calculator. The $C_j(n)$ and $D(n)$ are shown in Table 4.

TABLE 4 $A_j(n) \equiv C_j(n)/D(n)$

n	$C_4(n)$	$-C_5(n)$	$C_6(n)$	n
11	4 33523 26144	276 00097 65625	4690 09522 11456	11
12	2963 53792	1 85546 87500	30 76065 53856	12
13	41 05720 29952	2539 20898 43750	41364 52401 80736	13
14	3 11594 84416	190 91796 87500	3070 02326 69664	14
15	73 47078 43072	4469 00781 25000	71153 79028 62336	15
16	299 91003 75040	18137 20703 12500	2 86535 50491 68640	16
17	98713 24662 98880	59 41749 02343 75000	932 85993 94858 59840	17
18	202 53664 87040	12144 04296 87500	1 89697 11668 10720	18
19	5 14830 54912 30720	307 69771 72851 56250	4786 28907 41312 96256	19
20	33 51761 38752	1997 80546 87500	30967 19121 46176	20
21	2980 34491 55584	1 77232 05566 40625	27 39045 93898 29120	21
22	2557 65176 64768	1 51794 04296 87500	23 39958 38615 45424	22
23	11486 99740 73344	6 80571 09375 00000	104 68292 53359 69792	23
24	517 62000 03584	30621 58203 12500	4 70115 09625 81248	24
25	18277 34070 23104	10 79839 46875 00000	165 50605 49407 21152	25
30	47 47239 09632	2791 67890 62500	42538 14131 63616	30
35	19 94826 94088 78592	1169 85066 73242 18750	17764 31856 19235 88096	35
40	55614 22677 27872	32 55672 19140 62500	493 29102 44672 84736	40
45	1 10518 08503 88992	64 61947 71523 43750	977 63645 22693 67296	45
50	314 67009 69601 92512	18382 81470 45703 12500	2 77821 11083 92383 05056	50
55	12260 77786 28307 51744	7 15812 21397 61250 00000	108 09688 47762 36090 65472	55
60	2 10143 48218 04032	122 62754 30078 12500	1850 73872 20503 06816	60
65	1 99146 82341 59643 68896	116 16710 49246 78906 25000	1752 43060 92065 47783 18848	65
70	2 63102 15141 94944	153 42833 85078 12500	2313 69106 74489 04572	70
75	27053 53085 29138 40128	15 77253 55481 87207 03125	237 77936 68838 83951 47264	75
80	1138 17615 39236 16768	66344 10184 61484 37500	9 99933 39136 72116 19584	80
85	1 98436 64585 46096 70144	115 64971 16095 43261 71875	1742 71873 87812 89262 36672	85
90	21 91251 31803 36128	1276 89772 02734 37500	19238 30729 92601 52264	90
95	17 01791 62058 22944 21504	991 56366 90583 06093 75000	14937 24378 14420 81022 44352	95
100	27624 63417 03513 53856	16 09416 13444 06640 62500	242 41878 50975 44215 71328	100
200	17 09077 65240 57655 37792	995 03809 78214 71484 37500	14975 38393 17401 02043 49696	200
500	82 10561 52187 30411 82835 34336	4779 34787 72793 93251 99609 37500	71912 84793 65397 78913 65856 13568	500
1000	186 00631 75612 24045 59830 53028 72064	10827 09010 57365 44610 56489 80078 12500	1 62905 31822 30989 18580 33518 15589 49632	1000

