Note on "Approximation of Curves by Line Segments"

By N. Ream

The problem of obtaining a best fit of broken line segments to a curve over a given range has recently been investigated by Stone [1] who has prepared a general computer program to solve the least-squares equations.

The problem arose previously in designing diode function-generators for analog computers [2], [3], [4]. If $f(x)$ is the given curve and (u_0, u_N) is the range to be fitted by N segments, and if $f(x)$ may be approximated by a parabola in each segment, then it may be shown [4] that the unweighted least-squares fit yields the following equation for the breakpoints $u_1, \cdots, u_{N-1}$:

$$\int_{u_0}^{u_j} \{f''(x)\}^{0.4}\, dx = \frac{j}{N} \int_{u_0}^{u_N} \{f''(x)\}^{0.4}\, dx, \tag{1}$$

and that the ordinate v_j of each breakpoint is given by

$$v_j - f(u_j) = -\frac{1}{12N^2} \{f''(u_j)\}^{0.2} \left[\int_{u_0}^{u_N} \{f''(x)\}^{0.4}\, dx\right]^2. \tag{2}$$

For $f(x) = e^{-cx}$ fitted over $(0, 3)$, equations (1) and (2) become

$$1 - e^{-0.4cu_j} = \frac{j}{N}(1 - e^{-1.2c}), \tag{3}$$

$$v_j - e^{-cu_j} = -\frac{25}{48N^2} e^{-0.2cu_j}(1 - e^{-1.2c})^2. \tag{4}$$

Table 1 gives values of u_1 and maximum error $E_{\max}$ computed from (3) and (4) for $N = 2$; Stone's values are shown in parentheses. $E_{\max}$ occurs at $x = 0$. The table also gives values of the r.m.s. error R which the least-squares analysis aims to minimize; R is computed from the formula

$$(u_N - u_0)R^2 = (1/720N^4) \left[\int_{u_0}^{u_N} \{f''(x)\}^{0.4}\, dx\right]^5, \tag{5}$$

which for the chosen function becomes

$$R = (6c)^{-0.5} E_{\max}. \tag{6}$$

The derivation of equations (1), (2), and (5) involves expanding $f(x)$ in a Taylor series about the center of each segment and retaining the first three terms. Hence (i) the formulas are exact for a parabola—it follows immediately that the best fit to a parabola has equally-spaced breakpoints; (ii) the method fails where $f''(x) = 0$.

It may be mentioned that if the "best fit" is required to minimize the maximum

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TABLE 1

$f(x) = e^{-cx}$ *fitted with 2 segments over* $(0,3)$

c	u_1		$E_{\max}$		R	E
0.1	1.454	(1.385)	0.00166	(0.0016)	0.00215	0.00121 ± 0
0.2	1.410	(1.400)	0.00593	(0.0059)	0.00541	0.00420 ± 0
0.3	1.366	(1.360)	0.0119	(0.0119)	0.00887	0.00821 ± 1
0.4	1.322	(1.316)	0.0189	(0.0189)	0.0122	0.0127 ± 0
0.5	1.278	(1.276)	0.0265	(0.0265)	0.0153	0.0174 ± 1
0.6	1.236	(1.235)	0.0343	(0.0344)	0.0181	0.0221 ± 1
0.7	1.194	(1.196)	0.0420	(0.0423)	0.0205	0.0264 ± 2
0.8	1.153	(1.155)	0.0496	(0.0500)	0.0226	0.0305 ± 3
0.9	1.113	(1.116)	0.0568	(0.0574)	0.0244	0.0343 ± 5
1.0	1.074	(1.080)	0.0636	(0.0645)	0.0260	0.0377 ± 7
1.2	1.001	(1.008)	0.0758	(0.0774)	0.0283	0.0435 ± 13
1.5	0.900	(0.912)	0.0907	(0.0936)	0.0302	0.0500 ± 26

error, the breakpoints are given by equations (1) with the index 0.4 replaced by 0.5 and $f''(x)$ replaced by its absolute value. The maximum error E is then given by

$$E = \left[\frac{1}{4N}\int_{u_0}^{u_N} |f''(x)|^{0.5}\,dx\right]^2. \tag{7}$$

For the function under discussion (7) becomes

$$E = \frac{1}{4N^2}\,(1 - e^{-1.5c})^2, \tag{8}$$

and the error δE in E due to the approximations used in deriving (8) may be shown to be given by

$$\delta E \cong \tfrac{1}{9}E^2 e^{1.5c}. \tag{9}$$

Values of E and δE are included in the table.

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1. H. STONE, "Approximation of curves by line segments," *Math. Comp.*, v. 15, 1961, p. 40–47.

2. H. HAMER, "Optimum linear-segment function generation," *Trans. Amer. Inst. Elec. Engrs.*, v. 75, 1956, p. 518–520.

3. M. E. FISHER, "The optimum design of quarter-squares multipliers with segmented characteristics," *J. Sci. Instrum.*, v. 34, 1957, p. 312–316.

4. N. REAM, "Approximation errors in diode function-generators," *J. Electronics Control*, v. 7, 1959, p. 83–96.