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Bi-Variate, Rectangular, Optimum-Interval Interpolation

By Ferdinand Freudenstein

1. Definition of "Optimum-Interval" Interpolation. Let the real function $f(x, y)$, defined in the interval $-h \leq x \leq h$, $-k \leq y \leq k$, $h > 0$, $k > 0$, and assumed to have continuous partial derivatives of order up to $(m + n + 2)$ in the rectangle, be approximated by the polynomial

$$(1) \qquad P_{mn}(x, y) = \sum_{r=0}^{m} \sum_{s=0}^{n} a_{rs} x^r y^s$$

so that at the $(m + 1)(n + 1)$ *precision points*, (x_u, y_v), $u = 0, 1, 2, \cdots m$; $v = 0, 1, 2, \cdots n$,

$$(2) \qquad P_{mn}(x_u, y_v) = f(x_u, y_v).$$

In that case,

$$(3) \qquad f(x, y) = P_{mn}(x, y) + R_{mn}$$

where the *remainder*, R_{mn}, is given by the expression (see [1])

$$
\begin{aligned}
(4) \qquad R_{mn} = {} & \frac{1}{(m + 1)!} \prod_{u=0}^{m} (x - x_u) \frac{\partial^{m+1} f(\xi, y)}{\partial \xi^{m+1}} + \frac{1}{(n + 1)!} \prod_{v=0}^{n} (y - y_v) \frac{\partial^{n+1} f(x, \eta)}{\partial \eta^{n+1}} \\
& - \frac{1}{(m + 1)!(n + 1)!} \prod_{u=0}^{m} (x - x_u) \prod_{v=0}^{n} (y - y_v) \frac{\partial^{(m+1)+(n+1)} f(\xi' \eta')}{\partial \xi'^{m+1} \partial \eta'^{n+1}}
\end{aligned}
$$

where $\xi, \xi'(\eta, \eta')$ lie between the greatest and least of the numbers $x, x_u(y, y_v)$.

Consider the set, S, of all precision points (x_u, y_v) within the rectangle $(-h, h)$ $(-k, k)$—called (h, k) from now on. S will be called the spacing. Let $h = k\delta$, where δ is a positive number defining the proportions of the rectangle. Furthermore, we restrict ourselves to functions for which the partial derivatives $\dfrac{\partial^{m+1} f(x,y)}{\partial x^{m+1}}, \dfrac{\partial^{n+1} f(x,y)}{\partial y^{n+1}},$ $\dfrac{\partial^{(m+1)+(n+1)} f(x,y)}{\partial x^{m+1} \partial y^{n+1}}$ are not equal to zero within (h, k). Let R be the largest lower limit on the maximum absolute value of R_{mn} regardless of spacing. It is assumed

that $f(x, y)$ is not of the form $P_{mn}(x, y)$; thus $R > 0$. A particular spacing, S_0, will be determined with the property

$$\lim_{h \to 0} \left(\frac{\text{Maximum absolute value of } R_{mn} \text{ with spacing } S_0}{R} \right) = 1,$$

δ and all $\frac{x_u}{h}$, $\frac{y_v}{k}$ remaining fixed. The spacing S_0 is called an optimum spacing. It

should be observed that for finite values of h, the maximum absolute value of the remainder will in general not be minimized exactly, because of the unpredictable nature of the variation of the partial derivatives in equation (4).

2. Optimum-Interval Determination. Let

$$c_\xi = \frac{1}{(m + 1)!} \frac{\partial^{m+1} f(\xi, y)}{\partial \xi^{m+1}} ; \qquad c_\eta = \frac{1}{(n + 1)!} \frac{\partial^{n+1} f(x, \eta)}{\partial \eta^{n+1}} ;$$

$$\text{and} \quad c_{\xi'\eta'} = \frac{1}{(m + 1)!(n + 1)!} \frac{\partial^{(m+1)+(n+1)} f(\xi'\eta')}{\partial \xi'^{m+1} \partial \eta'^{n+1}} .$$

