Stability Analysis and Integration of the Viscous Equations of Motion

By L. Filler & H. F. Ludloff

1. Introduction. Techniques are established for the numerical solution of the time-dependent, one-dimensional equations of motion of a viscous, heat-conducting fluid. The equations of motion are approximated by suitable finite difference equations and the stability of the difference equations is investigated by using von Neumann's error analysis. Both explicit and implicit finite difference schemes are studied. The implicit equations are solved by an iteration scheme that is formulated with the requisite that its convergence does not place conditions on the mesh width ratio $\frac{\Delta t}{\Delta x}$.

The dimensionless equations of motion to be treated are written in divergence form [1]

$$\frac{\partial \rho}{\partial t} + \frac{\partial M}{\partial x} = 0$$

$$\frac{\partial M}{\partial t} + \frac{\partial R}{\partial x} = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial T}{\partial x} = 0$$

where ρ is the density, M the mass flow vector, E the total energy, and R and T are vectors which are certain non-linear combinations of ρ, M, and E, namely:

$$R = (\gamma - 1)E + \frac{3 - \gamma}{2} \frac{M^2}{\rho} - \frac{4}{3} \mu \frac{\partial(M/\rho)}{\partial x}$$

$$T = \gamma \frac{ME}{\rho} - \frac{\gamma - 1}{2} \frac{M^3}{\rho^2} - \left(\frac{4}{3} - \frac{\gamma}{\sigma}\right) \mu \frac{M}{\rho} \frac{\partial(M/\rho)}{\partial x} - \frac{\gamma}{\sigma} \mu \frac{\partial(E/\rho)}{\partial x}$$

with the usual notation that γ is the ratio of specific heats (constant $= \frac{7}{5}$), μ the coefficient of viscosity which is assumed to vary linearly with temperature, and σ the Prandtl number (constant $= \frac{3}{4}$).

This system of differential equations is replaced by a system of difference equations as follows. Derivatives with respect to time $\frac{\partial f}{\partial t}$ are approximated by the forward difference quotient

$$\frac{1}{\Delta t} \{ f_{m,n+1} - f_{m,n} \}$$

where $f_{m,n}$ denotes the value of f at the lattice point $x = m\Delta x$ and $t = n\Delta t$. Space

Received November 7, 1960. This work was supported by the Office of Naval Research.

derivatives $\frac{\partial^k f}{\partial x^k}$ are approximated by the difference quotients

$$(\Delta x)^{-k} \sum_{\lambda=0}^{k} (-1)^{\lambda} \binom{k}{\lambda} f_{m+k'-\lambda, n+\omega}$$

where $\binom{k}{\lambda}$ denotes the binominal coefficient and k' is the largest integer $\leqq \frac{k}{2}$. The index $\omega = 0$ for the explicit difference scheme formulation and $\omega = 1$ for the implicit scheme. Note that this approximation yields backward first differences in space and centered second differences in space.

Both the explicit and implicit difference approximations are used to compute the formation of a shock wave from a finite amplitude compression pulse. The computations were carried out on an IBM 650 digital computer at the Watson Scientific Computing Laboratory of Columbia University.

2. Explicit Difference Scheme. The explicit difference equations are with the notation $\Delta f_{m-1,n} = f_{m,n} - f_{m-1,n}$

$$\rho_{m,n+1} = \rho_{m,n} - \frac{\Delta t}{\Delta x} \Delta M_{m-1,n}$$

$$M_{m,n+1} = M_{m,n} - \frac{\Delta t}{\Delta x} \Delta R_{m-1,n}$$

$$E_{m,n+1} = E_{m,n} - \frac{\Delta t}{\Delta x} \Delta T_{m-1,n}$$

where, for example,

$$\Delta R_{m-1,n} = (\gamma - 1)\Delta E_{m-1,n} + \frac{3-\gamma}{2} \Delta \left(\frac{M^2}{\rho}\right)_{m-1,n} - \frac{4}{3} [\Delta \mu_{m-1,n}] \left[\Delta \left(\frac{M}{\rho}\right)_{m-1,n}\right] - \tfrac{4}{3} \mu_{m,n} \Delta^2 \left(\frac{M}{\rho}\right)_{m-1,n}.$$

The linear variation of the coefficient of viscosity with temperature is written in terms of the dependent variables as

$$\mu_{m,n} = \gamma(\gamma - 1) \left[\left(\frac{E}{\rho}\right)_{m,n} - \frac{1}{2}\left(\frac{M}{\rho}\right)^2_{m,n}\right].$$

The computation of the explicit difference equations proceeds directly, since all terms on the right-hand side of the equations are known values taken at the previous time cycle. The unknowns on the left-hand side are computed at each lattice point of the mesh in the new time cycle.

