# Table of a Weierstrass Continuous Non-Differentiable Function

By Herbert E. Salzer and Norman Levine

Many studies have been made of continuous non-differentiable functions [1], the most famous of which is Weierstrass's $W(a, b, x)$ defined by

$$(1)\qquad W(a, b, x) = \sum_{n=1}^{\infty} a^n \cos (b^n \pi x), \qquad 0 < a < 1,\ b \text{ an odd integer.}$$

It is shown in some books [1], [2] that for

$$(2)\qquad ab > 1 + \frac{3\pi}{2},$$

$W(a, b, x)$ is continuous everywhere and has no derivative anywhere, but Bromwich [3] improved this condition to

$$(3)\qquad ab > 1 + \frac{3\pi}{2}(1 - a),$$

which, according to Hardy [4] is the sharpest result (as of 1916) for no derivative, *finite or infinite*. (Hardy showed $b > 1$, $ab \geqq 1$ sufficient to establish the non-existence of any *finite* derivative. He also showed that those same conditions, together with $a(b + 1) < 2$ for $b = 4k + 1$, permitted the existence of an *infinite* derivative at certain points.) To illustrate the difference between (2) and (3) for $a = \frac{1}{2}$, (2) requires $b \geqq 13$, while (3) permits $b = 7$. However, as far as the authors know there may be considerable work to be done in the direction of lowering the bound of $1 + \frac{3\pi}{2}(1 - a)$ in (3) for the case of no derivative, finite or infinite.

Owing to the unusual nature of $W(a, b, x)$ and the absence of any previous table, or even graph, despite the countless number of theoretical papers, it was believed that an extensive table of this Weierstrass function for some typical pair of parameters $a$ and $b$ might be of value as more than a mere curiosity, namely for suggesting or motivating further research, and for its interest to workers in numerical analysis. Thus, in this last connection, it might be of interest to determine empirically what results in numerical integration and possibly interpolation are available from the continuity alone. That $W(a, b, x)$ is integrable follows from its continuity, and one might be curious to see the results of applying standard numerical integration formulas where the usual derivative formulas for the remainder would be inapplicable. Likewise, one might be curious to test out standard Lagrangian interpolation, where the remainder is often expressed in terms of derivatives. (We can write down interpolation and numerical integration formulas, avoiding derivatives in the remainder terms by employing divided differences and integrals with divided differences in the integrand, respectively. However, one usually estimates divided differences in terms of derivatives.) Finally, one's curiosity might extend as far as

Received February 23, 1960; revised July 28, 1960.

glancing at the results of standard numerical differentiation and interpretation of the results in the light of the knowledge that $W(a, b, x)$ has no derivative.

For tabulation of any $W(a, b, x)$, it is immediately apparent from (1) that

$$W(a, b, 1 + x) = - W(a, b, x), \tag{4}$$

so that the range of $x$ need not go outside (0, 1). From (1),

$$\begin{aligned} W(a, b, 0) &= -W(a, b, 1) = a/(1 - a); \\ W(a, b, \tfrac{1}{2}) &= 0. \end{aligned} \tag{5}$$

From the trigonometric identity

$$\cos(m\pi(\tfrac{1}{2} \pm t)) = \mp(-1)^{(m-1)/2} \sin m\pi t, \; m \text{ odd}, \tag{6}$$

we have

$$W(a, b, \tfrac{1}{2} + t) = -W(a, b, \tfrac{1}{2} - t), \tag{7}$$

so that for complete tabulation of any $W(a, b, x)$ it suffices for $x$ to range from 0 to $\frac{1}{2}$.

In connection with the choice of $a$ and $b$, it is apparent that for $a$ close to 1, we can choose $b$ as low as 3, but the convergence of the series in (1) would be too slow for practical calculation of $W(a, b, x)$ to high accuracy. Making $a$ very small would give rapid convergence, but for accuracy fixed at a certain number of decimal places as $a$ tends to get very small, say

$$a = \epsilon, \qquad b^n > N = \left\{1 + \frac{3\pi}{2}(1 - \epsilon)\right\}^n \Big/ \epsilon^n$$

becomes enormous and $W(\epsilon, b, x)$ becomes essentially the first term of (1), $\epsilon \cos (b^n \pi x)$, whose graph would appear like that of a very highly oscillatory function of small amplitude. As a compromise between these two extreme types, we took $a = \frac{1}{2}$ and $b = 7$. The choice $a = \frac{1}{2}$ did not lead to too many terms of (1), 50 terms giving a truncating error $< \frac{1}{2} \cdot 10^{-15}$, and yet there were sufficient terms beyond the first few to give a graph that is characteristic of $W(a, b, x)$ rather than a predominantly sinusoidal type of curve. The $b = 7$ barely satisfies (3), thus tending to minimize the oscillatory behavior of $W(a, b, x)$ and to facilitate graphing. We shall denote $W(a, b, x)$ which is tabulated here for $a = \frac{1}{2}$ and $b = 7$ by $W(x)$.

This present table of $W(x)$, $x = 0(.001)1$ to 12D, was printed out and rounded from a preliminary calculation on the IBM 704 to several more places. Two separate and independent print-outs, supposedly identical, were proofread against each other, with just a single print-out error turning up. Naturally, no differencing check could be made upon the correctness of this table of $W(x)$, but every value underwent the following final functional check:

$$W(7x) = 2W(x) - \cos (7\pi x), \tag{8}$$

which was performed by desk calculation upon $W(x)$ on one of the preliminary print-outs. The results showed $W(x)$ to be correct to around 14D. In employing (8), $W(7x)$ was found in the table as $\pm W(x')$ for some suitable $x'$, $0 \leqq x' \leqq \frac{1}{2}$, according to (4) and (7), and $\cos (7\pi x)$, after reduction of $7\pi x$ to the first quadrant, was