Then

$$(5) \qquad \frac{c_{\xi'\eta'}}{c_\xi \, c_\eta} R_{mn} = P + Q + PQ$$

where

$$(6) \qquad \begin{aligned} P &= P(x, \xi', \eta', \eta) = \frac{c_{\xi'\eta'}}{c_\eta} \prod_{u=0}^{m} (x - x_u); \\ Q &= Q(y, \xi', \eta', \xi) = \frac{c_{\xi'\eta'}}{c_\xi} \prod_{v=0}^{n} (y - y_v). \end{aligned}$$

As was assumed above, the partial derivatives are not equal to zero, so that their bounds can be identified as follows:

$$a_1 \leqq |c_\xi| \leqq a_2 \, ; \, b_1 \leqq |c_\eta| \leqq b_2 \, ; \, c_1 \leqq |c_{\xi'\eta'}| \leqq c_2.$$

Since

$$(7) \qquad \prod_{u=0}^{m} (x - x_u) \leqq (2h)^{m+1} \text{ and } \prod_{v=0}^{n} (y - y_v) \leqq (2k)^{n+1}$$

the maximum absolute value, M, of (PQ) within (h, k) regardless of spacing, satisfies the inequality

$$(8) \qquad M \leqq \frac{c_2^2}{a_1 \, b_1} (2h)^{m+1} (2k)^{n+1}.$$

It will be shown that an optimum spacing, S_0, is that spacing obtained when x_u, y_v are given by the roots of $T_{m+1}(x/h) = 0$, $T_{n+1}(y/k) = 0$, where $T_k(z)$ is the polynomial of degree k in z, with leading coefficient unity, deviating least from zero within the interval $(-1, 1)$ [2]. These roots are:

$$(9) \qquad x_u = -h \cos \frac{(2u + 1)\pi}{(2m + 2)} ; \qquad y_v = -k \cos \frac{(2v + 1)\pi}{2n + 2} .$$

This spacing will also be called *Chebyshev spacing*. Thus, if $W^* = |P| + |Q|$

$$(10) \qquad \left(\frac{c_1}{b_2} 2^{-m} h^{m+1} + \frac{c_1}{a_2} 2^{-n} k^{n+1} \right) \leqq W \leqq \left(\frac{c_2}{b_1} 2^{-m} h^{m+1} + \frac{c_2}{a_1} 2^{-n} k^{n+1} \right)$$

where W is equal to $W^*_{\max}$ with Chebyshev spacing.

It follows from relations (8) and (10) that

$$(11) \qquad \lim_{h \to 0} \frac{M}{W} = 0, \qquad \delta \text{ and all } x_u/h, y_v/k \text{ remaining fixed.}$$

We can now determine a non-zero lower limit on the maximum absolute value of R_{mn} as follows:

We restrict ourselves to such rectangles (h, k) for which M/W is less than unity, which, using relations (8) and (10), is certainly true when

$$h < \left\{ \frac{a_1 b_1 c_1}{2^{m+n+2} c_2{}^2 \left(\frac{b_2 2^m}{\delta^{n+1}} + a_2 2^n \right)} \right\}^{\frac{1}{1+\mu}} \qquad \text{where} \quad \begin{array}{l} \mu = m \text{ if } m < n \\ \mu = n \text{ if } n < m. \end{array}$$

For any spacing whatsoever, consider equation (5) at the point (x^*, y^*) at which W^* has its maximum value within (h, k):

$$(12) \qquad \left| \frac{c_{\xi'\eta'} R_{mn}}{c_\xi c_\eta} \right|_{(x^*, y^*)} = W^*_{\max} \pm \lambda M \qquad (0 \leqq \lambda \leqq 1)$$

Since the extreme values of $|P|$ and $|Q|$ occur independently, it follows that $W^*_{\max} \geqq W_{\min}$, where $W_{\min}$ is the lower bound for W in equation (10), i.e., $W_{\min} = \frac{c_1}{b_2} 2^{-m} h^{m+1} + \frac{c_1}{a_2} 2^{-n} k^{n+1}$.