The analysis of the stability of the difference equations follows the procedure of von Neumann [2]. However, the original system of equations is non-linear, and the variational equation obtained for the error has non-constant coefficients. In order to apply von Neumann's method of stability analysis the coefficients are systematically set constant, and all derivatives that appear in the *coefficients* are set to zero. Setting the derivatives that appear in the coefficients of the variational equation to zero is arbitrary. One could, with as much justification, consider them constant. However, the stability analysis is greatly simplified if only those terms not having derivatives as coefficients are retained. This does not seem to impair the results of the analysis when checked against actual computation. Indeed, when such a treatment is applied to the non-viscous hyperbolic equations of motion, the correct result is obtained: that stability is determined in terms of the characteristic directions [3]. Thus, the variational equation obtained from the system of difference equations may be written

$$U_{m,n+1} = [I - A]U_{m,n} + BU_{m+1,n} + CU_{m-1,n}$$

where

$$U_{m,n} = \begin{pmatrix} \rho \\ M \\ E \end{pmatrix}_{m,n}$$

Substituting for the error the Fourier term $\epsilon_0 e^{\alpha t} e^{i\beta x}$ leads to the result:

$$\lambda U_0 = [I - A(1 - \cos\theta) + iD \sin\theta]U_0 . \tag{2.1}$$

Here $\lambda = e^{\alpha\Delta t}$ and $\theta = \beta\Delta x$

and the matrix $D = B - C$. Note that $A = B + C$. Let $H = -A(1 - \cos\theta) + iD\sin\theta$ and write the spectral equation (2.1) as

$$[H - KI]U_0 = 0$$

with $K = \lambda - 1.$

The elements of H are, with $\Gamma = \dfrac{\Delta t}{\Delta x}(1 - \cos\theta)$, $\psi = \dfrac{\Delta t}{\Delta x}\sin\theta$, and in combination $\Phi = \Gamma + i\psi$

$$h_{11} = 0$$

$$h_{12} = -\Phi$$

$$h_{13} = 0$$

$$h_{21} = \frac{3-\gamma}{2}u^2\Phi + \frac{8}{3}u\frac{\nu}{\Delta x}\Gamma$$

$$h_{22} = -(3-\gamma)u\Phi - \frac{8}{3}\frac{\nu}{\Delta x}\Gamma$$

$$h_{23} = -(\gamma - 1)\Phi$$

$$h_{31} = -\left[(\gamma-1)u^3 - \gamma\frac{E}{\rho}u\right]\Phi + 2\left[\left(\frac{4}{3} - \frac{\gamma}{\sigma}\right)u^2 + \frac{\gamma}{\sigma}\frac{E}{\rho}\right]\frac{\nu}{\Delta x}\Gamma$$

$$h_{32} = -\left[\gamma\frac{E}{\rho} - \frac{3}{2}(\gamma-1)u^2\right]\Phi - 2\left(\frac{4}{3} - \frac{\gamma}{\sigma}\right)u\frac{\nu}{\Delta x}\Gamma$$

$$h_{33} = -\gamma u\Phi - 2\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma.$$

Here ν is the kinematic viscosity coefficient μ/ρ. Now, K the eigenvalues of H satisfy the cubic equation

$$K^3 - [h_{22} + h_{33}]K^2 - [h_{12}h_{21} + h_{23}h_{32} - h_{22}h_{33}]K - [h_{12}h_{23}h_{31} - h_{12}h_{21}h_{33}] = 0$$

and it may be verified by direct substitution that

$$K = \frac{1}{\gamma} h_{33} = -\left[u\Phi + 2\frac{\nu}{\sigma\Delta x}\Gamma\right] \tag{2.2}$$

is a root of the cubic equation if $\sigma = \frac{3}{4}$. Factoring this root, the roots of the remaining quadratic are

$$K = -\left[u\Phi + \frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma\right] \pm \sqrt{a^2\Phi^2 + \left(\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\right)^2\Gamma^2} \tag{2.3}$$

where

$$a^2 = \gamma(\gamma - 1)\left[\frac{E}{\rho} - \tfrac{1}{2}u^2\right].$$

Thus, the requirement for stability $|\lambda| = |K + 1| \leqq 1$ yields respectively,

$$\left|1 - \left[u\Phi + 2\frac{\nu}{\sigma\Delta x}\Gamma\right]\right| \leqq 1 \tag{2.4}$$

$$\left|1 - \left\{u\Phi + \frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma + \sqrt{a^2\Phi^2 + \left(\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\right)^2\Gamma^2}\right\}\right| \leqq 1 \tag{2.5}$$

and

$$\left|1 - \left\{u\Phi + \frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma - \sqrt{a^2\Phi^2 + \left(\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\right)^2\Gamma^2}\right\}\right| \leqq 1. \tag{2.6}$$

Each of the above inequalities must be considered to determine the conditions for the numerical stability of the difference scheme. For example, it may be shown that inequality (2.4) has a simple geometrical interpretation, viz., the absolute value of the locus of the ellipse in the complex η, ξ plane is less than or equal to one, where

$$\lambda = \eta + i\xi$$

$$\eta = 1 - \Gamma\left[u + 2\frac{\nu}{\sigma\Delta x}\right]$$

and $\xi = u\psi$.
Thus, the stability condition obtained from (2.4) is

$$\frac{\Delta t}{\Delta x} \leqq \frac{1}{u + 2\dfrac{\nu}{\sigma\Delta x}}. \tag{2.7}$$

