A Note on the Convergence of Alternating Direction Methods

By Milton Lees

1. Introduction. Convergence of the single-parameter alternating direction methods of Douglas, Peaceman, and Rachford [1], [2], [3] has been proved for a wide class of elliptic difference equations; see, for example, Birkhoff and Varga [4]. The proof consists in showing that a certain matrix, similar to the defining matrix, has spectral radius less than one. Since the defining matrices for these methods are symmetric only when they are induced by a proper discretization of Laplace's equation in a rectangular region, an estimate for their spectral radii does not imply a corresponding norm estimate for their rate of convergence.

The purpose of this note is to present another proof of the convergence of the two basic alternating direction methods which, at the same time, provides a norm estimate for their rate of convergence. First, for simplicity, we shall consider Laplace's equation in an arbitrary, bounded lattice region. We shall prove that it is possible to select the acceleration parameter so that a suitable norm of the error is reduced after each iteration by the amount $1 - ch + 0(h^2)$, where h is the uniform node spacing for the lattice, and $c > 0$ depends only on the minimal eigenvalue for the Laplace operator. We shall indicate only briefly the extension of these results to elliptic equations with variable coefficients.

Our proof of the convergence of the alternating direction methods was motivated by two considerations; first, the close resemblance of the iteration equations to parabolic difference equations, and second, an integral estimate technique for proving that, under suitable conditions, the solutions of parabolic differential equations decay exponentially.

2. Formulation of the Problem. Let $\mathcal{L}$ denote the (uniform) lattice of side h determined by the nodes $(\alpha h, \beta h)$, α and β being integers, positive, negative, or zero. Denote by $\bar{S}$ any finite subset of $\mathcal{L}$. In the usual way [5], we decompose $\bar{S}$ into two disjoint subsets: S, the interior nodes of $\bar{S}$, and ∂S, the boundary nodes of $\bar{S}$.

If Q is any subset of $\mathcal{L}$, we denote by $\mathcal{C}(Q)$ the linear space of all real-valued functions on $\mathcal{L}$ whose support is contained in Q. (In general, the support of a function f is the closure of the set $\{p \mid f(p) \neq 0\}$.) Note that $\mathcal{C}(S)$ is finite-dimensional with dimension equal to the number of nodes in S. The advantage of thus trivially extending functions on Q to all of $\mathcal{L}$ will become apparent later.

We denote by Δ the usual "five-point" Laplace difference operator, i.e.,

$$\Delta u = u_{x\bar{x}} + u_{y\bar{y}}, u \in \mathcal{C}(Q),$$

where the subscripts x and y denote forward difference quotients, and $\bar{x}$ and $\bar{y}$ denote backward difference quotients [6].

We shall be concerned, first, with the Dirichlet problem for the elliptic difference equation

(1.1) $$\Delta u = f \quad \text{in} \quad S,$$

where $f \; \epsilon \; \mathcal{C}(S)$. More precisely, we seek a function $u \; \epsilon \; \mathcal{C}(\bar{S})$ that satisfies (1.1) and assumes arbitrary prescribed values on ∂S.

It is easy to see that this problem is equivalent to the problem of solving a linear system of algebraic equations of order equal to the number of nodes in S. By means of the familiar maximum principle for the difference operator Δ, one can show that this system of linear equations always has a unique solution. In this paper we shall be concerned with the two basic, single-parameter alternating direction methods for the inversion of this system of equations.

3. The Alternating Direction Methods. Let $w(0) \; \epsilon \; \mathcal{C}(\bar{S})$ be a function that agrees with u on ∂S, and otherwise arbitrary in S; $w(0)$ will serve as the initial approximation to u. Now, for $n \geq 1$, we determine two sequences of functions $\{w(n)\}$ and $\{w^*(n)\}$ in $\mathcal{C}(\bar{S})$ such that

(2.1) $$w(n) = w^*(n) = u \quad \text{on} \quad \partial S,$$

and

(2.2)
(a) $\quad \rho^{-1}[w^*(n) - w(n-1)] = w_{x\bar{x}}^*(n) + w_{y\bar{y}}(n-1) - f$

(b) $\quad \rho^{-1}[w(n) - w^*(n)] = w_{y\bar{y}}(n) - w_{y\bar{y}}(n-1)$

in S. Here $\rho > 0$, the so-called acceleration parameter, is to be selected later. These equations define what is usually referred to as the Douglas-Rachford alternating direction method for solving $\Delta u = f$.

