Gaussian Quadrature with Weight Function x^n on the Interval $(-1, 1)$

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Most of the existent literature dealing with Gaussian quadrature formulas is based on the assumption that the relevant weight function is of constant sign in the interval of integration. This note deals with the weight function x^n on the interval $(-1, 1)$ and indicates the extent to which the existent theory can be generalized in that case.

Gaussian quadrature formulas on the interval $(-1, 1)$ have the form

$$\int_{-1}^{1} w(x)f(x)\,dx = \sum_{k=1}^{m} W_k f(x_k) + E \tag{1}$$

where the weights and abscissas are chosen to ensure a degree of precision $2m - 1$ (i.e., exactness for all polynomials of degree not exceeding $2m - 1$).

One method of determining the abscissas x_k involves obtaining a set of polynomials, $\phi_1, \phi_2, \phi_3, \cdots$, such that each is orthogonal to all polynomials of inferior degree relative to the weight function $w(x)$ over the interval $(-1, 1)$. That is, for the mth-degree polynomial $\phi_m(x)$,

$$\int_{-1}^{1} w(x)\phi_m(x)q_{m-1}(x)\,dx = 0 \tag{2}$$

where q_{m-1} is an arbitrary polynomial of degree $m - 1$ or less. The m abscissas x_k then are the zeros of the polynomial ϕ_m.

Upon defining

$$w(x)\phi_m(x) = \frac{d^m U_m(x)}{dx^m} \tag{3}$$

the requirement that ϕ_m be of degree m implies that $U_m(x)$ must satisfy the differential equation

$$\frac{d^{m+1}}{dx^{m+1}}\left[\frac{1}{w(x)}\frac{d^m U_m(x)}{dx^m}\right] = 0 \tag{4}$$

in the interval $(-1, 1)$. From expression (2), the $2m$ boundary conditions

$$U_m(\pm 1) = U_m'(\pm 1) = \cdots = U_m^{(m-1)}(\pm 1) = 0 \tag{5}$$

can be obtained. When $w(x) = x^n$, the solution of (4) is found to be of the form

$$U_m(x) = x^m w(x)[c_0 + c_1 x + \cdots + c_m x^m] + d_0 + d_1 x + \cdots + d_{m-1}x^{m-1}$$

from which ϕ_m can be obtained. It is convenient to impose the additional normalizing property

$$\phi_m(1) = 1. \tag{6}$$

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From the theory as given in [1], the weight corresponding to the abscissas x_k for the N-point formula is

$$W_{N,k} = \frac{1}{\phi_N'(x_k)} \int_{-1}^{1} w(x) \frac{\phi_N(x)\, dx}{x - x_k}. \tag{7}$$

When the weight function $w(x)$ is of constant sign, (7) reduces to

$$W_{N,k} = \frac{A_N\, \gamma_{N-1}}{A_{N-1}\, \phi_N'(x_k)\phi_{N-1}(x_k)} = -\frac{A_{N+1}\, \gamma_N}{A_N\, \phi_N'(x_k)\phi_{N+1}(x_k)} \tag{8}$$

where A_N is the coefficient of x^N in ϕ_N and

$$\gamma_N = A_N \int_{-1}^{1} w(x) x^N \phi_N(x)\, dx. \tag{9}$$

The error function for the N-point formula can be expressed as

$$E_N = \int_{-1}^{1} w(x) f\, [x_1, x_1, \cdots, x_N, x_N, x] \pi_N^2(x)\, dx \tag{10}$$

with $\pi_N(x) = (x - x_1)(x - x_2) \cdots (x - x_N)$ where the x_i are the zeros of ϕ_N. The expression (10) reduces in the case when $w(x)$ is of constant sign to

$$E_N = \frac{\gamma_N f^{(2N)}(\eta)}{A_N^2 (2N)!} \tag{11}$$

where $f(x)$ is assumed to have $2N$ continuous derivatives in $(-1, 1)$ and η is some value in that interval.

When $w(x) = x^{2n}$, the procedure outlined above can be employed to obtain the following

$\Phi_0 = 1 \qquad \Phi_1(x) = x$

$$\Phi_2(x) = \frac{1}{2} [(2n + 3)x^2 - (2n + 1)]$$

$$\Phi_3(x) = \frac{1}{2} [(2n + 5)x^3 - (2n + 3)x]$$

$$\Phi_4(x) = \frac{1}{2!\, 2^2} [(2n + 7)(2n + 5)x^4 - 2(2n + 5)(2n + 3)x^2 + (2n + 3)(2n + 1)]$$

$$\Phi_5(x) = \frac{1}{2!\, 2^2} [(2n + 9)(2n + 7)x^5 - 2(2n + 7)(2n + 5)x^3 + (2n + 5)(2n + 3)x].$$