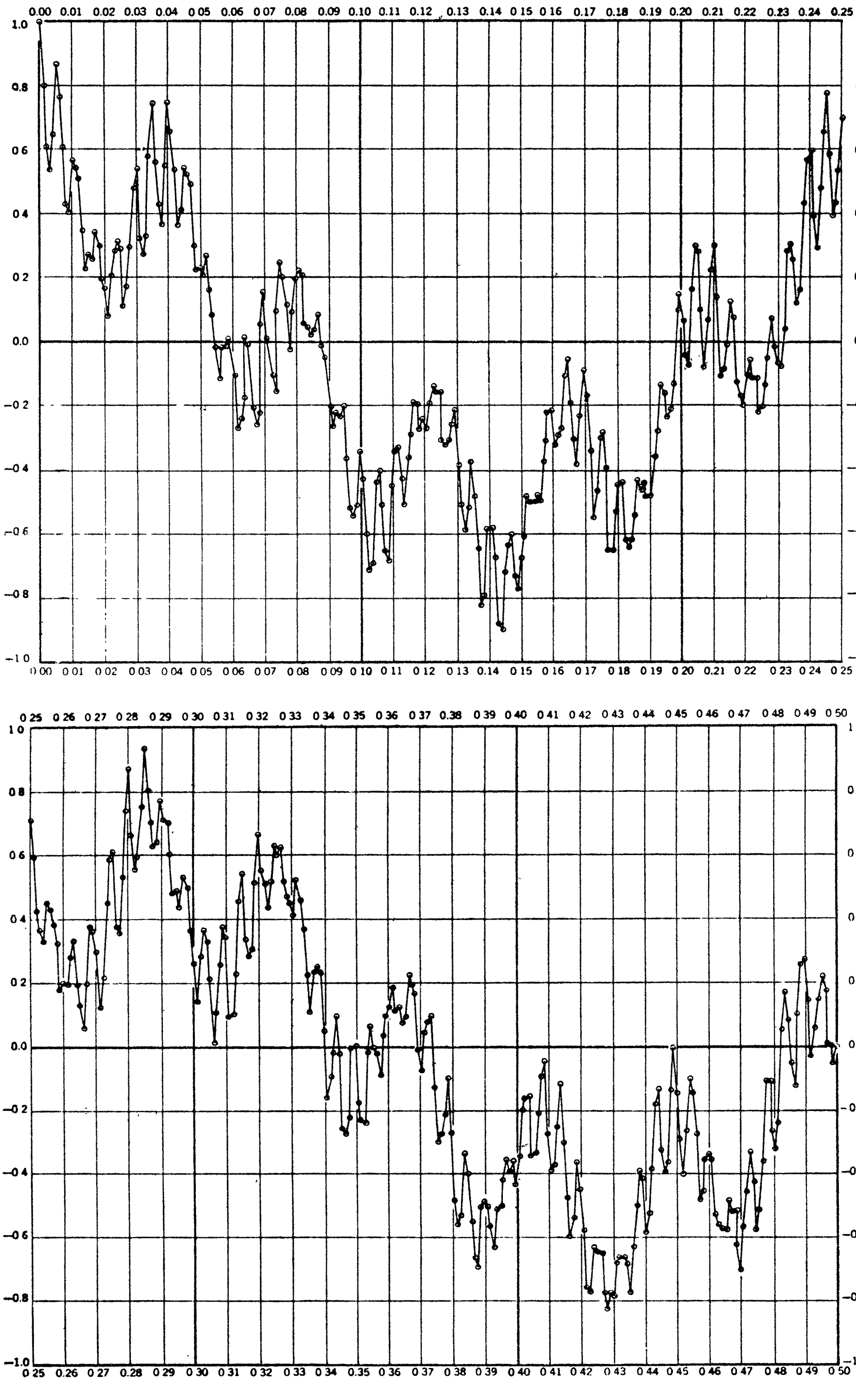


FIG. 1.—Illustration of a Weierstrass, Everywhere-Continuous Nowhere-Differentiable Function, $W(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$ $a = \frac{1}{2}$; $b = 7$; $x = 0(0.001)0.500$

looked up in a well-known 15-place table at intervals of 0.01° [5]. The final 12-decimal table was checked by reading it several times against one of the print-outs, and it is believed to be correct to well within a unit in the 12th decimal.

The purpose of the accompanying figure, which is merely a broken line graph of the table of $W(x)$, is to furnish at a glance a view of the peculiar behavior of $W(x)$. Of course, the graphical picture would be more complete if the time and means were available for calculating $W(a, b, x)$ as a function of $a$ also, and for a sequence of permissible odd integral values of $b$ (according to (3)) to correspond to each $a$. Although no offhand justification could be found for drawing anything smoother than a broken line connecting these 500 points, one still finds its ripples of irregularity, superposed upon a broader pattern of smoothness, to be quite revealing as to the nature of $W(x)$ and how it might appear under repeated "magnification" (i.e., subtabulation).

To establish (8), replace $x$ by $7x$, in $W(x) = \sum_{n=1}^{\infty} \cos(7^n \pi x)/2^n$, to get

$$W(7x) = 2\sum_{n=1}^{\infty} \cos(7^{n+1}\pi x)/2^{n+1} = 2\sum_{n'=2}^{\infty} \cos(7^{n'}\pi x)/2^{n'} = 2W(x) - \cos(7\pi x).$$