Hence,

$$(13) \qquad \left. \left| \frac{c_{\xi'\eta'}}{c_\xi c_\eta} \right| |R_{mn}| \right]_{(x^*, y^*)} \geqq W_{\min} - M > 0.$$

We are thus led to define R by the equation

$$(14) \qquad \frac{c_2}{a_1 b_1} R = W_{\min} - M.$$

We know then that for any spacing, there exists a point (x^*, y^*) within (h, k) for which $|R_{mn}| \geqq R$. Let $R_0 = |R_{mn}|_{\max}$ for Chebyshev spacing.

$$(15) \qquad \lim_{h \to 0} \left(\frac{R_0}{R} \right) = \lim_{h \to 0} \left(\frac{W + \lambda M}{W_{\min} - M} \right) = \lim_{h \to 0} \left(\frac{1 + \lambda \left(\frac{M}{W} \right)}{1 - \left(\frac{M}{W_{\min}} \right)} \right) = 1,$$

δ and all x_u/h, y_v/k remaining fixed. Thus, optimum spacing (as defined above) is obtained when x_u, y_v are respectively the zeros of the Chebyshev polynomials $T_{m+1}(x/h)$ and $T_{n+1}(y/k) = 0$.

3. Bi-variate Polynomials $P_{mn}(x, y)$ With Leading Coefficient Unity Deviating Least From Zero Within (h, k).

Let $P_{mn}(x, y)$, defined as in equation (1), but with leading coefficient unity, $(a_{mn} = 1)$, be written as follows:

(16) $P_{mn}(x, y) = x^m \phi_m(y) + x^{m-1} \phi_{m-1}(y) + \cdots + x \phi_1(y) + \phi_0(y)$

where $\phi_k(y) = \sum_{j=0}^n a_{kj} y^j (k = 0, 1, 2 \cdots m)$ and $\phi_m(y)$ has leading term y^n. Let $y = y_m^*$ be a value of y for which $| \phi_m(y) |$ reaches its extreme value within (h, k). Then

(17) $$| P_{mn}(x, y) |_{\max} > | \phi_m(y_m^*) | \, | Q_{mn}(x, y_{mn}^*) |_{\max}$$

where

(18) $$Q_{mn}(x, y_m^*) = x^m + \sum_{k=0}^{n-1} \frac{\phi_k(y_m^*)}{\phi_m(y_m^*)} \cdot x^k.$$

The least possible values of $| \phi_m(y_m^*) |$ and $| Q_{mn} |_{\max}$ are obtained if

(19) $$\begin{cases} \phi_m(y) = k^n T_n(y/k) \\ Q_{mn}(x, y_m^*) = h^m T_m(x/h) \end{cases}.$$

For this to be the case it is necessary that

(20) $$\frac{\phi_k(y_m^*)}{\phi_m(y_m^*)} = \alpha_k \qquad \begin{array}{l} k = 0, 1, 2 \cdots (m-1) \\ m^* = 1, 2, \cdots (n+1) \end{array}$$

and α_k is the coefficient of x^k in the Chebyshev polynomial $h^m T_m(x/h)$. Any of the equations (20) is an algebraic equation in y_m^* of degree n or less, having at most n real roots. However, $\phi_m(y_m)$ in equation (19) has an identical extreme value (except for sign variation) at least at $(n + 1)$ points. Hence for equation (20) to hold at these $(n + 1)$ points it must represent a set of identities, i.e. $\phi_k = \alpha_k k^n T_n(y/k)$, hence

(21) $$P_{mn}(x, y) = h^m k^n T_m \left(\frac{x}{h} \right) T_n \left(\frac{y}{k} \right).$$

This polynomial deviates least from zero within (h, k) with maximum deviation, L, given by $L = 4 h^m k^n 2^{-(m+n)}$. Note that the *lines* $x = x_u$, $y = y_v$ have replaced the *points* x_u in the one-dimensional theory and that the symmetry $P_{mn}(\pm x, \pm y) = P_{mn}(x, y) = P_{mn}(y, x)$ is preserved. Generalization to n-dimensions follows at once.

The above developments arose in the course of work on the synthesis of mechanisms having two degrees of freedom and the author has also utilized with benefit and appreciation the works on interpolation by Biermann [3], Hildebrand [4] and Kunz [5].

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