In [3] it is shown that (2.1) and (2.2) determine uniquely (alternately) the functions $w^*(n)$ and $w(n)$ in terms of $w(n-1)$ and the values of u on ∂S. This operation taking $w(n-1)$ into $w(n)$ involves only the inversion of approximately $2\sqrt{p}$ tridiagonal matrices, where p is the number of nodes in S.

Following Douglas and Rachford, we eliminate from (2.2) the auxiliary function $w^*(n)$. Adding (2.2a) to (2.2b), we obtain the equation

(2.3) $$Dw(n) = w_{x\bar{x}}^*(n) + w_{y\bar{y}}(n) - f,$$

where we have introduced the notation

$$Dw(n) = \rho^{-1}[w(n) - w(n-1)].$$

Solving (2.2b) for w^*, we get

$$w^*(n) = w(n) - \rho^2 Dw_{y\bar{y}}(n).$$

Hence,

$$w_{x\bar{x}}^*(n) = w_{x\bar{x}}(n) - \rho^2 \Lambda Dw(n),$$

where Λ is defined as follows:

$$\Lambda u = u_{x\bar{x}y\bar{y}} \, .$$

Putting the value just obtained for $w_{x\bar{x}}^*(n)$ into (2.3), we find that w is a solution of the fifth-order difference equation

(2.4) $$Dw = \Delta w - \rho^2 \Lambda Dw - f, \quad \text{in} \quad S, \quad \text{for} \quad n \geq 1.$$

The other alternating direction method, the Peaceman-Rachford method, is very similar to this; it is characterized by the equations

$$\rho^{-1}[w^*(n) - w(n-1)] = w_{x\bar{x}}^*(n) + w_{y\bar{y}}(n-1) - f$$

and

$$\rho^{-1}[w(n) - w^*(n)] = w_{x\bar{x}}^*(n) + w_{y\bar{y}}(n) - f.$$

After eliminating from these equations the auxiliary function $w^*(n)$, we find that w is a solution of the fifth-order difference equation

(2.5) $$Dw = 2\Delta w - \rho \Delta Dw - \rho^2 \Lambda Dw - 2f \quad \text{in} \quad S, \quad \text{for} \quad n \geq 1.$$

Our problem, now, is to show that $w(n) \to u$, as $n \to \infty$.

4. Preparation. We define on $\mathcal{C}(S)$ an inner product $\langle u, v \rangle$ as follows:

$$\langle u, v \rangle = h^2 \sum_{P \in \mathcal{L}} u(P)v(P).$$

The norm $\langle v, v \rangle^{1/2}$ induced by this inner product will be denoted by $\| v \|$. We require two additional norms for $\mathcal{C}(S)$:

$$\| v \|_1 = (\| v_{\bar{x}} \|^2 + \| v_{\bar{y}} \|^2)^{1/2}$$

and

$$\| v \|_2 = (\| v \|^2 + \rho^2 \| v_{\bar{x}\bar{y}} \|^2)^{1/2}.$$

A subscript S will be placed on these quantities whenever we want to indicate that the sum is to be extended only over S.

In [7] it is proved that the minimal eigenvalue λ of the Laplace difference operator Δ, relative to S, can be characterized as follows:

$$\lambda = \inf_{0 \neq v \in \mathcal{C}(S)} \frac{\| v \|_{1,S}^2}{\| v \|^2}.$$

Since $\| v \|_{1,S} \leq \| v \|_1$, it follows immediately that

(3.1) $$\lambda \| v \|^2 \leq \| v \|_1^2,$$

for all $v \in \mathcal{C}(S)$.