By repeated application of (8),

$$W(7^n x) = 2W(7^{n-1}x) - \cos(7^n \pi x) = 4W(7^{n-2}x) - 2\cos(7^{n-1}\pi x) - \cos(7^n\pi x)$$
$$= 8W(7^{n-3}x) - 4\cos(7^{n-2}\pi x) - \cdots \quad \text{etc. until we reach}$$

$$W(7^n x) = 2^n W(x) - \sum_{r=0}^{n-1} 2^r \cos(7^{n-r}\pi x). \tag{9}$$

From (9), for $x = 1/7^n$, $W(1) = -1 = 2^n W(1/7^n) - \sum_{r=0}^{n-1} 2^r \cos(\pi/7^r)$, from which

$$W(1/7^n) = -1/2^{n-1} + \sum_{r=1}^{n-1} \cos(\pi/7^r)/2^{n-r}. \tag{10}$$

Letting $n \to \infty$ in (10), we see at once that

$$\lim_{n\to\infty} \left\{ \sum_{r=1}^{n-1} 2^r \cos(\pi/7^r) \right\} \Big/ 2^n = 1. \tag{11}$$

To test the value of standard numerical integration formulas upon $W(x)$, whose integral is given by

$$\int_0^x W(t)\,dt = \frac{1}{\pi} \sum_{n=1}^{\infty} \sin(7^n \pi x)/14^n, \tag{12}$$

the values of $\int_0^{0.1} W(t)dt, \int_{0.1}^{0.2} W(t)dt, \cdots, \int_{0.4}^{0.5} W(t)dt$ were computed analytically from (12), and then were computed numerically by both trapezoidal and Simpson's rules at intervals of 0.001, with the following results:

| Interval | True Value | Trapezoidal Rule | Deviation | Simpson's Rule | Deviation |
|---|---|---|---|---|---|
| 0 to 0.1 | 0.01899 29 | 0.01898 76 | −0.00000 53 | 0.01901 44 | +0.00002 15 |
| 0.1 to 0.2 | −0.04145 65 | −0.04143 80 | +0.00001 85 | −0.04145 43 | +0.00000 22 |
| 0.2 to 0.3 | 0.03084 62 | 0.03084 43 | −0.00000 19 | 0.03085 14 | +0.00000 52 |
| 0.3 to 0.4 | 0.00337 70 | 0.00342 54 | +0.00004 84 | 0.00340 27 | +0.00002 57 |
| 0.4 to 0.5 | −0.03298 02 | −0.03300 67 | −0.00002 65 | −0.03288 27 | +0.00009 75 |

The results show no recognizable advantage in Simpson's rule. In fact, the sum of the absolute values of the above deviations in the trapezoidal rule is around $10^{-4}$, while the sum of the absolute values of the Simpson deviations is around $1\frac{1}{2}\cdot 10^{-4}$. This may indicate that no higher-point formula will improve over the trapezoidal formula.

Lagrangian polynomial interpolation at intervals of 0.002 was tried for the 2- through 7-point cases, for a mid-interval (i.e., already tabulated) value of $W(x)$ at two different places, $x = 0.007$ and $x = 0.037$, where the true value to 5D is 0.60807 and 0.43362 respectively. At each place the error in almost all cases ranged from around 0.01 to 0.05. More specifically, for $x = 0.007$ the error fluctuated between 0.01 for every even-point interpolation and 0.014 to 0.049 for various odd-point interpolations, and for $x = 0.037$ there were deviations of 0.032 and 0.055 for respective 2-point and 3-point interpolation and deviations ranging from 0.001 to 0.021 in the higher-point interpolation. On the basis of these two tests alone it would appear that one could not really count upon any systematic improvement beyond linear interpolation.

Finally, out of pure curiosity, 2- through 7-point Lagrangian differentiation, for the "first derivative," was tried out at the tabular interval of 0.001, for $x = 0.002$, and surprisingly enough, outside of the 2-point answer of $-74$ and the 3-point answer of $-133$, the remaining four cases all came within 6 units of $-150$.

From a casual look at the graph of $W(x)$, it is apparent that in place of the derivative there is a general directional trend from any point $x_0$ if we do not go too far away from $x_0$, and we might seek a suitable quantitative estimate for an "average slope" between $x_0$ and $x_0 + \cdot h$. (The discussion here is concerned with a suitable generalization of the left- or right-hand derivative, rather than the derivative.) One suggestion that would appear natural for $W(x, a, b)$, or any other continuous function, would be to investigate the possibilities of the average of the difference quotient $\{f(x) - f(x_0)\}/(x - x_0)$, which exists and is itself continuous for every $x$ except $x_0$ in the open interval $(x_0, x_0 + h)$. This average difference quotient or $\mathfrak{D}_h f(x_0)$ might have the following definition (assuming that it exists in the first place):

$$(13) \qquad \mathfrak{D}_h f(x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} \{[f(x) - f(x_0)]/(x - x_0)\}\, dx.$$

That (13) may be a suitable generalization follows from the fact that when $f'(x_0)$ exists, (13) exists, and

$$(14) \qquad \lim_{h \to 0} \mathfrak{D}_h f(x_0) = f'(x_0).$$

This is seen at once from the replacement of $\{f(x) - f(x_0)\}/(x - x_0)$ by $f'(x_0) + \epsilon(x)$ in (13) and the continuity of $\epsilon(x)$ in the *closed* set $(x_0, x_0 + h)$ which makes $\epsilon(x)$ integrable. Thus (13) exists and

$$\left| \frac{1}{h} \int_{x_0}^{x_0+h} \cdot \epsilon(x)\, dx \right| \to 0 \quad \text{as} \quad h \to 0,$$

which implies (14).

