Numerical Integration Using Sums of Exponential Functions

By F. C. Ledsham

Let y be a function of the independent variable x, and let

$$q = \frac{dy}{dx}. \tag{1}$$

Suppose that the numerical values of q be known for the $n + 1$ discrete values of x given by $x = x_0 + rh$, where r runs through the integers from zero to n and h is some fixed increment. Let $f(x)$ be some function of x chosen to coincide with q at these particular x-values. For convenience we shall denote $q(x_0 + rh)$ by q_r, with a similar convention for other functions. In order to obtain a numerical estimate of the increment of y over any chosen range of x, say $y_k - y_m$, one may then use the approximation

$$y_k - y_m = \int_{x_0+mh}^{x_0+kh} q\, dx \cong \int_{x_0+mh}^{x_0+kh} f\, dx, \tag{2}$$

the accuracy of which will depend upon the choice of the function $f(x)$. Note that k and m may have any real values (positive, negative or zero) and, in particular, are not confined to the integers nor to the region from zero to n.

It is customary to take $f(x)$ to be the lowest order polynomial satisfying the conditions specified above. As is well known, it is not then necessary to determine this polynomial, and it is possible to write

$$\int_{x_0+mh}^{x_0+kh} f\, dx = h \sum_{r=0}^{n} A(k, m, n; r)\, f_r = h \sum_{r=0}^{n} A(k, m, n; r)\, q_r, \tag{3}$$

where the A's are constants satisfying the equation

$$\sum_{r=0}^{n} A(k, m, n; r) = k - m, \tag{4}$$

and which may be tabulated once and for all as functions of k, m, n and r.

If $m = 0$ and $k = n$, then this last procedure leads to the Newton-Cotes series of formulae, of which Simpson's rule (for $n = 2$) is probably the best known. If we put $k = m + 1$, and also let this quantity equal $n + 1$ or n, then we get, respectively, extrapolation and check formulae which may be used for step-by-step numerical integration of first order differential equations. Other combinations of m and k also lead to useful formulae, and a selection of these (for various values of n) is included in a paper by Bickley [1].

It has been pointed out by Greenwood [2] and by Brock and Murray [3], that a

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polynomial form of $f(x)$ does not always provide the best approximation to $q(x)$. In particular, these authors give examples in which $f(x)$ is better taken as a sum of exponential functions, in the form

$$f(x) = \sum_{i=0}^{n} a_i \exp(\alpha_i x), \tag{5}$$

where the α_i are chosen to suit the problem concerned, and may be complex or zero.

Greenwood considers two particular forms of equation (5), both involving real values only of the α_i. For his first type he puts $\alpha_i = i$; while, for the second, he puts $\alpha_i = i - \frac{1}{2}n$ where i, in both cases, runs through the integers from zero to n. The second (symmetric) case is only applied by Greenwood to even values of n. He also takes $m = 0$ and $k = n$, to produce formulae analogous to those of the Newton-Cotes series.

Brock and Murray are concerned with the step-by-step numerical integration of first order differential equations. They consider more general cases in which the α_i are complex, and tailored to fit the particular problem to hand—with the help of earlier (and less accurate) solutions of the differential equations concerned.

The papers of Greenwood and Brock and Murray both contain practical examples of uses of the ideas expressed above, and discuss the magnitudes of the errors involved.

Using equation (5), with particular values given to the coefficients α_i, we may obtain an equation of the form

$$\int_{x_0+mh}^{x_0+kh} f\,dx = \sum_{r=0}^{n} B(k, m, n; h, r) f_r = \sum_{r=0}^{n} B(k, m, n; h, r) q_r . \tag{6}$$

Unlike the corresponding equation (3), h does not occur naturally as a factor on the right hand side of this equation, and the coefficients B have to be recalculated for every change in this quantity. Incidentally, Brock and Murray, in their work, do take out a factor h—and consequently calculate coefficients equivalent to B/h in our notation.

If one of the α_i be zero, then we have the relationship

$$\sum_{r=0}^{n} B(k, m, n; h, r) = h(k - m), \tag{7}$$

corresponding to equation (4). If none of the α_i be zero, then this last equation does not hold exactly—though it remains approximately true, and serves as a useful check during the calculation of the B's.

If one of the α_i be zero then let this be α_s. We then have

$$\begin{aligned} \int_{x_0+mh}^{x_0+kh} f\,dx &= a_s h(k - m) \\ &\quad + \sum_{\substack{i=0 \\ (i \neq s)}}^{n} \frac{a_i}{\alpha_i} [\exp\{\alpha_i(x_0 + kh)\} - \exp\{\alpha_i(x_0 + mh)\}]. \end{aligned} \tag{8}$$

If none of the α_i be zero then the first term on the right hand side of this equation drops out, and the restriction $i \neq s$ is omitted from the summation.