TABLE 4—*Continued*

n	$-C_7(n)$	$C_8(n)$	$-C_9(n)$	n
11	32215 59596 03275	1 05827 99417 34400	1 76151 30190 31997	11
12	203 46692 18547	628 35371 54048	932 01747 03873	12
13	2 66745 13455 15117	7 92802 85920 78848	11 04949 07557 46163	13
14	19442 06520 05475	56307 02125 05600	75493 41510 13713	14
15	4 44575 22425 25195	12 63888 61612 85120	16 50602 94005 59083	15
16	17 71857 77781 80125	49 67921 56241 92000	63 66611 34021 56463	16
17	5721 68313 61299 25965	15871 91324 13418 53696	20054 82572 16792 85845	17
18	11 55895 58305 65507	31 79469 79994 82880	39 73673 74457 90385	18
19	29008 53338 24768 97375	79250 25550 94548 48000	98190 51035 15985 81345	19
20	1 86850 45656 98995	5 07620 03723 05920	6 24536 43402 04353	20
21	164 65175 07237 67125	445 23348 97725 44000	544 62440 88208 18755	21
22	140 21515 98577 29411	377 66930 76995 60448	459 75489 79673 51217	22
23	625 57532 85960 23526	1679 36327 39394 84672	2036 05740 52839 83961	23
24	28 02756 84854 84925	75 02445 18166 52800	90 64378 27268 21583	24
25	984 70190 61233 70365	2629 30613 59658 59840	3167 20040 82195 30606	25
30	2 51281 64849 05545	6 65213 12473 90720	7 93146 86729 95923	30
35	1 04501 17314 93605 18895	2 75278 96205 65685 04320	3 26324 47322 86967 02113	35
40	2894 18470 98839 01945	7600 17480 22670 13120	8976 92198 97040 61283	40
45	5725 63039 44139 07145	15004 08135 79219 76320	17679 14149 53778 25463	45
50	16 25027 50478 80584 46845	42 52100 27375 85566 51520	50 01673 53498 68138 89743	50
55	631 69035 64270 86020 26140	1651 10814 77140 03986 02240	1939 75352 82211 59219 40191	55
60	10807 62370 15787 49045	28225 74314 90409 26720	33129 03631 38264 76023	60
65	10227 95435 17712 52380 23885	26694 90068 27877 68605 08160	31309 54142 10812 86737 65769	65
70	13497 88556 74941 83765	35211 73936 27440 74240	41275 01812 78067 39991	70
75	1386 70343 48980 87560 96555	3616 01493 44620 31514 82880	4236 72540 71523 70324 98717	75
80	58 29847 70830 98366 73455	151 97106 54462 40406 73280	177 99104 37154 06907 98677	80
85	10158 07546 77808 62362 83515	26472 62121 05369 52358 50240	30995 56816 28717 48010 98541	85
90	1 12115 49286 84852 73055	2 92113 94047 04369 86880	3 41934 10964 90406 39717	90
95	87035 63171 86878 79918 38615	2 26725 41556 61183 16566 11840	2 65335 47692 58504 18153 45331	95
100	1412 31425 76716 92152 80985	3678 43631 10728 17912 21760	4304 05053 46689 91848 03459	100
200	87160 02410 86950 16496 14895	2 26755 16422 07280 30737 20320	2 64979 65188 21093 77455 55413	200
500	4 18434 18289 57096 52184 08777 48535	10 88253 25190 80068 46170 98172 82560	12 71247 00401 14305 40998 13477 44429	500
1000	9 47848 63514 29742 74436 93853 54534 22965	24 65030 27747 54660 71683 78425 93408 61440	28 79387 20044 35172 31507 54843 52414 47671	1000