The form of the inequality (2.5) suggests the conservative approximation

$$\left|1 - \left\{(u + a)\Phi + 2\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma\right\}\right| \leqq 1$$

which gives the stability condition

$$\frac{\Delta t}{\Delta x} \leqq \frac{1}{u + a + 2\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}} . \tag{2.8}$$

The inequality 2.6 is approximated by

$$\left| 1 - \left\{ u\Phi + \frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma \right\} + \frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma \left\{ 1 + \frac{1}{2}\left(\frac{a\Phi}{\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma} \right)^2 \right\} \right| \leqq 1 .$$

This condition is critical when $u = 0$, since it can be fulfilled only if

$$\operatorname{Re}\left\{ 1 + \frac{1}{2}\left(\frac{a\Phi}{\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma} \right)^2 \right\} \to 1$$

and

$$\operatorname{Im}\left\{ 1 + \frac{1}{2}\left(\frac{a\Phi}{\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}\Gamma} \right)^2 \right\} \to 0 .$$

Consequently, we require

$$\frac{\gamma}{\sigma}\frac{\nu}{\Delta x} \gg a \,|_{u=0} .$$

For practical purposes, it has been verified when calculations were carried out that a reasonable estimate is

$$\frac{\gamma}{\sigma}\frac{\nu}{\Delta x} \geqq \frac{3}{2} a \,|_{u=0} . \tag{2.9}$$

The most stringent restriction on the mesh width ratio is

$$\frac{\Delta t}{\Delta x} \leqq \frac{1}{u + a + 2\frac{\gamma}{\sigma}\frac{\nu}{\Delta x}} \tag{2.8}$$

which together with (2.9) give the conditions for stability of the explicit difference equations.

It may be noted from (2.2) and (2.3) that in the limit as the viscosity tends to zero the eigenvalues K degenerate properly so that the correct stability conditions for the hyperbolic equations of motion are obtained.

3. Implicit Difference Scheme. The implicit difference scheme equations are written out in full

$$\rho_{m,n+1} = \rho_{m,n} - \frac{\Delta t}{\Delta x} \Delta M_{m-1,n+1}$$

$$M_{m,n+1} = M_{m,n} - (\gamma - 1)\frac{\Delta t}{\Delta x} \Delta E_{m-1,n+1}$$

$$- \frac{3 - \gamma}{2} \frac{\Delta t}{\Delta x} \Delta \left(\frac{M^2}{\rho}\right)_{m-1,n+1}$$

$$+ \frac{4}{3} \mu \frac{\Delta t}{(\Delta x)^2} \Delta^2 \left(\frac{M}{\rho}\right)_{m-1,n+1}$$

$$E_{m,n+1} = E_{m,n} - \gamma \frac{\Delta t}{\Delta x} \Delta \left(\frac{ME}{\rho}\right)_{m-1,n+1}$$

$$+ \frac{\gamma - 1}{2} \frac{\Delta t}{\Delta x} \Delta \left(\frac{M^3}{\rho^2}\right)_{m-1,n+1}$$

$$+ \left(\frac{4}{3} - \frac{\gamma}{\rho}\right) \mu \frac{\Delta t}{(\Delta x)^2} \left\{ \left[\Delta \left(\frac{M}{\rho}\right)_{m-1,n+1} \right]^2 \right.$$

$$\left. + \left(\frac{M}{\rho}\right)_{m,n+1} \times \Delta^2 \left(\frac{M}{\rho}\right)_{m-1,n+1} \right\}$$

$$+ \frac{\mu\gamma}{\sigma} \frac{\Delta t}{(\Delta x)^2} \Delta^2 \left(\frac{E}{\rho}\right)_{m-1,n+1} .$$

Note that the implicit difference equations are written with the assumption of constant coefficient of viscosity μ to facilitate solution by an iterative method.

An inspection of the implicit difference equations reveals that the continuity equation occupies a preferred position in the system of equations. The $\rho_{m,n+1}$ to be computed depends only on unknown M's at the lattice points $m, n + 1$ and $m - 1, n + 1$. The computations are started from a left-hand boundary, or from a region where conditions are known to be constant, so that the point $m - 1, n + 1$ falls on the boundary or in the region of constant value. Thus the values of ρ, M, and E at the lattice point $m - 1, n + 1$ are known. Then, for the continuity equation, the only unknown is the $M_{m,n+1}$. To obtain an approximate value of $M_{m,n+1}$ the momentum equation is set up for solution by an iteration procedure. When the $M_{m,n+1}$ has been approximately determined, it is inserted into the continuity equation and an approximate value of $\rho_{m,n+1}$ may be directly computed. Proceeding in this manner along a line of computation $m + 1$, the $M_{m+1,n+1}$ is next calculated by iteration, after inserting the previously determined values of M and ρ at the lattice point $m, n + 1$. In this way approximate values of M and ρ are computed at all lattice points on the line $n + 1$. When these values have been determined, the energy equation may be set up as a three-point recursion formula for $E_{m-1,n+1}$, $E_{m,n+1}$, and $E_{m+1,n+1}$ which may be solved by a standard reduction procedure. When the first set of computations has been completed we have approximate values of ρ, M, and E at all lattice points on the computation line $n + 1$. The calculations are repeated until convergence is achieved.*