On the strength of this inequality, we now prove the

LEMMA 1. *Any function $v \in \mathcal{C}(S)$ satisfies the inequality*

(3.2) $$\| v \|_1^2 \geq \lambda_0 \| v \|_2^2,$$

where

(3.3) $$\lambda_0 = \frac{\lambda}{1 + 2\lambda \rho^2 h^{-2}}.$$

Proof. We have from (3.1) and the definition of $\| v \|_2$ that

$$\| v \|_1^2 - \lambda \| v \|_2^2 \geq -\lambda \rho^2 \| v_{\bar{x}\bar{y}} \|^2.$$

To this inequality we apply the elementary inequality [8]

$$h^2 \| v_{\bar{x}\bar{y}} \|^2 \leqq 2 \| v \|_1^2,$$

valid for all $v \, \epsilon \, \mathfrak{C}(S)$, and we find that

$$\| v \|_1^2 - \lambda \| v \|_2^2 \geqq -2\lambda\rho^2 h^{-2} \| v \|_1^2,$$

which is equivalent to (3.2).

5. Convergence of the Alternating Direction Methods. Let $u \, \epsilon \, \mathfrak{C}(\bar{S})$ be the solution of the Dirichlet problem for $\Delta u = f$ in S, and let $w(n)$ be the corresponding solution of the Douglas-Rachford equation (2.4). Then, by linearity, the error function $v(n) = u - w(n)$ belongs to the linear space $\mathfrak{C}(S)$, and satisfies, for $n \geqq 1$, the difference equation

(4.1) $$Pv \equiv Dv - \Delta v + \rho^2 \Lambda Dv = 0$$

in S. Let $\phi(n) = (1 + \rho\mu)^{-n}$, where $\mu > 0$. One verifies immediately that ϕ is a solution of the difference equation

(4.2) $$D\phi = -\mu\phi.$$

THEOREM 1. *If $\mu \leqq \lambda_0$, then the error function $v(n)$ for the Douglas-Rachford method satisfies the inequalities*

$$\| v(n) \|_i \leqq \phi(n) \| v(0) \|_i \qquad\qquad (i = 1, 2)$$

Proof. We set

(4.3) $$M = 2\rho \sum_{\xi=1}^{n} \phi^{-2}(\xi) \, \langle v(\xi), Pv(\xi) \rangle.$$

Since $v(\xi) \, \epsilon \, \mathfrak{C}(S)$, we see that $M = 0$.

Introduce the function $z(\xi) \, \epsilon \, \mathfrak{C}(S)$ by means of the formula

(4.4) $$v(\xi) = \phi(\xi)z(\xi).$$

Then we have that

$$Dv = \phi Dz + zD\phi - \rho D\phi \cdot Dz$$

which, in view of (4.2) becomes

$$Dv = \phi[aDz - \mu z],$$

where $a = \phi^{-1}(1) > 0$.

From this and (4.1) it follows that

(4.5) $$Pv = \phi[aDz - \mu z - \Delta z + a\rho^2 \Lambda Dz - \rho^2 \mu \Lambda z] = \phi[aDz + Bz],$$

where B is defined in the obvious way.

Because of (4.5), (4.3) becomes

(4.6) $$M = \rho \sum_{\xi=1}^{n} 2 \, \langle z(\xi), aDz(\xi) + Bz(\xi) \rangle.$$

We now recall from [8] the following quadratic difference identities

(4.7) $$2 \langle z, Dz \rangle = D\| z \|^2 + \rho\| Dz \|^2,$$

(4.8) $$2 \langle z, \Lambda Dz \rangle = D\| z_{\bar{x}\hat{y}} \|^2 + \rho\| Dz_{\bar{x}\hat{y}} \|^2,$$

(4.9) $$\langle z, \Delta z \rangle = -\| z \|_1^2,$$

and

(4.10) $$\langle z, \Lambda \dot{z} \rangle = \| z_{\bar{x}\hat{y}} \|^2.$$

Using these identities in (4.6), we obtain, after some slight rearrangement, the identity

(4.11) $$M = \rho \sum_{\xi=1}^{n} \{aD\| z \|_2^2 + 2(\| z \|_1^2 - \mu \| z \|_2^2) + a^2\| Dz \|_2^2 + \mu a\| Dz \|_1^2\}.$$

Since $\| z \|_1^2 \geq \lambda_0 \| z \|_2^2$, by Lemma 1, and $\mu \leq \lambda_0$, we have from (4.11) that

$$M \geq \rho \sum_{\xi=1}^{n} aD\| z \|_2^2 = a\| z(n) \|_2^2 - a\| z(0) \|_2^2.$$

But, since $M = 0$, we obtain from this and (4.4) that

$$\| v(n) \|_2 = \phi(n)\| z(n) \|_2 \leq \phi(n)\| z(0) \|_2 = \phi(n)\| v(0) \|_2,$$

which proves the second inequality of the theorem.