It is not difficult to find examples of continuous functions $f(x)$ where $f'(x_0)$ does not exist and (a) also $\mathfrak{D}_h f(x_0)$ does not exist, or (b) $\mathfrak{D}_h f(x_0)$ exists but $\lim_{h\to 0} \mathfrak{D}_h f(x_0)$ does not exist. But we may also have (c) no $f'(x_0)$, with both $\mathfrak{D}_h f(x_0)$ and $\lim_{h \to 0} \mathfrak{D}_h f(x_0)$ existing. In other words the existence of $\lim_{h\to 0} \mathfrak{D}_h f(x_0)$ still does not imply the existence of $f'(x_0)$. Such a counter-example,* which is due to the referee, is the following. Let $x_0 = 0$ and

$$f(x) = x \sin \frac{1}{x} \qquad (x \neq 0)$$
$$f(0) = 0.$$

This continuous function has no derivative at $x = 0$, but

$$\lim_{h \to 0} \mathfrak{D}_h f(0) = 0.$$

First

$$\mathfrak{D}_h = \frac{1}{h} \int_0^h \sin\left(\frac{1}{x}\right) dx$$

exists since the integrand is bounded and continuous except at one point. This suffices. To estimate $\mathfrak{D}_h$ we let

$$I_n = \int_{1/(n+1)\pi}^{1/n\pi} \sin\left(\frac{1}{x}\right) dx = \int_{n\pi}^{(n+1)\pi} \frac{1}{y^2} \sin y \, dy.$$

By the mean value theorem

$$I_n = (-1)^n \cdot 2/\theta_n^2$$

where

$$n\pi < \theta_n < (n+1)\pi.$$

Suppose that $h = 1/(n+a)\pi, 0 \leqq a < 1$. Then

$$\mathfrak{D}_h = (n+a)\pi \left[ \int_{(n+a)\pi}^{(n+1)\pi} y^{-2} \sin y \, dy + I_{n+1} + I_{n+2} + \cdots \right],$$

and therefore $| \mathfrak{D}_h | < (n+a)\pi \, | I_n | < 2(n+a)\pi/n^2\pi^2$.

Therefore as $h \to 0$, $\mathfrak{D}_h$ also $\to 0$.

The authors wish to acknowledge the assistance of Mrs. Charlene M. Janos in checking the entire table of $W(x)$ by the functional check (8).

Convair-Astronautics
San Diego, California

1. A. N. Singh, *The Theory and Construction of Non-Differentiable Functions*, Lucknow University Studies, Faculty of Science, no. 1, 1935, reprinted in *Squaring the Circle and Other Monographs*, Chelsea Publishing Co., New York, 1953.

2. E. Goursat, *A Course in Mathematical Analysis*, Vol. 1, translated by E. R. Hedrick, Ginn & Co., Boston, 1904, p. 423–425.

3. T. Bromwich, *An Introduction to the Theory of Infinite Series*, Macmillan & Co., Ltd., London, 1908, p. 490–491. *Note*: The proof of the sufficiency of $ab > 1 + \frac{3\pi}{2}(1 - a)$ is not contained in the later 1926 edition.

4. G. H. Hardy, "Weierstrass's non-differentiable function," *Trans. Amer. Math. Soc.*, v. 17, 1916, p. 301–325.

5. Nat. Bur. Standards Appl. Math. Ser. No. 5, *Table of Sines and Cosines to Fifteen Decimal Places at Hundredths of a Degree*, U. S. Government Printing Office, Washington 25, D. C., 1949.

* Another counter-example found after that of the referee is the following: $f(x) = x\phi(x)$, $x \neq 0$, $f(0) = 0$, where $\phi(x) = 1$ except in the intervals $[(1/n - 1/n^3), 1/n]$, within which $\phi(x) = 0$. Now $f(x)$ is continuous at $x = 0$ and has no derivative there. But $1/h \int_0^h \phi(x)dx \to 1$ as $h \to 0$, because the "dipped-out" area becomes an infinitesimal fraction of the whole (also infinitesimal) area between 0 and $h$, since as $h \sim 1/n$, we remove $\sum_{m=n}^{\infty} 1/m^3 \sim 1/2n^2 \sim 0(h)$.