The procedure for computing the momentum equation takes the form of a systematic relaxation method. Setting up the difference operator

$$G^j_{m,n+1} = \frac{4}{3} \mu \frac{\Delta t}{(\Delta x)^2} \Delta^2 \left(\frac{M}{\rho}\right)^j_{m-1,n+1}$$

---

* This procedure was adopted upon the suggestion of H. Keller, Institute of Mathematical Sciences, New York University.

$$- \left\{ (M^j_{m,n+1} - M_{m,n}) + (\gamma - 1) \frac{\Delta t}{\Delta x} \Delta E^j_{m-1,n+1} + \frac{3-\gamma}{2} \frac{\Delta t}{\Delta x} \Delta \left(\frac{M^2}{\rho}\right)^j_{m-1,n+1} \right\}$$

where the iteration number is given by j. We then write the following equation for $M^{j+1}_{m,n+1}$

$$M^{j+1}_{m,n+1} = M^j_{m,n+1} + \delta G^j_{m,n+1} \,. \tag{3.1}$$

Approximations to the unknowns at iteration number j are inserted into the expression $G^j_{m,n+1}$ leaving a remainder or residue, since the exact values satisfy $G^j_{m,n+1} = 0$. The residue is systematically reduced by the factor δ until convergence is obtained.

The remaining equations are written so that they may be solved directly once the approximations to the momentum equation are available.

$$\rho^{j+1}_{m,n+1} = \rho_{m,n} - \frac{\Delta t}{\Delta x} \Delta M^{j+1}_{m-1,n+1} \tag{3.2}$$

$$\begin{aligned}
&\left[1 + \gamma \frac{\Delta t}{\Delta x} \left(\frac{M}{\rho}\right)^{j+1}_{m,n+1} + 2 \frac{\gamma}{\sigma} \frac{\mu}{\rho^{j+1}_{m,n+1}} \frac{\Delta t}{(\Delta x)^2}\right] E^{j+1}_{m,n+1} \\
&- \frac{\gamma}{\sigma} \frac{\mu}{\rho^{j+1}_{m+1,n+1}} \frac{\Delta t}{(\Delta x)^2} E^{j+1}_{m+1,n+1} - \left[\gamma \frac{\Delta t}{\Delta x} \left(\frac{M}{\rho}\right)^{j+1}_{m-1,n+1} \right. \\
&\left. + \frac{\gamma}{\sigma} \frac{\mu}{\rho^{j+1}_{m-1,n+1}} \frac{\Delta t}{(\Delta x)^2}\right] E^{j+1}_{m-1,n+1} = E_{m,n} \\
&+ \frac{\gamma - 1}{2} \frac{\Delta t}{\Delta x} \Delta \left(\frac{M^3}{\rho^2}\right)^{j+1}_{m-1,n+1} \\
&+ \mu \left(\frac{4}{3} - \frac{\gamma}{\sigma}\right) \frac{\Delta t}{(\Delta x)^2} \left\{ \left[\Delta \left(\frac{M}{\rho}\right)^{j+1}_{m-1,n+1}\right]^2 \right. \\
&\left. + \left(\frac{M}{\rho}\right)^{j+1}_{m,n+1} \Delta^2 \left(\frac{M}{\rho}\right)^{j+1}_{m,n+1} \right\}
\end{aligned} \tag{3.3}$$

In (3.2) the $M^{j+1}_{m-1,n+1}$ is known and the $M^{j+1}_{m,n+1}$ is inserted from the solution of (3.1). In (3.3) the coefficients of the $E^{j+1}_{m's,n+1}$ may be computed after the values of M and ρ have been found from (3.1) and (3.2). The right-hand side of the equation is similarly given. Equation (3.3) is thus a linear system of equations in the unknowns $E^{j+1}_{m's,n+1}$. This system is tridiagonal and may be written as follows:

$$\sum_{q=1}^{k} a_{pq} E^{j+1}_{q,n+1} = b_p \; ; \; p = 1, 2, \cdots k \tag{3.4}$$

The matrix (a_{pq}) is triangularized by letting

$$a'_{pq} = \frac{a_{pq}}{a_{p(p-1)}} \; ; \qquad p \neq 1$$

and

$$a_{pp}^{*} = a'_{pp} - \frac{a'_{(p-1)p}}{a^{*}_{(p-1)(p-1)}}; \qquad p \neq 1.$$

Equation (3.4) becomes

$$\sum_{q=1}^{k} a_{pq}^{*} E_{q,n+1}^{j+1} = b_p; \qquad p = 1, 2, \cdots k \tag{3.5}$$

with the triangular matrix a_{pq}^{*}. Equation (3.5) is easily inverted to yield the $E_{m's,n+1}^{j+1}$.

The convergence of the iteration procedure used for the momentum equation,

$$M_{m,n+1}^{j+1} = M_{m,n+1}^{j} + \delta G_{m,n+1}^{j} \tag{3.1}$$

follows the techniques used to analyze the stability of the explicit difference scheme. Equation (3.1) is "linearized" and a von Neumann analysis of the "error" is made [4].