From these, the forms

$$\Phi_{2N}(x) = \frac{1}{N!\, 2^N} \sum_{i=0}^{N} \binom{N}{i} (-1)^i \prod_{j=1}^{N} [2n + 2(N - i) + 2j - 1]\, x^{2(N-i)} \qquad (m = 2N) \tag{12}$$

$$\Phi_{2N+1}(x) = \frac{1}{N!\, 2^N} \sum_{i=0}^{N} \binom{N}{i} (-1)^i \prod_{j=1}^{N} [2n + 2(N - i) + 2j + 1]\, x^{2(N-i)+1} \qquad (m = 2N + 1) \tag{13}$$

can be deduced [2] by an inductive procedure and shown to satisfy the desired properties (2) and (6). Since the weight function is of constant sign in the interval $(-1, 1)$, a known theorem [1, page 171–2] states that the zeros of the polynomials are all real, distinct, and lie in the interval $(-1, 1)$.

To display the formulas for the weights and error, the values A_N and γ_N must be obtained. From (12) and (13), it is easily seen that

$$(14)\qquad \begin{aligned} A_{2N} &= \frac{1}{N!\,2^N}\prod_{j=1}^{N}(2n+2N+2j-1) \\ A_{2N+1} &= \frac{1}{N!\,2^N}\prod_{j=1}^{N}(2n+2N+2j+1). \end{aligned}$$

The value

$$(15)\qquad \gamma_N = \frac{2}{2n+2N+1}$$

can be established by an inductive proof [2]. Thus, the weights are now expressed in the form

$$\begin{aligned} W_{2N,k} &= \frac{1}{N\Phi_{2N}'(x_k)\Phi_{2N-1}(x_k)} = \frac{-2}{(2n+2N+1)\Phi_{2N}'(x_k)\Phi_{2n+1}(x_k)} \\ W_{2N+1,k} &= \frac{2}{(2n+2N+1)\Phi_{2N+1}'(x_k)\Phi_{2N}(x_k)} = \frac{-1}{(N+1)\Phi_{2N+1}(x_k)\Phi_{2N+2}(_k)}. \end{aligned}$$

Since $w(x)$ is of constant sign, the error is given by (11) which can be written in the form

$$\begin{aligned} E_{2N} &= \frac{2(N!\,2^N)^2 f^{(4N)}(\eta)}{(4N)!\,(2n+4N+1)\prod_{j=1}^{N}(2n+2N+2j-1)^2} \\ E_{2N+1} &= \frac{2(N!\,2^N)^2 f^{(4N+2)}(\eta)}{(4N+2)!\,(2n+4N+3)\prod_{j=1}^{N}(2n+2N+2j+1)^2} \quad (-1<\eta<1). \end{aligned}$$

When $w(x) = x^{2n+1}$, it is found [2] that the formulas for an odd number of abscissas do not exist, but that the polynomials of even degree satisfying properties (2) and (6) do exist, and have the form

$$(16)\qquad \theta_{2N}(x) = \frac{1}{N!\,2^N}\sum_{i=0}^{N}\binom{N}{i}(-1)^i\prod_{j=1}^{N}[2n+2(N-i)+2j+1]\,x^{2(N-i)}.$$

Since $x\theta_{2N}(x) = \Phi_{2N+1}(x)$, the zeros of θ_{2N} are all real and distinct and lie in the interval $(-1, 1)$, and, in fact, are the zeros of Φ_{2N+1} when the zero $x = 0$ is suppressed.

The weights for the $2N$-point formula involve the polynomial Φ_{2N+1} in the following manner

$$W_{2N,k} = \frac{1}{\theta_{2N}'(x_k)}\int_{-1}^{1} x^{2n+1}\frac{\theta_{2N}(x)}{x-x_k}\,dx$$

$$= \frac{1}{\theta'_{2N}(x_k)} \int_{-1}^{1} x^{2n} \frac{\Phi_{2N+1}(x)}{x - x_k}\, dx$$

which reduces to

$$W_{2N,k} = \frac{-A_{2N+2}\,\gamma_{2N+1}}{A_{2N+1}\theta'_{2N}(x_k)\Phi_{2N+2}(x_k)} = \frac{A_{2N+1}\,\gamma_{2N}}{A_{2N}\theta'_{2N}(x_k)\Phi_{2N}(x_k)} \tag{17}$$

where the A's and γ's refer to the polynomial Φ and are given by (14) and (15). After the substitution of (14) and (15), (17) reduces to

$$W_{2N,k} = \frac{-1}{(N+1)\theta'_{2N}(x_k)\Phi_{2N+2}(x_k)} = \frac{2}{(2n+2N+1)\theta'_{2N}(x_k)\Phi_{2N}(x_k)}\,.$$

From (10), the error term E_{2N} can be written as

$$E_{2N} = \int_{-1}^{1} x^{2n+1} f[x_1, x_1, \cdots, x_{2N}, x_{2N}, x]\, \pi_{2N}^2(x)\, dx$$