Table of $W(x) \equiv \sum_{n=1}^{\infty} \cos (7^n \pi x)/2^n$

| $x$ | $W(x)$ | | $x$ | $W(x)$ | |
|---|---|---|---|---|---|
| .000 | 1.00000 00000 00 | 1.000 | .050 | .23088 91433 53 | .950 |
| .001 | .80391 58298 49 | .999 | .051 | .20682 52628 39 | .949 |
| .002 | .61188 60438 58 | .998 | .052 | .27128 71570 31 | .948 |
| .003 | .53777 60375 27 | .997 | .053 | .16118 34941 71 | .947 |
| .004 | .64747 48039 38 | .996 | .054 | .08069 56769 70 | .946 |
| .005 | .87163 69853 23 | .995 | .055 | −.02066 12990 04 | .945 |
| .006 | .76687 71957 75 | .994 | .056 | −.11450 55193 73 | .944 |
| .007 | .60807 34552 61 | .993 | .057 | −.02295 65257 19 | .943 |
| .008 | .43502 94075 78 | .992 | .058 | −.01951 72464 55 | .942 |
| .009 | .40541 06494 76 | .991 | .059 | .01151 68818 80 | .941 |
| .010 | .56641 31472 93 | .990 | .060 | −.09698 14952 80 | .940 |
| .011 | .54275 36720 27 | .989 | .061 | −.27187 35472 72 | .939 |
| .012 | .50694 91215 98 | .988 | .062 | −.23653 00063 45 | .938 |
| .013 | .34801 25245 87 | .987 | .063 | −.16965 63244 21 | .937 |
| .014 | .22473 91530 39 | .986 | .064 | .01498 43634 87 | .936 |
| .015 | .27196 45026 68 | .985 | .065 | −.00239 70885 80 | .935 |
| .016 | .25665 87904 18 | .984 | .066 | −.20181 95236 74 | .934 |
| .017 | .34500 48434 78 | .983 | .067 | −.25856 31395 23 | .933 |
| .018 | .29744 09740 20 | .982 | .068 | −.21932 57817 04 | .932 |
| .019 | .19896 02842 26 | .981 | .069 | .05550 33815 80 | .931 |
| .020 | .16232 54753 01 | .980 | .070 | .15690 95326 47 | .930 |
| .021 | .07772 75335 97 | .979 | .071 | .01436 83992 20 | .929 |
| .022 | .20584 44795 34 | .978 | .072 | −.09812 03304 88 | .928 |
| .023 | .28363 56796 36 | .977 | .073 | −.15074 56668 50 | .927 |
| .024 | .31741 47365 60 | .976 | .074 | .09240 88499 35 | .926 |
| .025 | .28730 16038 97 | .975 | .075 | .24890 58340 07 | .925 |
| .026 | .11054 29341 42 | .974 | .076 | .20632 59257 00 | .924 |
| .027 | .17279 94307 92 | .973 | .077 | .11462 33882 35 | .923 |
| .028 | .29881 59987 33 | .972 | .078 | −.02304 18919 60 | .922 |
| .029 | .48372 84610 58 | .971 | .079 | .09557 81653 49 | .921 |
| .030 | .54441 21945 09 | .970 | .080 | .19794 01773 19 | .920 |
| .031 | .32388 99122 78 | .969 | .081 | .22531 98834 20 | .919 |
| .032 | .26990 13283 84 | .968 | .082 | .20876 59176 94 | .918 |
| .033 | .33225 74462 04 | .967 | .083 | .05397 57757 43 | .917 |
| .034 | .58370 92580 29 | .966 | .084 | .04851 66043 63 | .916 |
| .035 | .74931 30151 91 | .965 | .085 | .02128 19742 78 | .915 |
| .036 | .56632 09611 85 | .964 | .086 | .03684 58507 18 | .914 |
| .037 | .43361 86486 58 | .963 | .087 | .08433 37682 49 | .913 |
| .038 | .36496 26383 50 | .962 | .088 | −.01214 33731 42 | .912 |
| .039 | .55435 98106 40 | .961 | .089 | −.05215 21171 01 | .911 |
| .040 | .75565 42269 37 | .960 | .090 | −.19576 22844 41 | .910 |
| .041 | .66141 61182 70 | .959 | .091 | −.26338 74226 81 | .909 |
| .042 | .54244 67604 80 | .958 | .092 | −.21893 60178 36 | .908 |
| .043 | .36553 53065 06 | .957 | .093 | −.22931 50089 58 | .907 |
| .044 | .41175 90597 74 | .956 | .094 | −.19289 54543 34 | .906 |
| .045 | .54502 46829 96 | .955 | .095 | −.36048 94459 76 | .905 |
| .046 | .52178 60105 03 | .954 | .096 | −.51624 68304 42 | .904 |
| .047 | .49248 88676 02 | .953 | .097 | −.54350 09214 87 | .903 |
| .048 | .30088 64437 50 | .952 | .098 | −.50350 10354 40 | .902 |
| .049 | .22797 19914 12 | .951 | .099 | −.33848 06786 76 | .901 |
| | $-W(x)$ | $x$ | | $-W(x)$ | $x$ |

TABLE OF *W*(*x*)—*Continued*

| x | W(x) | | x | W(x) | |
|---|---|---|---|---|---|
| .100 | −.42532 54041 76 | .900 | .150 | −.60928 87419 74 | .850 |
| .101 | −.60122 07728 99 | .899 | .151 | −.48052 65499 66 | .849 |
| .102 | −.71436 04664 23 | .898 | .152 | −.49741 97079 36 | .848 |
| .103 | −.69032 26595 69 | .897 | .153 | −.49479 75145 65 | .847 |
| .104 | −.43794 73064 53 | .896 | .154 | −.47354 23522 65 | .846 |
| .105 | −.40215 63534 91 | .895 | .155 | −.49291 96963 68 | .845 |
| .106 | −.50671 30937 48 | .894 | .156 | −.36979 28855 91 | .844 |
| .107 | −.65237 17461 67 | .893 | .157 | −.30677 09741 35 | .843 |
| .108 | −.68741 46475 51 | .892 | .158 | −.21917 49907 16 | .842 |
| .109 | −.44815 12393 09 | .891 | .159 | −.21531 69983 41 | .841 |
| .110 | −.33948 90492 28 | .890 | .160 | −.32774 87639 11 | .840 |
| .111 | −.32696 83696 91 | .889 | .161 | −.30751 75250 62 | .839 |
| .112 | −.42541 62768 18 | .888 | .162 | −.27020 56659 27 | .838 |
| .113 | −.50701 25026 15 | .887 | .163 | −.11494 78174 84 | .837 |
| .114 | −.36527 04053 46 | .886 | .164 | −.05693 76441 49 | .836 |
| .115 | −.28717 02983 14 | .885 | .165 | −.19542 29668 60 | .835 |
| .116 | −.19435 86539 72 | .884 | .166 | −.30152 83944 41 | .834 |
| .117 | −.20343 01549 89 | .883 | .167 | −.38366 39179 31 | .833 |
| .118 | −.27648 95287 96 | .882 | .168 | −.22909 39440 73 | .832 |
| .119 | −.24091 87061 61 | .881 | .169 | −.09500 91416 95 | .831 |
| .120 | −.27427 89580 66 | .880 | .170 | −.17303 80098 99 | .830 |
| .121 | −.19594 53705 68 | .879 | .171 | −.33806 67973 86 | .829 |
| .122 | −.14745 62719 19 | .878 | .172 | −.55064 02832 16 | .828 |
| .123 | −.16077 90760 02 | .877 | .173 | −.46584 78110 99 | .827 |
| .124 | −.16313 11716 57 | .876 | .174 | −.30129 83741 07 | .826 |
| .125 | −.30795 98441 70 | .875 | .175 | −.27803 69565 60 | .825 |
| .126 | −.32779 07918 31 | .874 | .176 | −.39280 65936 50 | .824 |
| .127 | −.30642 17906 98 | .873 | .177 | −.65182 79852 26 | .823 |
| .128 | −.25457 54992 71 | .872 | .178 | −.65182 39388 20 | .822 |
| .129 | −.20521 28645 40 | .871 | .179 | −.53351 02371 89 | .821 |
| .130 | −.38226 57006 18 | .870 | .180 | −.44268 44274 72 | .820 |
| .131 | −.51008 60535 12 | .869 | .181 | −.43578 01021 83 | .819 |
| .132 | −.58897 88293 07 | .868 | .182 | −.61985 87140 15 | .818 |
| .133 | −.51605 03218 22 | .867 | .183 | −.64922 21692 63 | .817 |
| .134 | −.37228 02555 49 | .866 | .184 | −.62277 87433 01 | .816 |
| .135 | −.48222 40135 76 | .865 | .185 | −.54441 65431 73 | .815 |
| .136 | −.64476 90942 44 | .864 | .186 | −.43416 51101 85 | .814 |
| .137 | −.82656 60891 90 | .863 | .187 | −.47153 43722 97 | .813 |
| .138 | −.78900 50242 13 | .862 | .188 | −.44187 85912 52 | .812 |
| .139 | −.58460 20582 18 | .861 | .189 | −.48324 34653 21 | .811 |
| .140 | −.58017 61018 65 | .860 | .190 | −.48100 45544 76 | .810 |
| .141 | −.67358 93294 65 | .859 | .191 | −.36210 24639 76 | .809 |
| .142 | −.88334 97740 78 | .858 | .192 | −.28385 07250 58 | .808 |
| .143 | −.90195 54475 25 | .857 | .193 | −.13957 34378 96 | .807 |
| .144 | −.71735 67984 31 | .856 | .194 | −.17132 73511 09 | .806 |
| .145 | −.63542 71888 15 | .855 | .195 | −.24380 91207 97 | .805 |
| .146 | −.60172 84929 47 | .854 | .196 | −.21870 72532 52 | .804 |
| .147 | −.73979 05811 32 | .853 | .197 | −.13838 71322 99 | .803 |
| .148 | −.77996 61358 53 | .852 | .198 | .09801 82360 71 | .802 |
| .149 | −.67821 23623 64 | .851 | .199 | .14666 52327 61 | .801 |
| | −W(x) | x | | −W(x) | x |