Next, we set

(4.12) $$N = \rho \sum_{\xi=1}^{n} \phi^{-2}(\xi)\| Pv(\xi) \|_S^2.$$

Then, by (4.4) and (4.5), (4.12) becomes

$$N = \rho \sum_{\xi=1}^{n} \| aDz(\xi) + Bz(\xi) \|_S^2.$$

But

$$\| aDz + Bz \|_S^2 = a^2\| Dz \|_S^2 + \| Bz \|_S^2 + 2a\langle Dz, Bz \rangle_S \geq a^2\| Dz \|^2 + 2a\langle Dz, Bz \rangle,$$

where we have dropped the subscript S, because $Dz \in \mathcal{C}(S)$. Hence,

(4.13) $$N \geq \rho \sum_{\xi=1}^{n} \{a^2\| Dz \|^2 + 2a\langle Dz, Bz \rangle\}.$$

Using the identities (4.7)–(4.10) again, we obtain from (4.13) the inequality

$$N \geq \rho \sum_{\xi=1}^{n} \{a^2\| Dz \|^2 + aD\| z \|_1^2 - \mu aD\| z \|_2^2 + a\rho\| Dz \|_1^2\}.$$

Therefore, by adding a combination of the above inequalities,

$$N + \mu M \geq \rho a \sum_{\xi=1}^{n} D\| z \|_1^2.$$

Again, $N + \mu M = 0$, so that

$$\| z(n) \|_1 \leq \| z(0) \|_1,$$

which, in view of (4.4), implies that

$$\| v(n) \|_1 \leq \phi(n)\| v(0) \|_1.$$

This completes the proof of the theorem.

Since $\rho\mu > 0$, Theorem 1 implies that the Douglas-Rachford method is convergent, but we also have the stronger result:

THEOREM 2. *If* $\mu = \lambda_0$ *and* $\rho = h/\sqrt{2\lambda}$, *then the error function* $v(n)$ *for the Douglas-Rachford alternating direction method satisfies the inequalities*

$$(4.14) \qquad \| v(n) \|_i \leq \left(1 + \frac{\sqrt{2\lambda}}{4} h \right)^{-n} \| v(0) \|_i \qquad (i = 1, 2).$$

Proof. The inequality (4.14) follows from Theorem 1, by minimizing $\phi(n)$ as a function of ρ.

In an analogous fashion, we can prove

THEOREM 3. *If* $\mu = 2\lambda_0(1 - \lambda_0\rho)^{-1}$ *and* $\rho = h/\sqrt{2\lambda}$, *then the error function* $v(n)$ *for the Peaceman-Rachford alternating direction method satisfies the inequality*

$$\| v(n) \|_1 \leq \left(\frac{4 - \sqrt{2\lambda}h}{4 + \sqrt{2\lambda}h}\right)^{n} \| v(0) \|_1 .$$

Note that the Peaceman-Rachford method has a smaller convergence factor.

The preceding results can be extended to more general elliptic difference equations, e.g., to those that come from elliptic differential equations of the form

$$\frac{\partial}{\partial x}\left(a(x, y) \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b(x, y) \frac{\partial u}{\partial y}\right) = f(x, y)$$

in a bounded, open set Ω. The functions a and b must belong to $c^1(\bar{\Omega})$. The main point that we wish to emphasize is that, because the inner product $\langle u, v \rangle$ is defined as a sum over all of $\mathfrak{L}$, one must be able to extend a and b in a continuously differentiable way to a slightly larger set $\omega \supset \bar{\Omega}$. This can be carried out, for instance, by means of the Whitney extension theorem [9], in a form given by Hörmander [10].

One can also show that similar results can be proved for the Douglas-Rachford method as applied to elliptic equations in n independent variables, although this is not the case with the Peaceman-Rachford method [8].

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