TABLE 1

Gaussian Quadrature with Weight x^{2n}

m	n	Weights	Abscissas	Error Coefficients
2	0	1.0000000	± .5773503	7.4×10^{-3}
	1	.3333333	± .7745967	1.9×10^{-3}
	2	.2000000	± .8451543	7.6×10^{-4}
	3	.1428571	± .8819171	3.7×10^{-4}
	4	.1111111	± .9045340	2.1×10^{-4}
	5	.09090909	± .9198662	1.3×10^{-4}
3	0	.8888889	0	6.3×10^{-5}
		.5555556	± .7745967	
	1	.1066667	0	2.5×10^{-5}
		.2800000	± .8451543	
	2	.03265306	0	1.2×10^{-5}
		.1836735	± .8819171	
	3	.01410935	0	7.1×10^{-6}
		.1358025	± .9045340	
	4	.007346189	0	4.4×10^{-6}
		.1074380	± .9198662	
	5	.004303389	0	2.9×10^{-6}
		.08875740	± .9309493	
4	0	.3478548	± .8611363	2.9×10^{-7}
		.6521452	± .3399810	
	1	.1945553	± .9061798	7.3×10^{-8}
		.1387780	± .5384693	
	2	.1343622	± .9290483	2.5×10^{-8}
		.06563784	± .6399973	
	3	.1024498	± .9429254	1.0×10^{-8}
		.04040730	± .7039226	
	4	.08273203	± .9522526	4.9×10^{-9}
		.02837808	± .7482524	
	5	.06935661	± .9589554	2.6×10^{-9}
		.02155248	± .7809074	

which reduces after one integration by parts to

$$E_{2N} = -\int_{-1}^{1} f[x_1, x_1, \cdots, x_{2N}, x_{2N}, x, x] \int_{-1}^{x} x^{2n+1}\pi_{2N}^2(x)\, dx\, dx.$$

Now consider the function

$$A(x) = \int_{-1}^{x} x^{2n+1}\pi_{2N}^2(x)\, dx.$$

Since $A'(x)$ is negative for $x < 0$ and positive for $x > 0$ and $A(\pm 1) = 0$, it follows that $A(x)$ is of constant sign in $(-1, 1)$ and hence

$$\begin{aligned} E_{2N} &= -f[x_1, \cdots, x_{2N}, \xi, \xi] \int_{-1}^{1}\int_{-1}^{x} x^{2n+1}\pi_{2N}^2(x)\, dx \qquad (-1 < \xi < 1) \\ &= \frac{f^{(4N+1)}(\eta)}{(4N+1)!}\int_{-1}^{1} x^{2n}K_{2N+1}^2(x)\, dx \qquad (-1 < \eta < 1) \end{aligned} \tag{18}$$

where $f(x)$ is assumed to have $4N + 1$ continuous derivatives in $(-1, 1)$. The function K_{2N+1} is the monic polynomial consisting of the linear factors of Φ_{2N+1} and therefore (18) reduces to

$$E_{2N} = \frac{f^{(4N+1)}(\eta)\gamma_{2N+1}}{(4N+1)!\,A_{2N+1}^2}$$

i. e.

$$E_{2N} = \frac{2(N!\,2^N)^2 f^{(4N+1)}(\eta)}{(4N+1)!(2n+4N+3)\prod_{j=1}^{N}(2n+2N+2j+1)^2}.$$

TABLE 2

Gaussian Quadrature with Weight x^{2n+1}

m	n	Weights	Abscissas	Error coefficients
2	0	±.4303315	±.7745967	3.8×10^{-4}
	1	±.2366432	±.8451543	1.5×10^{-4}
	2	±.1619848	±.8819171	7.5×10^{-5}
	3	±.1228380	±.9045340	4.2×10^{-5}
	4	±.09882860	±.9198662	2.6×10^{-5}
	5	±.08262864	±.9309493	1.7×10^{-5}
4	0	±.2577268	±.5384693	8.1×10^{-9}
		±.2146984	±.9061798	
	1	±.1025596	±.6399973	2.8×10^{-9}
		±.1446234	±.9290483	
	2	±.05740305	±.7039226	1.1×10^{-9}
		±.1086511	±.9429254	
	3	±.03792714	±.7482524	5.5×10^{-10}
		±.08688035	±.9522526	
	4	±.02759928	±.7809074	2.9×10^{-10}
		±.07232517	±.9589554	
	5	±.02137648	±.8060023	1.6×10^{-10}
		±.06192242	±.9640060	

In Tables 1 and 2, the weights and abscissas are given for $m = 2, 3, 4$, for $w(x) = x^{2n}$ and for $m = 2, 4$, in the case $w(x) = x^{2n+1}$. The values for the weights and abscissas are correct to the 7 figures given. The coefficient of $f^{(s)}(\eta)$ in the error term is also given. These error coefficients are correct to the two figures given.

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1. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.
2. H. A. Rothmann, "Numerical integration over the interval $(-1, 1)$ with the weight function x^n," Unpublished M.S. Thesis, Massachusetts Institute of Technology, 1960.