TABLE OF $W(x)$—*Continued*

| $x$ | $W(x)$ | | $x$ | $W(x)$ | |
|---|---|---|---|---|---|
| .200 | .06366 10018 75 | .800 | .250 | .70710 67811 87 | .750 |
| .201 | −.04175 22364 04 | .799 | .251 | .59986 16383 91 | .749 |
| .202 | −.07476 70750 18 | .798 | .252 | .42999 19525 71 | .748 |
| .203 | .16401 55220 04 | .797 | .253 | .36660 93235 94 | .747 |
| .204 | .29971 35815 61 | .796 | .254 | .32795 62544 06 | .746 |
| .205 | .28236 84375 00 | .795 | .255 | .45218 11134 91 | .745 |
| .206 | .10035 12674 93 | .794 | .256 | .43000 78607 24 | .744 |
| .207 | −.08006 36504 29 | .793 | .257 | .38270 70934 21 | .743 |
| .208 | .06602 53228 00 | .792 | .258 | .32431 23791 08 | .742 |
| .209 | .22194 83909 79 | .791 | .259 | .18040 34507 72 | .741 |
| .210 | .29866 93766 43 | .790 | .260 | .20082 17490 15 | .740 |
| .211 | .14347 00102 07 | .789 | .261 | .19502 92282 86 | .739 |
| .212 | −.11156 27605 97 | .788 | .262 | .28277 11533 62 | .738 |
| .213 | −.08927 61047 33 | .787 | .263 | .33124 94143 10 | .737 |
| .214 | −.00503 43683 93 | .786 | .264 | .19535 83704 39 | .736 |
| .215 | .12393 18196 44 | .785 | .265 | .13130 01934 73 | .735 |
| .216 | 07237 02520 68 | .784 | .266 | .05923 97001 64 | .734 |
| .217 | −.12870 66881 32 | .783 | .267 | .20320 81841 97 | .733 |
| .218 | − 17190 63119 85 | .782 | .268 | .38097 30175 89 | .732 |
| .219 | −.20539 46224 39 | .781 | .269 | .36488 72388 22 | .731 |
| .220 | −.10396 93551 29 | .780 | .270 | .30069 58598 63 | .730 |
| .221 | −.05670 84900 53 | .779 | .271 | .12888 96220 58 | .729 |
| .222 | −.11058 04979 98 | .778 | .272 | .21930 88571 52 | .728 |
| .223 | −.11053 47920 41 | .777 | .273 | .45450 66523 44 | .727 |
| .224 | − 22263 98542 44 | .776 | .274 | .58788 37027 28 | .726 |
| .225 | −.20433 20524 80 | .775 | .275 | .61063 78772 02 | .725 |
| .226 | −.13899 14773 88 | .774 | .276 | .37897 11709 32 | .724 |
| .227 | − 05338 52274 30 | .773 | .277 | .35491 00906 37 | .723 |
| .228 | .07179 91975 44 | .772 | .278 | .53317 01360 59 | .722 |
| .229 | −.01896 62898 75 | .771 | .279 | .74080 38401 87 | .721 |
| .230 | − 07113 29711 74 | .770 | .280 | .87388 44641 26 | .720 |
| .231 | − 08351 49064 96 | .769 | .281 | .66344 41290 97 | .719 |
| .232 | .03967 20418 75 | .768 | .282 | .55360 44310 88 | .718 |
| .233 | .28470 23445 34 | .767 | .283 | .59858 96747 52 | .717 |
| .234 | .30373 06421 48 | .766 | .284 | .75311 92971 19 | .716 |
| .235 | .25331 56946 85 | .765 | .285 | .93575 68089 02 | .715 |
| .236 | 12247 07877 84 | .764 | .286 | .80593 31523 57 | .714 |
| .237 | .16091 46178 36 | .763 | .287 | .70250 54785 63 | .713 |
| .238 | 43416 31698 35 | .762 | .288 | .62769 78735 12 | .712 |
| .239 | .57254 03757 22 | .761 | .289 | .64051 31524 34 | .711 |
| .240 | .60020 74057 93 | .760 | .290 | .76998 70795 56 | .710 |
| .241 | .39293 37880 48 | .759 | .291 | .71343 52538 50 | .709 |
| .242 | 29091 21091 20 | .758 | .292 | .70111 02427 36 | .708 |
| .243 | .47723 86339 28 | .757 | .293 | .59700 86384 17 | .707 |
| .244 | .65452 02702 15 | .756 | .294 | .48196 39792 70 | .706 |
| .245 | .78049 97326 19 | .755 | .295 | .48841 28610 21 | .705 |
| .246 | .58771 13946 18 | .754 | .296 | .43820 32711 49 | .704 |
| .247 | .39392 08421 96 | .753 | .297 | .53246 89138 76 | .703 |
| .248 | .43534 55892 80 | .752 | .298 | .50028 47645 04 | .702 |
| .249 | .53704 70311 88 | .751 | .299 | .36415 33185 00 | .701 |
| | $-W(x)$ | $x$ | | $-W(x)$ | $x$ |