From the $n + 1$ equations

$$f_r = q_r = \sum_{i=0}^{n} a_i \exp\{\alpha_i(x_0 + rh)\}, \tag{9}$$

the a_i, or rather $a_i \exp(\alpha_i x_0)$, may be found in terms of the q_r, enabling us to put equation (8) into the form of equation (6). In previous applications of these ideas [2, 3] this step, or its equivalent, has been carried out numerically. It is therefore considered that there may be some interest in analytical formulae into which one might substitute directly in order to obtain the required coefficients $B(k, m, n; h, r)$, such as those given below.

Write

$$\left.\begin{aligned} b_i &= a_i \exp(\alpha_i x_0), \\ t_i &= \exp(\alpha_i h) \quad , \end{aligned}\right\} \tag{10}$$

so that equations (9) may be written in the form

$$f_r = q_r = \sum_{i=0}^{n} b_i t_i^r, \qquad (r = 0, 1, \cdots n). \tag{11}$$

For any value of n, it may be verified that the solution of these $n + 1$ equations leads to the following symbolic equations for the b_i:

$$b_i = \frac{\prod_{j \neq i} (t_j - q)}{\prod_{j \neq i} (t_j - t_i)}, \tag{12}$$

where, after expansion, the powers of the q's are lowered to represent suffixes, and the term of the numerator originally independent of q is taken as the coefficient of q_0. For example, if $n = 2$ we would have

$$b_0 = \frac{(t_1 - q)(t_2 - q)}{(t_1 - t_0)(t_2 - t_0)} \quad \text{(symbolically)},$$

which would be interpreted as giving

$$b_0 = \frac{t_1 t_2 q_0 - (t_1 + t_2)q_1 + q_2}{(t_1 - t_0)(t_2 - t_0)},$$

and similarly for b_1 and b_2.

If, as in equation (8), α_s be zero then $a_s = b_s$ and that equation becomes

$$\int_{x_0+mh}^{x_0+kh} f\,dx = b_s h(k - m) + \sum_{\substack{i=0 \\ (i \neq s)}}^{n} \frac{b_i}{\alpha_i} (t_i^k - t_i^m). \tag{13}$$

Substituting into equation (13) the values of b_i calculated from equations (12) leads immediately to the required form (6). It will be noted that the b_i do not involve k and m, so that they do not have to be recalculated if integrations are required over ranges corresponding to more than one pair of values of these parameters.

If two of the α_i be complex conjugates, then so also will be the corresponding terms of the summation on the right hand side of equation (13). Instead of calculating both those terms completely, it is therefore only necessary to find the real part of one of them and then to double it to obtain the sum of the two.

Equations (12) and (13) have been used to recalculate some of the coefficients quoted in the papers mentioned above. The first example taken was Greenwood's symmetrical case with $n = 6$. The second was taken from the paper by Brock and Murray, with $n = 3$ and the α_i consisting of two pairs of complex conjugates. Using the full 10-figure capacity of a desk machine, cancellation reduced the accuracy of the results obtained to some four significant figures in the first example and six in the second. The procedure given here should, however, be readily adaptable to electronic digital computers, and the increased capacity of those machines should enable the coefficients $B(k, m, n; h, r)$ to be calculated to any accuracy likely to be required in practice.

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1. W. G. Bickley, "Formulae for numerical integration," *Math. Gaz.*, v. 23, 1939, p. 352–359.
2. R. E. Greenwood, "Numerical integration for linear sums of exponential functions," *Ann. Math. Statist.* v. 20, 1949, p. 608–611.
3. P. Brock & F. J. Murray, "The use of exponential sums in step by step integration," *MTAC*, v. 6, 1952, p. 63–78 and 138–150.

New Factors of Mersenne Numbers

By Edgar Karst

I have tested for prime factors all the Mersenne numbers $2^p - 1$ corresponding to prime exponents p in the interval $3000 < p < 3500$. The limit of the search for factors was $9p^2$ when no factor was previously known; otherwise the limit was $3p^2$.

The nineteen new prime factors of Mersenne numbers found by this search are displayed in the following table. Factors corresponding to smaller values of p have been listed in a paper by Brillhart and Johnson [1].

p	New factors of $2^p - 1$
3037	145 777
3041	5 565 031
3067	22 063 999
3083	15 914 447
3119	230 807 · 14 222 641
3121	31 509 617
3167	12 237 289
3181	127 241
3191	40 895 857
3253	46 452 841
3257	4 032 167
3299	19 873 177
3329	665 801 · 1 005 359 · 26 225 863
3391	1 519 169
3433	5 952 823 · 12 688 369

An extensive table by Riesel [2] includes smaller prime factors of Mersenne numbers corresponding to p = 3037, 3041, 3119, 3121, 3181, 3257, 3299, 3329, 3391, and 3433 in the preceding table.

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1. John Brillhart & G. D. Johnson, "On the factors of certain Mersenne numbers," *Math. Comp.*, v. 14, 1960, p. 365–369.
2. H. Riesel, "Mersenne numbers," *MTAC*, v. 12, 1958, p. 207–213.

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