TABLE 4—*Continued*

n	$C_{10}(n)$	$D(n)$	n
11	1 43000 00000 00000	44879 52578 71103 = (1 90333) (23579 47691)	11
12	568 75000 00000	90 55414 51776 = (1 31072) (69 08733)	12
13	6 00600 00000 00000	60575 02131 84506 = (7 42586) (8157 30721)	13
14	38542 96875 00000	2796 73094 52992 = (14144) (19773 26743)	14
15	8 10409 60000 00000	45878 30507 81250 = (19 53125) (2348 96922)	15
16	30 43906 25000 00000	1 42056 90230 86592 = (1 65376) (85899 34592)	16
17	9406 54000 00000 00000	376 37396 52504 25606 = (539 54566) (69757 57441)	17
18	18 37214 84375 00000	64870 92642 37248 = (1 86048) (34867 84401)	18
19	44894 85000 00000 00000	1429 80143 77167 40006 = (15995 59954) (8938 71739)	19
20	2 83032 75000 00000	8268 80000 00000 = 82688 × 10^{9}	20
21	245 05000 00000 00000	6 65527 25103 02199 = (33 65793) (19773 26743)	21
22	205 64098 24218 75000	5 24745 22458 77512 = (22 25432) (23579 47691)	22
23	906 19200 00000 00000	21 91462 44320 20321 = (64 36343) (34048 25447)	23
24	40 17406 25000 00000	92727 44466 18624 = (69 08733) (1342 17728)	24
25	1398 70016 00000 00000	30 99441 52832 03125 = (50 78125) (61035 15625)	25
30	3 46064 46875 00000	6643 01250 00000 = 664 30125 × 10^{6}	30
35	1 41427 09838 40000 00000	2494 83022 65195 31250 = (12617 18750) (19773 26743)	35
40	3874 39102 05000 00000	64 74956 80000 00000 = 647 49568 × 10^{9}	40
45	7609 15064 00000 00000	122 58226 40976 56250 = (351 56250) (34867 84401)	45
50	21 48569 72361 28125 00000	33721 92382 81250 00000 = (5525) (61035 15625) × 10^{6}	50
55	832 08552 92288 00000 00000	12 81116 27096 61894 53125 = (19 53125) (30 59969) (2143 58881)	55
60	14196 11500 45000 00000	215 41075 20000 00000 = 2154 10752 × 10^{9}	60
65	13405 46811 80201 60000 00000	201 12800 04714 18007 81250 = (19 53125) (126 23962) (8157 30721)	65
70	17660 88429 01562 50000	262 61370 80546 87500 = (1328 12500) (19773 26743)	70
75	1811 88546 04312 00000 00000	26 74891 94869 99511 71875 = (3 90625) (11 21931) (61035 15625)	75
80	76 08787 21924 75000 00000	1 11669 14969 60000 00000 = (13) (85899 34592) × 10^{9}	80
85	13245 41781 61905 60000 00000	193 44878 97051 94082 03125 = (19 53125) (241 37569) (4103 38673)	85
90	1 46077 09919 92812 50000	2124 75924 43593 75000 = (6093 75000) (34867 84401)	90
95	1 13325 68971 63248 00000 00000	1642 69466 64944 16367 18750 = (39 06250) (470 45881) (8938 71739)	95
100	1837 89011 11730 25000 00000	26 56250 00000 00000 00000 = 2 65625 × 10^{16}	100
200	1 12987 07515 96336 75000 00000	1600 00000 00000 00000 00000 = 16 × 10^{22}	200
500	5 41841 72385 59042 30937 75000 00000	7629 39453 12500 00000 00000 00000 = (2048) (61035 15625)^{2} × 10^{6}	500
1000	12 27206 94867 61018 36245 33412 25000 00000	17265 62500 00000 00000 00000 00000 00000 = 172 65625 × 10^{27}	1000

The coefficients $A_j(n) \equiv C_j(n)/D(n)$ were calculated directly from the formula

$$(8) \qquad A_j(n) = \frac{j^{12}}{n^{12}} \frac{\prod_{k=4}^{10}{}' (n^2 - k^2)}{\prod_{k=4}^{10}{}' (j^2 - k^2)},$$

where $k = j$ is absent from $\prod'$. Both the calculation of $A_j(n)$ and the determination of $D(n)$ was facilitated by expressing each of the factors in the right member of (8) in terms of powers of primes.

To facilitate the use of (8) for desired values of n other than in this present table, we notice that we may express $A_j(n)$ as

$$(9) \qquad A_j(n) = B_j \cdot \frac{\prod_{k=4}^{10}{}' (n^2 - k^2)}{n^{12}}, \quad \text{where}$$

$$(10) \qquad B_j = \frac{j^{12}}{\prod_{k=4}^{10}{}' (j^2 - k^2)}$$

is independent of n. The exact, as well as 30 decimal, values of the fundamental quantities B_j are given in the following Schedule 1.