TABLE OF $W(x)$—*Continued*

| $x$ | $W(x)$ | | $x$ | $W(x)$ | |
|---|---|---|---|---|---|
| .300 | .26286 55560 60 | .700 | .350 | .00778 77869 67 | .650 |
| .301 | .14582 98589 87 | .699 | .351 | −.17204 22086 50 | .649 |
| .302 | .28563 97407 01 | .698 | .352 | −.23029 21241 86 | .648 |
| .303 | .36633 50157 87 | .697 | .353 | −.23647 30864 44 | .647 |
| .304 | .33282 81740 47 | .696 | .354 | −.01762 25274 42 | .646 |
| .305 | .21458 96315 30 | .695 | .355 | .06669 22195 13 | .645 |
| .306 | .00939 20926 45 | .694 | .356 | .00075 62690 21 | .644 |
| .307 | .10710 80078 80 | .693 | .357 | −.02063 75236 12 | .643 |
| .308 | .25624 91704 22 | .692 | .358 | −.08880 13434 64 | .642 |
| .309 | .37838 97945 27 | .691 | .359 | .03980 77688 40 | .641 |
| .310 | .34385 20085 52 | .690 | .360 | .10006 60841 69 | .640 |
| .311 | .09875 62075 19 | .689 | .361 | .12219 89553 34 | .639 |
| .312 | .10737 46483 70 | .688 | .362 | .18737 55558 83 | .638 |
| .313 | .23160 40973 59 | .687 | .363 | .10954 21317 00 | .637 |
| .314 | .45535 62002 52 | .686 | .364 | .12819 87330 70 | .636 |
| .315 | .54102 65479 93 | .685 | .365 | .07610 38829 89 | .635 |
| .316 | .33736 28357 70 | .684 | .366 | .09741 56111 37 | .634 |
| .317 | .28355 86706 20 | .683 | .367 | .22835 66376 41 | .633 |
| .318 | .30968 52189 51 | .682 | .368 | .19796 35878 29 | .632 |
| .319 | .51427 37371 70 | .681 | .369 | .16937 55518 30 | .631 |
| .320 | .66458 80166 07 | .680 | .370 | −.00562 33217 53 | .630 |
| .321 | .55383 02222 36 | .679 | .371 | −.07189 57360 01 | .629 |
| .322 | .51306 08367 00 | .678 | .372 | .04604 08396 66 | .628 |
| .323 | .43864 14186 98 | .677 | .373 | .07904 58408 70 | .627 |
| .324 | .52349 24425 12 | .676 | .374 | .09708 84764 28 | .626 |
| .325 | .63004 29627 66 | .675 | .375 | −.12756 11441 22 | .625 |
| .326 | .59308 72965 01 | .674 | .376 | −.30043 50117 72 | .624 |
| .327 | .62794 97613 07 | .673 | .377 | −.27256 67762 10 | .623 |
| .328 | .51805 41271 03 | .672 | .378 | −.21256 78517 98 | .622 |
| .329 | .47323 47350 89 | .671 | .379 | −.09426 94456 39 | .621 |
| .330 | .45296 76932 02 | .670 | .380 | −.26632 63801 78 | .620 |
| .331 | .41365 95766 99 | .669 | .381 | −.48045 19086 93 | .619 |
| .332 | .52433 35362 36 | .668 | .382 | −.55645 23195 61 | .618 |
| .333 | .45999 71920 26 | .667 | .383 | −.52637 52089 99 | .617 |
| .334 | .37014 12343 31 | .666 | .384 | −.33212 51324 29 | .616 |
| .335 | .22552 02235 19 | .665 | .385 | −.39479 74417 87 | .615 |
| .336 | .10904 55459 52 | .664 | .386 | −.54657 07923 33 | .614 |
| .337 | .23421 37763 08 | .663 | .387 | −.66192 84709 16 | .613 |
| .338 | .25303 23429 37 | .662 | .388 | −.69040 53292 05 | .612 |
| .339 | .23376 32937 51 | .661 | .389 | −.49978 37896 32 | .611 |
| .340 | .05089 90862 54 | .660 | .390 | −.48100 38625 93 | .610 |
| .341 | −.15716 90669 72 | .659 | .391 | −.50444 79258 90 | .609 |
| .342 | −.09044 72949 99 | .658 | .392 | −.56183 05832 68 | .608 |
| .343 | −.01752 94215 89 | .657 | .393 | −.62318 58803 80 | .607 |
| .344 | .09699 75179 51 | .656 | .394 | −.50778 26196 53 | .606 |
| .345 | −.01688 93672 23 | .655 | .395 | −.49647 33948 32 | .605 |
| .346 | −.25654 09168 79 | .654 | .396 | −.41302 52891 40 | .604 |
| .347 | −.27382 88597 64 | .653 | .397 | −.35589 92135 83 | .603 |
| .348 | −.21463 82850 53 | .652 | .398 | −.38867 82210 92 | .602 |
| .349 | .00184 25835 82 | .651 | .399 | −.35791 63481 26 | .601 |
| | $-W(x)$ | $x$ | | $-W(x)$ | $x$ |