Convergence of (3.1) implies convergence of the continuity and energy equations. To establish convergence of the iteration scheme we consider convergence of the $M_{m,n+1}$ iterates with predetermined values of ρ and E obtained at each lattice point from (3.2) and (3.5) respectively.

Thus,

$$\begin{aligned} M_{m,n+1}^{j+1} = M_{m,n+1}^{j} + \delta \Bigg\{ & \frac{4}{3} \frac{\nu}{\Delta x} \frac{\Delta t}{\Delta x} \Delta^2 M_{m-1,n+1}^{j} \\ & - \frac{4}{3} u \frac{\nu}{\Delta x} \frac{\Delta t}{\Delta x} \Delta^2 \rho_{m-1,n+1}^{j} - \Bigg[(M_{m,n+1}^{j} - M_{m,n}) \\ & + (\gamma - 1) \frac{\Delta t}{\Delta x} \Delta E_{m-1,n+1}^{j} \\ & + (3 - \gamma) u \frac{\Delta t}{\Delta x} \Delta M_{m-1,n+1}^{j} \\ & - \frac{3-\gamma}{2} u^2 \frac{\Delta t}{\Delta x} \Delta \rho_{m-1,n+1}^{j} \Bigg] \Bigg\} \end{aligned} \tag{3.6}$$

Here $\nu = \frac{\mu}{\rho}$, $u = \frac{M}{\rho}$.

Before introducing an error for the iterations we note that $M_{m,n}$ is considered an errorless input, and by virtue of holding ρ and E fixed at each lattice point they will also be errorless inputs. Therefore, the error term considered is

$$M_{m,n+1}^{j} = e^{\alpha j} e^{i\beta x}.$$

This term is introduced into (3.6) with fixed ρ and E, giving

$$e^{\alpha} = p + q \cos \theta + ir \sin \theta$$

where

$$p = 1 - \delta \left\{ 1 + \left[\frac{8}{3} \frac{\nu}{\Delta x} + (3 - \gamma) u \right] \frac{\Delta t}{\Delta x} \right\}$$

$$q = \delta \left[\frac{8}{3} \frac{\nu}{\Delta x} + (3 - \gamma) u \right] \frac{\Delta t}{\Delta x}$$

$$r = -\delta (3 - \gamma) u \frac{\Delta t}{\Delta x} .$$

Since $q^2 > r^2$ the convergence requirement is

$$| p | + | q | \leqq 1 \tag{3.7}$$

which yields upon substitution of p and q

$$0 \leqq \delta \leqq \frac{1}{1 + \left[\frac{8}{3} \frac{\nu}{\Delta x} + (3 - \gamma) u \right] \frac{\Delta t}{\Delta x}} \tag{3.8}$$

if $u \geqq 0$. The inequality (3.8) identically satisfies (3.7). However, (3.7) may be satisfied with

$$\delta > \frac{1}{1 + \left[\frac{8}{3} \frac{\nu}{\Delta x} + (3 - \gamma) u \right] \frac{\Delta t}{\Delta x}} . \tag{3.9}$$

If (3.9) holds we get from (3.7) that at most

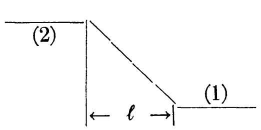
$$\delta \leqq \frac{1}{\frac{1}{2} + \left[\frac{8}{3} \frac{\nu}{\Delta x} + (3 - \gamma) u \right] \frac{\Delta t}{\Delta x}} . \tag{3.10}$$

The least restrictive condition on δ that covers all cases is obtained from the inequality (3.10).

Thus, the mesh width ratio may be arbitrarily chosen for the implicit scheme but convergence of the iteration procedure is obtained by choosing δ according to (3.10).

4. The Formation of a Shock Wave. The formation of a shock wave from a finite amplitude pulse has been numerically treated in three ways. First, a technique due to Lax [5] was used, where the equations of motion of an ideal fluid are used to compute flow fields in which shocks may develop. Second, the explicit difference formulation of the Navier-Stokes equations has been applied to the problem, with restrictions on the mesh-width ratio as previously determined in Section 2. Third, the implicit difference scheme approximation with the previously described iteration technique has been applied. An isentropic initial field is prescribed consisting of two homogeneous states of different velocity, pressure, and density connected by a simple compression wave. The problem is then formulated by asking for the development of this field in time as governed by the equations of motion. In particular we look for the time required for the shock to form, the shock's final shape, and especially how the entropy profile, characteristic of shocks, develops.