SCHEDULE 1

j	$B_j = j^{12} \Big/ \prod_{k=4}^{10}{}' (j^2 - k^2)$	
4	$\frac{65536}{60\ 81075}$ =	0.01077 70418 88152 99926 41103 75221
5	$-\frac{97\ 65625}{155\ 67552}$ =	−0.62730 63998 75844 32028 87647 33209
6	$\frac{2\ 36196}{25025}$ =	9.43840 15984 01598 40159 84015 98402
7	$-\frac{1\ 38412\ 87201}{2520\ 46080}$ =	−54.91570 11329 03951 53140 25117 94669
8	$\frac{2684\ 35456}{18\ 79605}$ =	142.81482 33272 41627 89522 26664 64497
9	$-\frac{3\ 13810\ 59609}{1881\ 15200}$ =	−166.81830 92541 16626 40764 80794 74705
10	$\frac{390\ 62500}{5\ 49423}$ =	71.09731 48193 65042 96325 41775 64463

6. Illustrations of Partial Summation.

A. *Example* 5. Suppose that in Example 1 above, instead of passing to the limit as $j \to \infty$ to obtain π, we wished to calculate S_{20}, or the semi-perimeter of a 20-sided regular polygon from the semi-perimeters of the 4- through 10-sided regular polygons. We have $S_{20} = 20 \sin 9°$, whose value to 20D is 3.12868 93008 04617

38020. Using the same values of S_j as in Example 1, we find by the earlier method of $(1/j)$-extrapolation [2] $S_{20} = 3.12868\ 93076\ \ldots$ which is correct to around a unit in the 8th decimal. But use of the present tables for $(1/j^2)$-extrapolation in (6) or (7), for $n = 20$, yields the highly accurate $S_{20} = 3.12868\ 93008\ 04617\ 359\ \ldots$, correct to about 2 units in the 17th decimal, showing a gain of around 9 places.

B. *Example* 6. As an illustration of a different type of problem that does not correspond to one in complete summation, consider the case where from the first few known zeros of some higher mathematical function, we wish to obtain the value of some later zero, say the nth. As will be seen below, there are circumstances when it is preferable to choose as the sequence S_j, $j \leqq j_0$, from which to extrapolate, some suitable even function of $1/j$ which may not be a function of the jth root, and yet from S_j, $j > j_0$, the jth root, is readily obtainable.

Consider the problem of finding the later zeros of the spherical Bessel functions $J_{2m+\frac{1}{2}}(z)$ from either tabulated earlier zeros or some other suitable function of m. In the general asymptotic formula for $z_\nu^{(n)}$, the nth zero of $J_\nu(z) \cos \alpha - Y_\nu(z) \sin \alpha$, namely,

$$(11)\qquad z_\nu^{(n)} = \left(n + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha - \frac{4\nu^2 - 1}{8\{(n + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha\}} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384\{(n + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha\}^3} - \cdots [11],$$

set $\alpha = 0$ and $\nu = 2m + \frac{1}{2}$. Then from (11) it is apparent that

$$(12)\qquad S_{n+m} \equiv (n + m)[z_{2m+\frac{1}{2}}^{(n)} - (n + m)\pi]$$

has a formal expansion in even powers of $1/(n + m)$, which could serve as the basis of an extrapolation formula.

However, after searching for ready-made tables of $z_{2m+\frac{1}{2}}^{(n)}$, none were found capable of testing the full potentialities of Table 4. To avoid extra labor, we shall first illustrate this principle of $(1/\nu^2)$-extrapolation with a smaller example limited to the available published 6D values of $z_{9/2}^{(n)}$ as far as $n = 6$ [12]. The problem is to calculate $z_{9/2}^{(n)}$ for $n = 6$, whose published value is 24.727566, from the four preceding values of $z_{9/2}^{(2)} = 11.704907$, $z_{9/2}^{(3)} = 15.039665$, $z_{9/2}^{(4)} = 18.301256$ and $z_{9/2}^{(5)} = 21.525418$. In other words, since $m = 2$, the problem is to find S_8 from S_4, S_5, S_6 and S_7, from which $z_{9/2}^{(6)}$ is found from (12). From (8), with $\prod'^{10}_{k=4}$ replaced by $\prod'^{7}_{k=4}$, we find $A_4(8) = -\frac{91}{2112}$, $A_5(8) = \frac{546875}{1081344}$, $A_6(8) = -\frac{19683}{11264}$ and $A_7(8) = \frac{823543}{360448}$ from which $S_8 = \sum_{j=4}^{7} A_j(8)S_j = -3.241393$. Finally, from (12), $z_{9/2}^{(6)}$ is found to be 24.727567, which deviates by only 10^{-6} from the published value.* Comparing with $(1/\nu)$-extrapolation based upon those same values of $S_4 - S_7$, and where $A_4(8) = -\frac{1}{8}$, $A_5(8) = \frac{125}{128}$, $A_6(8) = -\frac{81}{32}$, $A_7(8) = \frac{343}{128}$, we find $S_8 = -3.241225$, from which $z_{9/2}^{(6)}$ is found to be 24.727588, which deviates by 0.000022 from the published value.