TABLE OF $W(x)$—*Concluded*

| $x$ | $W(x)$ | | $x$ | $W(x)$ | |
|---|---|---|---|---|---|
| .400 | −.43633 89981 25 | .600 | .450 | −.14085 88911 07 | .550 |
| .401 | −.34108 64853 67 | .599 | .451 | −.28701 85703 94 | .549 |
| .402 | −.19995 66617 14 | .598 | .452 | −.40662 30552 60 | .548 |
| .403 | −.15624 84342 39 | .597 | .453 | −.26053 58919 44 | .547 |
| .404 | −.15344 31864 89 | .596 | .454 | −.09792 69986 33 | .546 |
| .405 | −.33660 42737 89 | .595 | .455 | −.14569 54090 97 | .545 |
| .406 | −.33007 06597 37 | .594 | .456 | −.26984 09362 85 | .544 |
| .407 | −.20452 46632 06 | .593 | .457 | −.47876 24067 97 | .543 |
| .408 | −.09102 42265 68 | .592 | .458 | −.44907 28535 16 | .542 |
| .409 | −.04111 67965 83 | .591 | .459 | −.34753 36936 00 | .541 |
| .410 | −.26774 44625 33 | .590 | .460 | −.33327 19438 23 | .540 |
| .411 | −.38274 33375 09 | .589 | .461 | −.35146 25764 11 | .539 |
| .412 | −.36998 51442 42 | .588 | .462 | −.52273 62074 43 | .538 |
| .413 | −.24689 45307 00 | .587 | .463 | −.55987 41267 67 | .537 |
| .414 | −.11736 78392 84 | .586 | .464 | −.57344 06332 45 | .536 |
| .415 | −.29765 17844 11 | .585 | .465 | −.57404 69550 42 | .535 |
| .416 | −.47494 21492 22 | .584 | .466 | −.48136 22660 42 | .534 |
| .417 | −.59655 84918 74 | .583 | .467 | −.51427 61901 27 | .533 |
| .418 | −.53275 36804 59 | .582 | .468 | −.51301 35662 51 | .532 |
| .419 | −.35968 31001 54 | .581 | .469 | −.61436 17729 69 | .531 |
| .420 | −.44265 28777 24 | .580 | .470 | −.69144 70666 05 | .530 |
| .421 | −.57367 68152 00 | .579 | .471 | −.56392 63767 92 | .529 |
| .422 | −.75568 10645 53 | .578 | .472 | −.45520 04387 35 | .528 |
| .423 | −.77343 16806 93 | .577 | .473 | −.32911 92328 10 | .527 |
| .424 | −.63242 61726 82 | .576 | .474 | −.42540 32355 39 | .526 |
| .425 | −.64210 94688 15 | .575 | .475 | −.57627 07637 41 | .525 |
| .426 | −.64792 12620 64 | .574 | .476 | −.51397 83689 36 | .524 |
| .427 | −.77107 83101 91 | .573 | .477 | −.35913 37922 98 | .523 |
| .428 | −.82369 79240 71 | .572 | .478 | −.10430 50805 99 | .522 |
| .429 | −.76886 58123 18 | .571 | .479 | −.10454 99179 66 | .521 |
| .430 | −.78295 98538 29 | .570 | .480 | −.26292 26878 67 | .520 |
| .431 | −.67178 95122 99 | .569 | .481 | −.31706 81976 54 | .519 |
| .432 | −.65728 68184 10 | .568 | .482 | −.24133 62343 53 | .518 |
| .433 | −.65957 56572 42 | .567 | .483 | .05762 50734 01 | .517 |
| .434 | −.67892 26214 60 | .566 | .484 | .17288 12030 15 | .516 |
| .435 | −.76752 41703 56 | .565 | .485 | .08627 79883 17 | .515 |
| .436 | −.62898 65506 20 | .564 | .486 | −.05153 93039 59 | .514 |
| .437 | −.49719 39800 06 | .563 | .487 | −.12044 48896 59 | .513 |
| .438 | −.38838 06903 26 | .562 | .488 | .10705 03214 66 | .512 |
| .439 | −.41153 59153 14 | .561 | .489 | .26694 06923 26 | .511 |
| .440 | −.58326 16692 24 | .560 | .490 | .28240 83062 16 | .510 |
| .441 | −.52360 70974 04 | .559 | .491 | .15028 16426 95 | .509 |
| .442 | −.38190 82986 12 | .558 | .492 | −.02361 75574 02 | .508 |
| .443 | −.17443 01231 01 | .557 | .493 | .06684 38933 29 | .507 |
| .444 | −.12778 79441 71 | .556 | .494 | .15875 42472 12 | .506 |
| .445 | −.32413 68662 58 | .555 | .495 | .23215 63219 75 | .505 |
| .446 | −.38999 44407 08 | .554 | .496 | .18367 46210 92 | .504 |
| .447 | −.35689 24535 27 | .553 | .497 | .01931 21600 07 | .503 |
| .448 | −.13266 84382 20 | .552 | .498 | .00378 55928 21 | .502 |
| .449 | .00059 06984 60 | .551 | .499 | −.04441 66347 11 | .501 |
| | | | .500 | .00000 00000 00 | .500 |
| | $-W(x)$ | $x$ | | $-W(x)$ | $x$ |