The numerical values used are



$$\rho_1 = 1 \qquad \rho_2 = 3.35$$
$$M_1 = 0 \qquad M_2 = 4.52$$
$$E_1 = 1.885 \qquad E_2 = 12.6$$

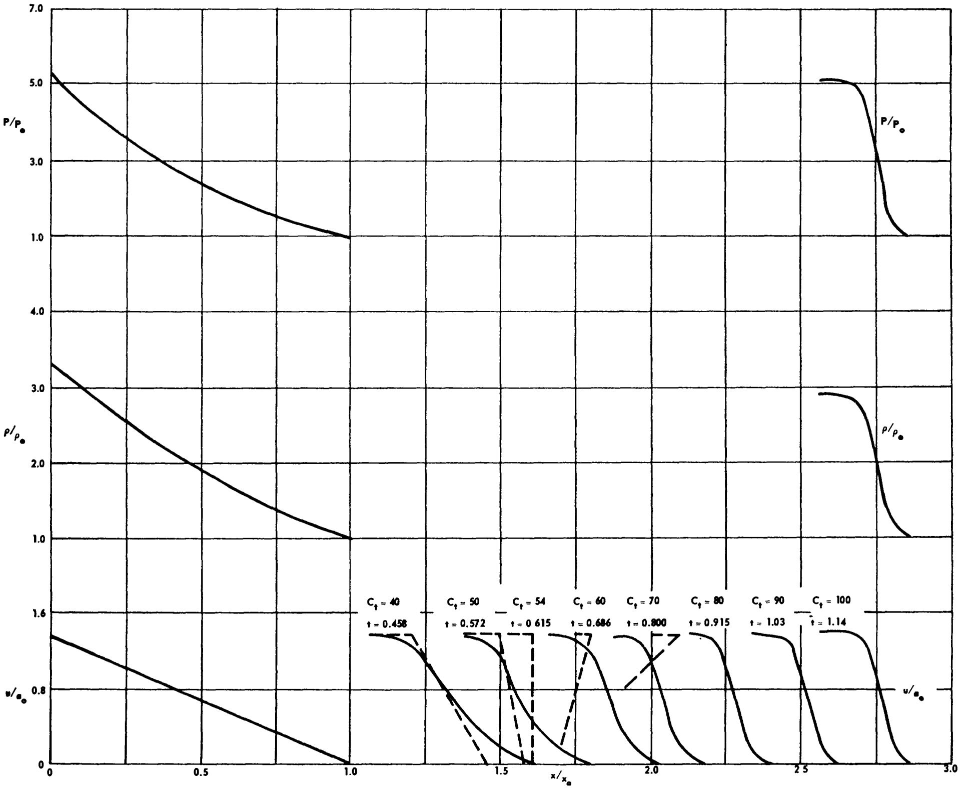


FIG. 1.—Non-Viscous Case.

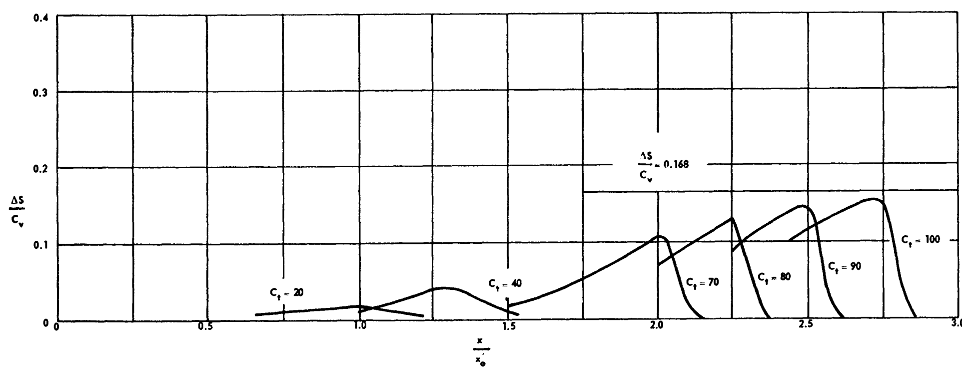


FIG. 2.—Entropy Growth Non-Viscous Case.

The initial velocity decrease from state (2) to state (1) is linear. The unit length, ℓ, was subdivided into 32 space intervals Δx. For the viscous equations a viscosity parameter $\frac{\nu}{\Delta x} = 1$ was taken for both the explicit and implicit difference schemes. This value is seen to satisfy the inequality (2.9) with $\gamma = \frac{7}{5}$, $\sigma = \frac{3}{4}$ and