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* Since we started with 6D values, it is not possible to estimate from this example the possibly higher theoretical accuracy in $(1/\nu^2)$-extrapolation, which is just the truncation error when the example is done with a sufficiently large number of places both initially and in the course of the work.

For a similar example employing Table 4, and revealing the full accuracy of (6) or (7), we choose a modification of S_{n+m}, say $\bar{S}_{n+m}$, where

$$\bar{S}_{n+m} = (n + m)[\bar{z}_{2m+\frac{1}{2}}^{(n)} - (n + m)\pi], \tag{13}$$

and where now $\bar{z}_{2m+\frac{1}{2}}^{(n)}$, instead of being the nth zero of $J_{2m+\frac{1}{2}}(x)$, is defined as a preassigned number of terms of the right member of (11) (for $\alpha = 0$, $\nu = 2m + \frac{1}{2}$) which is the same for every n. For the lowest values of n, there will be considerable deviation between the true value of the root $z_{2m+\frac{1}{2}}^{(n)}$ and the function $\bar{z}_{2m+\frac{1}{2}}^{(n)}$ which is $(n + m)\pi$ + an exact odd polynomial in $1/(n + m)$, making $\bar{S}_{n+m}$ an exact even polynomial in $1/(n + m)$. But at the inconvenience of having to compute $\bar{S}_{n+m}$ for the initial values of n, we may employ (6) or (7) to extrapolate for $\bar{S}_{n+m}$ for some larger n to get $\bar{z}_{2m+\frac{1}{2}}^{(n)}$ which will agree with the true value of the root $z_{2m+\frac{1}{2}}^{(n)}$ to very high accuracy. Taking (11) as far out as $1/\{(n + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha\}^9$, we have for $\alpha = 0$, $\nu = 2m + \frac{1}{2}$ and $\mu \equiv 4\nu^2 = (4m + 1)^2$,

$$\begin{aligned} \bar{S}_{n+m} = &-\frac{\mu - 1}{2^3\pi} - \frac{(\mu - 1)(7\mu - 31)}{3 \cdot 2^7\pi^3(n + m)^2} - \frac{(\mu - 1)(83\mu^2 - 982\mu + 3779)}{15 \cdot 2^{10}\pi^5(n + m)^4} \\ &- \frac{(\mu - 1)(6949\mu^3 - 153855\mu^2 + 1585743\mu - 6277237)}{105 \cdot 2^{15} \cdot \pi^7 \cdot (n + m)^6} \\ &- \frac{(\mu - 1)(70197\mu^4 - 24\,79316\mu^3 + 480\,10494\mu^2 - 5120\,62548\mu + 20921\,63573)^*}{40320 \cdot 2^{11} \cdot \pi^9(n + m)^8}. \end{aligned} \tag{14}$$

Suppose that the problem is to calculate the 14th zero of $J_{5/2}(z)$ or $z_{5/2}^{(14)}$. Then $m = 1$, and we should want to find $\bar{S}_{15}$ using Table 4 upon $\bar{S}_4 - \bar{S}_{10}$, after which we obtain $\bar{z}_{5/2}^{(14)}$ from (13).** From (14) and then (13), $\bar{z}_{5/2}^{(14)}$ which is equal to $z_{5/2}^{(14)}$ to around 14D, is found to be 47.06014 16127 6054. A quick examination of the ratios of successive terms in (14) indicates without having to compute the $1/(n + m)^{10}$ term that, to 14D, $z_{5/2}^{(14)}$ is actually 47.06014 16127 6053. Following are the calculated values of $\bar{S}_j$, for $j = n + 1 = 4(1)10$, to 16D (last figure approximate):

j	$\bar{S}_j$
4	−0.97371 85140 72535 8
5	−0.96680 12788 75286 8
6	−0.96311 73960 26803 8
7	−0.96092 04667 12625 8
8	−0.95950 42113 72512 2
9	−0.95853 75688 13022 8
10	−0.95784 82845 01448 5

Employing the older $(1/j)$-extrapolation, we find $\bar{S}_{15} = -0.95622\ 28677\ 507\ \ldots$ and from (13), $\bar{z}_{5/2}^{(14)} = 47.06014\ 16126\ 6\ \ldots$ which agrees with the true value of

* The coefficients through $1/(n + m)^6$ are from Watson [11], and the coefficient of $1/(n + m)^8$ is from Bickley and Miller [13].

