$$N = a^2 - b^2,$$

where

$$a = (2^3 \cdot 5 \cdot 3^2 \cdot 53^2)x + 11\ 150\ 802\ 925.$$

This representation of a can be deduced from theory presented by Kraitchik [1], combined with the fact that both -1 and 5 are quadratic residues of N, as established by suitable representations of N by quadratic forms.

Corresponding to $x = 102908$, $a^2 - N$ is the square of $b = 114674787084$. Hence, N is the difference of the squares of $a = 115215488845$ and of the preceding value of b. Thus

$$N = 540\ 701\ 761 \cdot 229\ 890\ 275\ 929.$$

The primality of each of these factors was determined in a similar manner. The factorization of $2^{159} - 1$ is, therefore, now complete.

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1. M. Kraitchik, *Théorie des Nombres*, Gauthier-Villars et Cie, Paris, 1922, p. 146.

Two Formulas Relating to Elliptic Integrals of the Third Kind

By J. Boersma

Using Legendre's notation, the normal elliptic integral of the third kind is defined by the equation

$$\prod(\phi, \alpha^2, k) = \int_0^\phi \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}.$$

For $k^2 < 1$, the following expansion holds uniformly over the closed interval $0 \leqq \theta \leqq \frac{\pi}{2}$:

$$\frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (-1)^m k^{2m} \sin^{2m} \theta,$$

where $\binom{-\frac{1}{2}}{m} = \frac{(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{1}{2} - m + 1)}{m!}$ for $m > 0$, and $\binom{-\frac{1}{2}}{0} = 1.$

The factor $\frac{1}{1 - \alpha^2 \sin^2 \theta}$ in the integrand is bounded for $-\infty < \alpha^2 < \frac{1}{\sin^2 \phi}$ and $0 \leqq \theta \leqq \phi$; consequently, the expanded integrand may be integrated term by term. Such integration yields the series

$$\prod (\phi, \alpha^2, k) = \sum_{m=0}^{\infty} b_m k^{2m},$$

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where

$$b_m = \binom{-\frac{1}{2}}{m}(-1)^m \int_0^\phi \frac{\sin^{2m}\theta}{1 - \alpha^2 \sin^2\theta}\, d\theta, m > 0,$$

and

$$b_0 = \int_0^\phi \frac{d\theta}{1 - \alpha^2 \sin^2\theta} = \frac{1}{\sqrt{1-\alpha^2}} \tan^{-1}[\sqrt{1-\alpha^2}\tan\phi], \text{for } -\infty < \alpha^2 < 1,$$

$$= \tan\phi, \text{for } \alpha^2 = 1,$$

$$= \frac{1}{\sqrt{\alpha^2 - 1}} \tanh^{-1}[\sqrt{\alpha^2 - 1}\tan\phi], \text{for } 1 < \alpha^2 < \frac{1}{\sin^2\phi}.$$

In general, the coefficients b_m satisfy the recurrence relation

$$2(m+1)\alpha^2 b_{m+1} = (-1)^{m+1}(2m+1)\binom{-\frac{1}{2}}{m} t_{2m}(\phi) + (2m+1)b_m,$$

where $t_{2m}(\phi) = \int_0^\phi \sin^{2m}\theta\, d\theta$.

Byrd and Friedman [1] give [formula (902.00)] the recurrence relation

$$t_{2m}(\phi) = \frac{2m-1}{2m} t_{2m-2}(\phi) - \frac{1}{2m}\sin^{2m-1}\phi\cos\phi$$

and explicit expressions for $t_0(\phi)$, $t_2(\phi)$, and $t_4(\phi)$. Corresponding to these we find

$$b_1 = \frac{b_0 - \phi}{2\alpha^2}$$

$$b_2 = \frac{1}{16\alpha^4}[3\alpha^2\sin\phi\cos\phi + 6b_0 - 3(2+\alpha^2)\phi]$$

$$b_3 = \frac{5}{128\alpha^6}[2\alpha^4\sin^3\phi\cos\phi + \alpha^2(3\alpha^2+4)\sin\phi\cos\phi + 8b_0 - (8 + 4\alpha^2 + 3\alpha^4)\phi].$$

When $\phi = \frac{\pi}{2}$, $-\infty < \alpha^2 < 1$, and $k^2 < 1$, we deduce the following expansion of the complete elliptic integral of the third kind:

$$\Pi(\alpha^2, k) \equiv \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) = \sum_{m=0}^{\infty} c_m k^{2m},$$

where

$$c_0 = \frac{\pi}{2\sqrt{1-\alpha^2}},$$

$$c_1 = \frac{\pi}{4\alpha^2}\left[\frac{1}{\sqrt{1-\alpha^2}} - 1\right],$$

$$c_2 = \frac{3\pi}{32\alpha^4}\left[\frac{2}{\sqrt{1-\alpha^2}} - 2 - \alpha^2\right],$$

$$c_3 = \frac{5\pi}{256\alpha^6}\left[- 4\alpha^2 - 3\alpha^4 - 8 + \frac{8}{\sqrt{1 - \alpha^2}}\right];$$

and, in general, the coefficients satisfy the recurrence formula

$$2(m + 1)\alpha^2 c_{m+1} = - \left(m + \frac{1}{2}\right) \pi \binom{-\frac{1}{2}}{m}^2 + (2m + 1)c_m ,$$

which follows from the recurrence formula for b_m when use is made of the definite integral

$$t_{2m}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \sin^{2m}\theta \, d\theta = \frac{1}{2} \cdot \frac{\Gamma(m + \frac{1}{2})\Gamma(\frac{1}{2})}{m!}$$
$$= \frac{\pi}{2}(- 1)^m \binom{-\frac{1}{2}}{m}.$$

The expansions obtained above for $\prod (\phi, \alpha^2, k)$ and $\prod (\alpha^2, k)$ constitute extensions and simplifications of formulas (906.01) and (906.00), respectively, in the book already cited, by Byrd and Friedman. Furthermore, the coefficient of α^2 has been corrected here in the expression for c_3 appearing in (906.00).

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1. PAUL F. BYRD & MORRIS D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, Berlin, 1954.