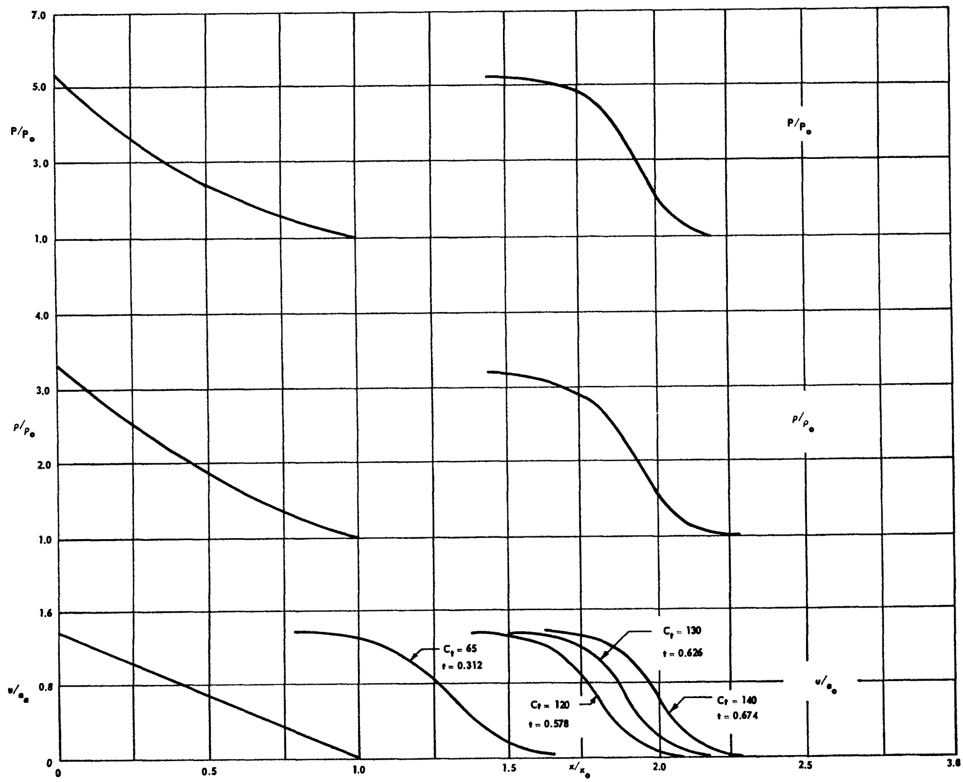


FIG. 3.—Viscous Case $\frac{\nu}{\Delta x} = 1.0$; $\bar{\mu} = 0.03125\ \mu$.

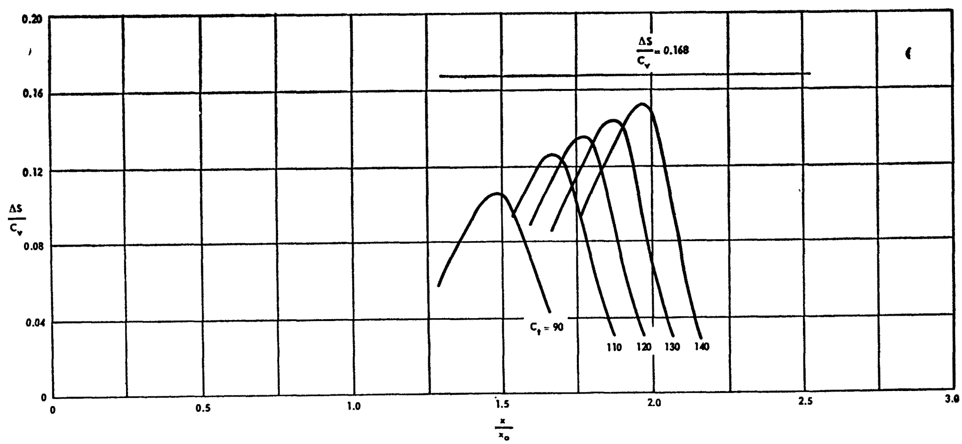


FIG. 4.—Entropy Change Viscous Case.