** This particular problem could, of course, be set up equally efficiently computationwise by writing $\bar{S}_{n+1} = a_0 + a_1/(n + 1)^2 + \cdots + a_r/(n + 1)^{2r}$, where a_i is independent of n. But this present method works as long as we know somehow the values of $\bar{S}_j$.

$z_{5/2}^{(14)}$ to a unit in the 10th decimal (12th significant figure). But the $(1/j^2)$-extrapolation yields $\bar{S}_{15} = -0.95622\ 28662\ 9517 \ldots$ and from (13), $\bar{z}_{5/2}^{(14)} = 47.06014\ 16127\ 6055 \ldots$ which almost agrees with the true value of $z_{5/2}^{(14)}$ to 14 decimals (16 significant figures).

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1. H. E. Salzer, "A simple method for summing certain slowly convergent series," *J. Math. Phys.*, v. 33, 1955, p. 356–359.
2. H. E. Salzer, "Formulas for the partial summation of series," *MTAC*, v. 10, 1956, p. 149–156.
3. L. F. Richardson, "The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam," *Philos. Trans. Roy. Soc. London, Ser. A*, v. 210, 1910, p. 307–357.
4. L. F. Richardson & J. A. Gaunt, "The deferred approach to the limit," *Philos. Trans. Roy. Soc. London, Ser. A*, v. 226, 1927, p. 299–361.
5. G. Blanch, "On the numerical solution of parabolic partial differential equations," *J. Res. Nat. Bur. Standards*, v. 50, 1953, p. 343–356.
6. H. C. Bolton & H. I. Scoins, "Eigenvalues of differential equations by finite-difference methods," *Proc. Cambridge Philos. Soc.*, v. 52, pt. 2, 1956, p. 215–229.
7. M. G. Salvadori, "Extrapolation formulas in linear difference operators," *Proc. First U. S. Nat. Congress of Applied Mech.*, Edwards Bros., 1952, p. 15–18.
8. Herrmann, "Bestimmung der trigonometrischen Functionen aus den Winkeln und der Winkel aus den Functionen, bis zu einer beliebigen Grenze der Genauigkeit," *Kaiserliche Akademie der Wissenschaften*, Wien, *Mathematisch-naturwissenschaftliche Classe, Sitzungsberichte*, 1, pt. IV, 1848, p. 176–177. Reprinted in *Table of Sines and Cosines to Fifteen Decimal Places at Hundredths of a Degree*, *Nat. Bur. Standards. Appl. Math. Ser.* No. 5, 1949, p. 92–93.
9. E. Whittaker & G. Robinson, *The Calculus of Observations*, 4th edition, Blackie and Son, London, 1954, p. 135.
10. H. T. Davis, *Tables of the Higher Mathematical Functions*, v. II, Principia Press, Bloomington, Indiana, 1935, p. 282, 284–285, 304. Davis cites the earlier work of J. W. L. Glaisher in relation to Catalan's constant in *Mess. of Math.*, v. 6, 1876, p. 71–76, *Proc. London Math. Soc.*, v. 8, 1877, p. 200–201, *Mess. of Math.*, v. 42, 1913, p. 35–58.
11. G. N. Watson, *Theory of Bessel Functions*, 2nd edition, Cambridge University Press, 1952, p. 506.
12. NBS, *Tables of Spherical Bessel Functions*, v. II, New York, Columbia University Press, 1947, p. 318.
13. W. G. Bickley & J. C. P. Miller, "Notes on the evaluation of zeros and turning values of Bessel functions, II, The McMahon Series," *Phil. Mag.*, s. 7, v. 36, 1945, p. 125.