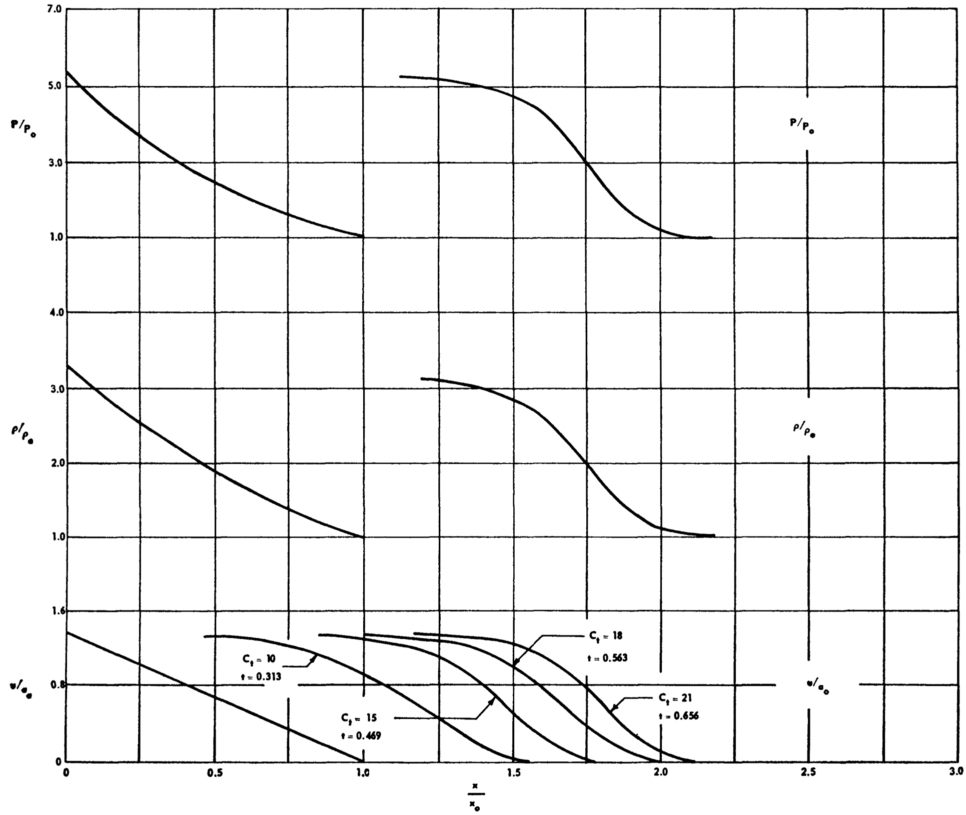


FIG. 5.—Viscous Case $\frac{\nu}{\Delta x} = 1.0; \bar{\mu} = 0.03125\ \mu.$

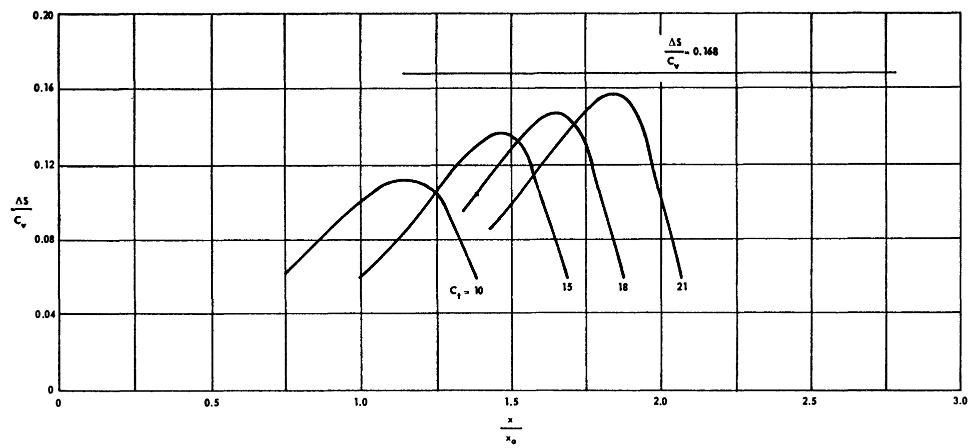


FIG. 6.—Entropy Change Viscous Case.

$a|_{u=0} = 1$. The stability criterion (2.8) for the explicit scheme with the specified initial field values is

$$\frac{\Delta t}{\Delta x} = 0.155 \quad \text{for the viscous equations}$$

$$\frac{\Delta t}{\Delta x} = 0.366 \quad \text{for the ideal equations.}$$

For the implicit difference scheme we obtain*

$$\delta = 0.167$$

from (3.10) and arbitrarily chosen $\dfrac{\Delta t}{\Delta x} = 1$.

The computation of the viscous implicit equations is somewhat arbitrary; the machine controls should be flexible so that the most economical way of solution may be found. The values of M and ρ should first be computed at each lattice point, then inserted back into the equations for M and ρ and computed a second time. Then these second computed values of M and ρ are used to compute E. The process is repeated until convergence is reached. Thus, two "inner" iterations for M and ρ per "outer" iteration for E was found to give the most rapid convergence (9 outer iterations per time cycle). The results of the machine computations are shown in Figures 1 and 2 for the ideal equations, Figures 3 and 4 for the viscous explicit equations, and Figures 5 and 6 for the viscous implicit equations. In each case, the time development has been continued until the form of the profiles becomes stationary, i.e., only a translation takes place. This was coincident with the time at which the entropy had built up to the value to be expected from stationary shock theory for a shock of the given pressure discontinuity.

The velocity profile, which falls off linearly within 32 mesh intervals in the initial field, steepens rapidly, especially in the ideal rather than in the viscous case.

In the ideal case it takes approximately 100 time cycles, in the viscous explicit case 140 time cycles, while in the viscous implicit case approximately 21 time cycles to reach the final state. But the absolute time for the shock formation, while essentially equal in the viscous cases ($t = 0.67$ explicit; $t = 0.66$, implicit) is longer in the ideal case ($t = 1.14$). This is, of course, to be expected since the entropy buildup is proportional to the viscosity present.

Note also that the slope of the entropy curves is steep in front, as it should be, but falls off to zero in the rear, indicating that the air particles at the rear have not been swept over by the fully developed shock.

5. Acknowledgements. The authors wish to express their appreciation to Dr. H. Keller for suggesting the method of solution used for the implicit difference formulation; also to Professor E. Isaacson for many stimulating discussions.

The Martin Company
Denver, Colorado, and

New York University
New York 3, N. Y.

* $\delta = 0.2$ was found in practice, to give convergence of the iteration scheme.

1. R. Courant & K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, Inc., New York, 1948.
2. G. G. O'Brien, M. A. Hyman & S. Kaplan, "A study of the numerical solution of partial differential equations," *J. Math. Phys.*, v. 29, 1951, p. 223–251.
3. R. Courant, E. Isaacson & M. Rees, "On the solution of non-linear hyperbolic equations by finite differences," *Comm. Pure Appl. Math.*, v. 5, no. 3, 1952, p. 243–255.
4. L. Filler, H. F. Ludloff & P. Truenfels, "A useful method for integrating the time-dependent, viscous equations of motion," *J. Aero. Sci.*, v. 24, 1957, p. 147.
5. Peter D. Lax, "Weak solutions of non-linear hyperbolic equations and their numerical computation," *Comm. Pure Appl. Math.*, v. 7, no. 1, 1954, p. 159–193.